ON COVERING HOMOTOPY THEOREMS

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(Received October 10, 1951)

The first and the second covering homotopy theorems [1; § 11. 3 and § 11. 7][1] and the results which follow from these play important rôles in the theory of fibre bundles. In this paper we shall consider covering homotopy theorems. In §1, we deal with bundles which have totally disconnected groups. §2 is concerned with bundles whose bundle spaces are compact.

1. Let X be an arcwise connected space, and let $\mathfrak{B}$ be a bundle over the base space X with fibre Y and totally disconnected group G. According to [1; § 13. 5], an equivalence class $\chi(\mathfrak{B})$ of homomorphisms of the fundamental group $\pi_1(X)$ into G under inner automorphisms of G is defined, which is called the characteristic class of $\mathfrak{B}$. First, we shall recall the definition and some properties of $\chi(\mathfrak{B})$.

Choose a reference point x_0 in X. Let Y_0 be the fibre over x_0, and choose an admissible map $\zeta : Y \to Y_0$. For each element $\alpha \in \pi_1(X, x_0)$, there exists a unique admissible map $\alpha : Y_0 \to Y_0$ (see [1; § 13. 1-3]). Therefore $\xi^{-1}\alpha\xi$ represents an element $\chi(\mathfrak{B}; x_0, \xi)(\alpha)$ of G. This correspondence

$$\chi(\mathfrak{B}; x_0, \xi) : \pi_1(X, x_0) \to G$$

is a homomorphism. The equivalence class of $\chi(\mathfrak{B}; x_0, \xi)$ under inner automorphisms of G is $\chi(\mathfrak{B})$. Let $\zeta : Y \to Y_0$ be another admissible map. Then the homomorphisms $\chi(\mathfrak{B}; x_0, \xi)$ and $\chi(\mathfrak{B}; x_0, \zeta)$ differ by an inner automorphism of G. That is, $\chi(\mathfrak{B})$ is independent upon the choice of ξ.

Suppose x_1 is another reference point, and $\xi_1 : Y \to Y_1$ is an admissible map. Since X is arcwise connected, there is a path D which binds x_0 to x_1. If C is a closed path from x_1, then DCD^{-1} is a closed path from x_0. As is well known, D induces in this way an isomorphism $D_* : \pi_1(X, x_1) \to \pi_1(X, x_0)$. Then homomorphisms $\chi(\mathfrak{B}; x_0, \xi) D$ and $\chi(\mathfrak{B}; x_1, \xi)$ differ by an inner automorphism of G. (See [1; § 13. 5]).

Suppose now X' is an arcwise connected space, and $f : X' \to X$ is a continuous map. If C is a closed path from a fixed point x_0' of X', then fC is a closed path from the point $f(x_0')$. In this way, f induces a natural homomorphism

$$f^{\#} : \pi_1(X', x_0') \to \pi_1(X, f(x_0')).$$

Let $f : f^{-1}\mathfrak{B} \to \mathfrak{B}$ be the induced bundle map [1; § 10] and $\zeta : Y \to Y_0'$ be an admissible map, where Y_0' denotes the fibre over $x_0' \in X'_0$. Then we have the

1) Numbers in square brackets refer to the references at the end of the paper.

following relation.

LEMMA 1.1. $\chi(f^{-1}\mathfrak{B};\ x_0',\zeta)=\chi(\mathfrak{B},f(x_0'),\ f\zeta)f^{\#}$.

PROOF. Let C' be a closed path from x_0' which represents an element α' of $\pi_1(X',\ x_0')$. Let h' be a homotopy of $\zeta: Y\to f^{-1}\mathfrak{B}$ such that the induced map is $C'(t)$. For each $t\in I$, we define a map $h_t': Y\to Y_t'$ by taking $h_t'(y)=h'(y,\ t)\,(y\in Y)$, where Y_t' denotes the fibre over the point $C'(t)$. Then the admissible map $\alpha': Y_0'\to Y_0'$ associated with the element α' is $C'^{\#}=h_0'h_1'^{-1}: Y_0'\to Y_0'$.

Hence, by the definition of χ, $\chi(f^{-1}\mathfrak{B};\ x_0',\ \zeta)(\alpha')$ reprensents a map $\zeta^{-1}C'^{\#}\zeta$, i. e.

$$\chi(f^{-1}\mathfrak{B};x_0',\ \zeta)\ (\alpha')=\zeta^{-1}C'^{\#}\zeta.$$

Since f is the bundle map, $h=fh'$ is the bundle map of $Y\times I$ into $\mathfrak{B}$, and $h(y,\ 0)=f\zeta(y)$, and the induced map of h is fC'. Therefore the admissible map associated with the element $f^{\#}(\alpha')\in\pi_1(X,f(x_0'))$ is given by

$$C^{\#}=h_0h_1^{-1}: Y_0\to Y_0,$$

where Y_0 denotes the fibre over $f(x_0')$, and $h_t(y)=h_t(y)=h(y,\ t)\,(y\in Y,\ t\in I)$. Hence

$$\begin{aligned}\chi(\mathfrak{B};f(x_0'),\ f\zeta)f^{\#}(\alpha') &= (f\zeta)^{-1}C^{\#}(f\zeta)\\ &= (f\zeta)^{-1}h_0h_1^{-1}(f(\zeta)\\ &= (f\zeta)^{-1}(fh_0')(fh_1')^{-1}(f\zeta)\\ &= \zeta^{-1}h_0'h_1'\zeta=\zeta^{-1}C'^{\#}\zeta.\end{aligned}$$

Thus we have

$$\chi(f^{-1}\mathfrak{B};x_0',\ \zeta)=\chi(\mathfrak{B};f(x_0'),\ f\zeta)f^{\#}\qquad \text{Q. E. D.}$$

THEOREM 1.2. *Let $\mathfrak{B}$ be a bundle over a space X with fibre Y and totally disconnected group G. Let X' be a locally arcwise connected space and let f, g be homotopic maps of X' into X. Then the induced bundles $f^{-1}\mathfrak{B}$ and $g^{-1}\mathfrak{B}$ are equivalent.*

PROOF. Denote by X_α' ($\alpha\in\Omega$; Ω denotes the set of indices) component of X' in the sense of arcwise connectedness. Let ${}_f\mathfrak{B}_\alpha$ and ${}_g\mathfrak{B}_\beta$ be the portions of $f^{-1}\mathfrak{B}$ and $g^{-1}\mathfrak{B}$ over X_α' respectively. It is sufficient to prove that for each $\alpha\in\Omega$, ${}_f\mathfrak{B}_\alpha$ and ${}_g\mathfrak{B}_\alpha$ are equivalent. In fact, if ${}_f\mathfrak{B}_\alpha\sim{}_g\mathfrak{B}_\alpha$, then there exist equivalencces $e_\alpha: {}_f\mathfrak{B}_\alpha\sim{}_g\mathfrak{B}_\alpha$. Denote by B_α' the bundle space of ${}_f\mathfrak{B}_\alpha$. We define a transformation e of the bundle space $\mathfrak{B}_f$ of $f^{-1}\mathfrak{B}$ onto the bundle space of $g^{-1}\mathfrak{B}$ by taking

$$e(b)=e_\alpha(b)\quad\text{if } b\in B_\alpha'.$$

Since X' is locally arcwise connected, each component X_α' is open in X', hence B_α' is open in B_f. Therefore e is continuous. Moreover it is clear that e is a bundle map and the induced map is identity. Hence the bundle map e defines an equivalence $\bar{f}^{-1}\mathfrak{B} \sim \bar{g}^{-1}\mathfrak{B}$.

Therefore we may assume that X' is arcwise connected. Then, since $\bar{f} \sim \bar{g}$, $\bar{f}(X')$ and $\bar{g}(X')$ belong to the same component X_0 of X, in the sense of arcwise connectedness. Let $\mathfrak{B}_0$ be the portion of $\mathfrak{B}$ over X_0. Then it is obvious that

$$\bar{f}^{-1}\mathfrak{B} = \bar{f}^{-1}\mathfrak{B}_0, \quad \bar{g}^{-1}\mathfrak{B} = \bar{g}^{-1}\mathfrak{B}_0.$$

Hence we can assume furthermore without any loss of generality that X is arcwise connected.

Choose a base point $x_0' \in X'$ and set $\bar{f}(x_0') = x_0$, $\bar{g}(x_0') = x_1$. Let Y_f' and Y_g' be the fibres of $\bar{f}^{-1}\mathfrak{B}$ and $\bar{g}^{-1}\mathfrak{B}$ over x_0' respectively. Let $\xi : Y \to Y_f'$ and $\eta : Y \to Y_g'$ be admissible maps. Then, by Lemma 1.1 we have

$$\chi(\bar{f}^{-1}\mathfrak{B}\,;\, x_0',\ \xi\,) = \chi(\mathfrak{B}\,;\, x_0,\ f\xi)\bar{f}^{\#},$$
$$\chi(\bar{g}^{-1}\mathfrak{B}\,;\, x_0',\ \eta) = \chi(\mathfrak{B}\,;\, x_1,\ g\eta)\bar{g}^{\#}.$$

Since $\bar{f} \sim \bar{g}$, there exists a homotopy $h : X' \times I \to X$ between $\bar{f}$ and $\bar{g}$. Define a path D from x_0 to x_1 by taking $D(t) = h(x_0',\ t)$ $(t \in I)$, Then, for $\alpha' \in \pi_1(X',\ x_0')$, $\bar{f}^{\#}(\alpha') \in \pi_1(X, x_0)$ and $\bar{g}^{\#}(\alpha') \in \pi_1(x,\ x_1)$ are homotopic along the path D. Hence, as is well known, we have the relation

$$\bar{f}^{\#}(\alpha') = D_{*}\,\bar{g}^{\#}(\alpha').$$

On the other hand, $\chi(\mathfrak{B}\,;\, x_0,\ f\xi)D_{*}$ and $\chi(\mathfrak{B}\,; x_1,\ g\eta)$ differ by an inner automorphism of G, hence there exists an element g_0 of G such that

$$\chi(\mathfrak{B}\,;\, x_0\,,\, f\xi)D_{*}(\alpha) = g_0^{-1}[\chi(\mathfrak{B}\,;\, x_1,\ g\eta)\ (\alpha)]g \quad (\alpha \in \pi_1(x, x_1)).$$

Hence

$$\begin{aligned}\chi\,(\bar{f}^{-1}\mathfrak{B}; x_0',\ \xi)\ (\alpha') &= \chi(\mathfrak{B}; x_0,\ f\xi)\bar{f}^{\#}(\alpha') \\ &= \chi(\mathfrak{B}; x_0,\ f\xi)D_{*}\bar{g}^{\#}(\alpha') \\ &= g_0^{-1}[\alpha(\mathfrak{B}; x_1,\ g\eta)\bar{g}^{\#}(\alpha')]g_0 \\ &= g_0^{-1}[\chi(\bar{g}^{-1}\mathfrak{B}\,;\, x_0',\, \eta)(\alpha')]g_0.\end{aligned}$$

Therefore, homomorphisms $\chi(\bar{f}^{-1}\mathfrak{B}\,;\, x_0', \xi)$ and $\chi(\bar{g}^{-1}\mathfrak{B}\,;\, x_0', \eta)$ differ by an inner automorphism of G and so we have $\chi(\bar{f}^{-1}\mathfrak{B}) = \chi(\bar{g}^{-1}\mathfrak{B})$. By virtue of [1 ; § 13.7 and 8.2], $\chi(\bar{f}^{-1}\mathfrak{B}) = \chi(\bar{g}^{-1}\mathfrak{B})$ implies $\bar{f}^{-1}\mathfrak{B} \sim \bar{g}^{-1}\mathfrak{B}$. Thus our theorem is proved.

As an immediate result from Theorem 1.2, we have

COROLLARY 1. 3. *If X is locally arcwise connected and contractible on itself to a point, then any bundle over X with totally disconnected group is equivalent to a product bundle.*

THEOREM 1. 4. *Let X be a locally arcwise connected space. Then any bundle $\mathfrak{B}$ over the product space $X \times I$ with a totally disconnected group is equivalent to a bundle of the form $\mathfrak{B}' \times I$.*

PROOF. Define a map $f_0:X\to X\times I$ by taking $f_0(x)=(x,0)$ and put $\mathfrak{B}'=f_0^{-1}\mathfrak{B}$. We shall prove that $\mathfrak{B}\sim\mathfrak{B}'\times I$.

Let $\pi:X\times I\to X$ be the projection. Then by [1, § 11. 1], $\mathfrak{B}'\times I$ is equivalent to the induced bundle $\pi^{-1}\mathfrak{B}'$. Hence

$$\mathfrak{B}'\times I\sim\pi^{-1}\mathfrak{B}'=\pi^{-1}f_0^{-1}\mathfrak{B}\sim(f_0\pi)^{-1}\mathfrak{B}.$$

It is easily seen that the map $f_0\pi:X\times I\to X\times I$ is homotopic to the identity map $e:X\times I\to X\times I$. Therefore by Theorem 1.2 $(f\pi)^{-1}\mathfrak{B}\sim e^{-1}\mathfrak{B}$. Since $\mathfrak{B}\sim e^{-1}\mathfrak{B}$, we have $\mathfrak{B}\sim\mathfrak{B}'\times I$. Q. E. D.

LEMMA 1.5. *Let $\mathfrak{B}$ be a bundle over the space $X\times I$. If $\mathfrak{B}$ is equvivalent to a bundle of the form $\mathfrak{B}'\times I$, and if there is given an arbitrary equivalence $\mu_0:\mathfrak{B}_0\to\mathfrak{B}'\times 0$, then there exists an equivalence $\mu:\mathfrak{B}\to\mathfrak{B}'\times I$ such that $\mu|\mathfrak{B}_0=\mu_0$, where $\mathfrak{B}_0$ is the portion of $\mathfrak{B}$ over $X\times 0$.*

PROOF. By the assumption, there exists an equivalence $\lambda:\mathfrak{B}\to\mathfrak{B}'\times I$. Set $\lambda_0=\lambda|\mathfrak{B}_0$, then $\nu=\mu_0\lambda_0^{-1}:\mathfrak{B}'\times 0\to\mathfrak{B}'\times 0$ is an equivalence. If we put $\nu(b',t)=(\nu(b'),t)$ (b' is a point of the bundle space of $\mathfrak{B}$), then, as is easily seen $\nu:\mathfrak{B}'\times I\to\mathfrak{B}'\times I$ is an equivalence. If we put $\mu=\nu\lambda$, then μ give an equivalence $\mathfrak{B}\sim\mathfrak{B}'\times I$ and $\mu|\mathfrak{B}_0=\nu\lambda|\mathfrak{B}_0=\nu_0\lambda_0=\mu_0$, hence $\mu=\nu\lambda$ is the desired equivalence. Q.E.D.

THEOREM 1.6. *Let $\mathfrak{B}$, $\mathfrak{B}'$ be two bundles with the same fibre Y and the same totally disconnected group G and with the base spaces X and X' respectively. We assume that X' is locally arcwise connected. Let $h_0:\mathfrak{B}'\to\mathfrak{B}$ be a bundle map, and let $\bar{h}:X'\times I\to X$ be a homotopy of the induced map $\bar{h}_0:X'\to X$. Then there exists a homotopy $h:\mathfrak{B}'\times I\to\mathfrak{B}$ of h_0 whose induced homotopy is $\bar{h}$.*

PROOF. Set $\check{\mathfrak{B}}=\bar{h}^{-1}\mathfrak{B}$, and let $\check{h}:\check{\mathfrak{B}}\to\mathfrak{B}$ be the induced bundle map. Let $\check{h}_0:\bar{h}_0^{-1}\mathfrak{B}\to\mathfrak{B}$ be the induced map. By [1 ; § 10. 3] there exists an equivalence $g_0:\mathfrak{B}'\to\bar{h}_0^{-1}\mathfrak{B}$ such that $h_0=\widetilde{h}_0g_0$.

Define a map $\bar{f}_0:X'\to X'\times 0$ by taking $\bar{f}_0(x')=(x',0)\,(x'\in X')$, then we have $(\bar{f}_0^{-1}\check{\mathfrak{B}}_0)\times I\sim\check{\mathfrak{B}}$ from Theorem 1.4. As is easily shown, $\mathfrak{B}'$ and $\bar{f}_0^{-1}\check{\mathfrak{B}}_0$ are equivalent. Therefore we get

$$\mathfrak{B}'\times I\sim\check{\mathfrak{B}}.$$

Set $\check{\mathfrak{B}}_0=\check{\mathfrak{B}}|X'\times 0$, $\check{h}_0=\check{h}|\check{\mathfrak{B}}_0$, $\bar{h}_0=\bar{h}|X'\times 0$, then $\check{\mathfrak{B}}_0$ is the induced bundle $'\bar{h}_0^{-1}B$ and the induced bundle map is $\check{h}_0$.

Let $f_0:\bar{f}_0^{-1}\check{\mathfrak{B}}=\bar{f}_0^{-1}\check{\mathfrak{B}}_0\to\check{\mathfrak{B}}_0$ be the induced map, then the induced map of $\check{h}_0f_0:\bar{f}_0^{-1}\check{\mathfrak{B}}_0\to\check{\mathfrak{B}}_0$ is $'\bar{h}_0\bar{f}=\bar{h}_0$. Hence by [1 ; 10. 3] there exists an equivalence $k_0:\bar{f}_0^{-1}\check{\mathfrak{B}}_0\to\bar{h}_0^{-1}\mathfrak{B}$ such that $\check{h}_0f_0=\widetilde{h}_0k_0$. Since $f_0k_0^{-1}g_0:\mathfrak{B}'\to\check{\mathfrak{B}}_0$ is an equivalence, we can define an equivalence $\mu_0:\mathfrak{B}'\times 0\to\check{\mathfrak{B}}_0$ by taking

$\mu_0(b'\times 0)=f_0k_0^{-1}g_0(b')$ (b' is a point in the bundle space of $\mathfrak{B}'$).

Then by Lemma 1. 5, there exists an equivalence $\mu : \mathfrak{B}' \times I \to \check{\mathfrak{B}}$ such that $\mu | \mathfrak{B}' \times 0 = \mu_0$. We put

$$h = \check{h}\mu.$$

Then it is obvious that h is the bundle map of $\mathfrak{B}' \times I$ into $\mathfrak{B}$ and its induced map is $\bar{h}$. On the other hand

$$h(b' \times 0) = \check{h}\mu(b' \times 0) = \check{h}_0\mu_0(b' \times 0)$$
$$= \check{h}_0 f_0 k_0^{-1} g_0(b') = (\widetilde{h_0 k_0})(k_0^{-1} g_0(b')) = \widetilde{h}_0 g_0(b') = h_0(b').$$

Hence h is the homotopy of h_0, thus h is the desired homotopy. Q. E. D.

THEOREM 1.7. *Let $\mathfrak{B}$ be a bundle over X with the totally disconnected group, and let X' be a locally arcwise connected space. Given a map $f_0 : X' \to \mathfrak{B}$, and a homotopy $\bar{f}$: $X' \times I \to X$ of $pf_0 = \bar{f}_0$, then there exists a homotopy $f : X' \times I \to \mathfrak{B}$ of f_0 which covers $\bar{f}$ (i. e. $pf = \bar{f}$), where p is the projection of $\mathfrak{B}$.*

PROOF. Let $\mathfrak{B}' = \bar{f}_0^{-1}\mathfrak{B}$, and let $h_0 : \mathfrak{B}' \to \mathfrak{B}$ be the induced map. By Theorem 1.6, there exists a homotopy $h : \mathfrak{B}' \times I \to \mathfrak{B}$ of h_0 which covers $\bar{f}$. Define a cross-section ϕ of $\mathfrak{B}'$, using the second definition of induced bundle [1 ; § 10. 2], as follows: $\phi(x') = (x', f_0(x'))$. Then $h_0\phi = f_0$. Define $f(x', t) = h(\phi(x'), t)$, It follows immediately that f is the desired homotopy.

REMARK. In Theorem 1. 5 and 1. 6 stationality of h is not satisfied. Theorems 1.2, 1.4 and 1.6 are equivalent.

2. In this section we assume that all spaces are T_1-spaces. If X is a compact space, then it has one and only one uniformity agreeing with its topology, and this uniformity is made up of all open coverings (see [4;4. 4, p. 60]). Let $\{\mathfrak{M}_\alpha\}$ $(\alpha \in \Omega)$ be the system of all coverings of X. For a covering $\mathfrak{M}$ of X and subset U of X, we shall denote by $S(U, \mathfrak{M})$ the uniform of all the sets of $\mathfrak{M}$ meeting U. Let $\mathfrak{M}$ be a covering of X, then we denote by $\mathfrak{M}^*$ a covering wich consists of the sets $S(U, \mathfrak{M})$, where $U \in \mathfrak{M}$. The symbol $\mathfrak{M}_\alpha < \mathfrak{M}_\beta$ means that the covering $\mathfrak{M}_\alpha$ is a refinement of $\mathfrak{M}_\beta$, i. e. each set of $\mathfrak{M}_\alpha$ is contained in some element of $\mathfrak{M}_\beta$.

Two maps f_0 and f_1 of a space X' into compact space X are said to be uniformly homotopic, if there is a homotopy h of f_0 and f_1 which satisfies the following condition:

For any index $\alpha \in \Omega$ there is a positive number δ such that if $x' \in X'$ and $|t - t'| < \delta$, then

$$h(x', t) \in S(h(x', t'), \mathfrak{M}_\alpha)$$

holds.

Čech [2] has shown that if X' is a completely regular space there exist a comact Hausdorff space $\beta(X')$ and a topological map ϕ of X' into $\beta(X')$ such that:

1) $\phi(X')$ is dense in $\beta(X')$.

2) Any bounded continuous real function f on X' has the form $f(x') = F\phi(x')$, where F is a similar function on $\beta(X')$.

By 1) the function F is determined uniquely.

Hence it follows from 2) that any map $f: X' \to X$ determines a map $F: \beta(X') \to X$ uniquely such that $f(x') = F\phi(x')$.

LEMMA 2.1. *Two map f_0 and f_1 of a countably compact space X′ into a compact space X are uniformly homotopic if and only if they are homotopic.*

PROOF. It follows immediately from the definitions that if f_0 and f_1 are uniformly homotopic they are homopic.

Let f_0 and f_1 be homotopic. Then there is a homotopy $h: X' \times I \to X$ such that $h(x', 0) = f_0(x')$, $h(x', 1) = f_1(x')$. For a given index α, $\mathfrak{M}_\alpha$ is a covering of the compact space X, hence it has a finite subcovering $\mathfrak{U}$. Hence, by [3; Lemma 9.1] there exist finite coverings $\mathfrak{V}$ and $\mathfrak{W}$ of X' and I respectively such that $\mathfrak{V} \times \mathfrak{W}$ is a refinement of $h^{-1}(\mathfrak{U})$, where $h^{-1}(\mathfrak{U})$ is a covering made up of all inverse images of elements of $\mathfrak{U}$ under the map h. Hence if $V \in \mathfrak{V}$, $W \in \mathfrak{W}$, there is an element $U \in \mathfrak{U}$ such that $h(V \times W) \subset U$.

Since I is a compact metric space there exists a positive number δ such that if $|t - t'| < \delta$ then t and t' are contained in the same element of $\mathfrak{W}$. Hencs, if $|t - t'| < \delta$, $h(x', t)$ and $h(x', t')$ are two points in the same element of $\mathfrak{M}_\alpha$. Thus, if $x' \in X'$, $|t - t'| < \delta$ then $h(x', t) \in S(h(x', t'), \mathfrak{M}_\alpha)$ holds good. Therefore f_0 and f_1 are uniformly homotopic. Q.E.D.

LEMMA 2.2. *Let f_0, f_1 be two maps of a completely regular space X' into a compact space X, and let F_0, F_1 be maps of $\beta(X')$ into X such that $f_i = F_i\phi$ $(i = 0, 1)$. Then f_0 and f_1 are uniformly homotopic if and only if F_0 and F_1 are homotopic.*

PROOF. Let F_0 and F_1 be homotopic. Since $\beta(X')$ is compact, by Lemma 2.1, F_0 and F_1 are uniformly homotopic, hence the same is true of f_0 and f_1.

Next, let f_0 and f_1 be uniformly homotopic. Let $g(x', t)$ be the map of $X' \times I$ into X which gives the uniform homotopy between f_0 and f_1. For a fixed, t, $g(x', t)$ is a map of X' into X which we call $g_t(x')$ and there is a corresponding map G_t of $\beta(X')$ into X such that $G_t\phi(x') = g_t(x')$ $(x' \in X')$. Defining $G(y, t)$ to be $G_t(y)$ for $y \in \beta(X')$ we have a transformation of $\beta(X') \times I$ into X with $G(y, 0) = F_0(y)$, $G(y, 1) = F_1(y)$. Hence, if we can show that $G(y, t)$ is continuous, F_0 and F_1 are homotopic.

Let (y_0, t_0) be a fixed point of $\beta(X) \times I$, and let M be an arbitrary open set in X which contains the point $G(y_0, t_0)$. Since the uniformity $\{\mathfrak{M}_\alpha; \alpha \in \Omega\}$ agrees with the topology of X, there is an index α such that $S(G(y_0, t_0), \mathfrak{M}_\alpha) \subset M$ (see [4; §3. p. 57]).

Now, since every compact space is fully normal [4; Theorem 4. 5, p. 60] there is an index $\beta \in \Omega$ such that $\mathfrak{M}_\beta^{**} < \mathfrak{M}_\alpha$. Hence we have inclusion relations

$$S(G(y_0,\ t_0),\ \mathfrak{M}_\beta^{**}) \subset S(G(y_0,\ t_0),\ \mathfrak{M}_\alpha) \subset M.$$

Since G_{t_0} is continuous, there is a neighborhood U of y_0 in $\beta(X')$ such that for $y \in U$

$$G_{t_0}(y) \in S(G_{t_0}(y_0),\ \mathfrak{M}_\beta).$$

Since $g(x',\ t)$ is a uniform homotopy there is a δ-neighborhood W of t_0 such that if $x' \in X'$ and $t \in W$,

$$g_t(x') \in S(g_{t_0}(x'),\ \mathfrak{M}_\beta) \text{ i. e. } G_t\phi(x') \in S(G_{t_0}\phi(x'),\ \mathfrak{M}_\beta).$$

Now let (y, t) be any point of the neighborhood $U \times W$ of (y_0, t_0) in $\beta(X') \times I$. Since G_t is continuous there is a neighborhood $U_1 \subset U$ of y such that if $y' \in U_1$, $G_t(y') \in S(G_t(y),\ \mathfrak{M}_\beta)$.

But, since $\phi(X')$ is dense in $\beta(X')$, there is a point $x' \in X'$ such that $\phi(x') \in U_1$. Therefore if $y \in U$ and $t \in W$, then there is a point $y' \in \phi(X')$ such that simultaneously

$$G_{t_0}(y') \in S(G_{t_0}(y_0),\ \mathfrak{M}_\beta),$$
$$G_t(y') \in S(G_{t_0}(y'),\ \mathfrak{M}_\beta),$$

and

$$G_t(y') \in S(G_t(y),\ \mathfrak{M}_\beta)$$

hold good. Hence we get

$$G(y,\ t) \in S(G(y_0,\ t_0),\ \mathfrak{M}_\beta^{**}) \subset M.$$

Thus it is proved that G is a continuous map. Q. E. D.

THEOREM 2.3. *Let $\mathfrak{B}, \mathfrak{B}'$ be two bundles having the same fibre and group. We assume that the bundle space of $\mathfrak{B}$ is compact and the base space X' of $\mathfrak{B}'$ is completely regular. Let $h_0 : \mathfrak{B}' \to \mathfrak{B}$ be a bundle map, and let $\bar{h} : X' \times I \to X$ be a uniform homotopy of the induced map $\bar{h}_0 : X' \to X$. Then there exists a uniform homotopy $h : \mathfrak{B}' \times I \to \mathfrak{B}$ of the map h_0 whose induced map is $\bar{h}$, and which is stationary with $\bar{h}$.*

PROOF. Let $\bar{H}_0 : \beta(X') \to X$ and $\bar{H} : \beta(X') \times I \to X$ be corrersponding maps of $\bar{h}_0$ *and* $\bar{h}$ respectively such that $\bar{h}_0 = \bar{H}_0\phi$, $\bar{h}(x',\ t) = \bar{H}(\phi(x'),\ t)$. Then $\bar{H}$ is a homotopy of $\bar{H}_0$, i. e. $\bar{H}(y, 0) = \bar{H}_0(y)$ $(y \in \beta(X'))$. Put $\mathfrak{B}'' = \bar{H}_0^{-1}\mathfrak{B}$, and let $H_0 : \mathfrak{B}'' \to \mathfrak{B}$ be the induced map.

Let $\tilde{\phi} : \phi^{-1}\mathfrak{B}'' \to \mathfrak{B}''$ be the induced map. Then $H_0\tilde{\phi}$ is a bundle map of $\mathfrak{B}'' \to \mathfrak{B}$ and its induced map is $\bar{H}_0\phi = \bar{h}_0$. Therefore, by [1; 10.3], there exists an equivalence $g_0 : \phi^{-1}\mathfrak{B}'' \to \bar{h}_0^{-1}\mathfrak{B}$ such that $H_0\tilde{\phi} = \tilde{h}_0 g_0$, where $\tilde{h}_0 : \bar{h}_0^{-1}\mathfrak{B} \to \mathfrak{B}$ is the induced map. On the other hand, by [1; 10.3], there exists an equivalence $\check{h}_0 : \mathfrak{B}' \to \bar{h}_0^{-1}\mathfrak{B}$ such that $h_0 = \tilde{h}_0\check{h}_0$. A bundle map $\lambda_0 : \mathfrak{B}' \to \mathfrak{B}''$ is defined to be $\lambda_0 = \tilde{\phi} g_0^{-1} \check{h}_0$, and put $\lambda(b', t) = \lambda_0(b') \times t$, then λ is a bundle map of $\mathfrak{B}' \times I$ into $\mathfrak{B}'' \times I$.

Now since the base space $\beta(X')$ of the bundle $\mathfrak{B}''$ is compact, by the first covering theorem [1; §11.3], there exists a homotopy $H:\mathfrak{B}''\times I\to\mathfrak{B}$ of H_0 whese induced homotopy is $\bar{H}$ and H is stationary with $\bar{H}$.

We put $h = H\lambda:\mathfrak{B}'\times I\to\mathfrak{B}$. Then h is the desired homotopy. In fact, we obtain

$$h(b',0)=H\lambda(b',0)=H(\lambda_0(b'),0)=H_0\lambda_0(b')=H_0\widetilde{\phi g_0^{-1}}\check{h}_0(b')=\widetilde{h_0}\check{h}_0(b')=h_0(b').$$

Hence h is a homotopy of h_0, and since H is uniform homotopy by Lemma 2.1, it is easily seen from λ that h is a uniform homotopy. The induced map of λ is $\bar{\lambda}\,(b',t)=(\phi(b'),t)$, and the induced map of H is $\bar{H}$ and so the induced map of H is $\bar{H}\bar{\lambda}=\bar{h}$. Thus the proof is completed. Q. E. D.

Using Theorem 2. 3 we can prove the following theorems. But these proof are similar to that of [1; § 11.4, §11.7], and so we shall omit here.

THEOREM 2.4. *Let $\mathfrak{B}$ be a bundle with a compact bundle space, and let X' be a completely regular space. If two maps f, $g:X'\to X$ are uniformly homotopic, then induced bundles $f^{-1}\mathfrak{B}$ and $g^{-1}\mathfrak{B}$ are equivalent.*

THEOREM 2.5. *Let $\mathfrak{B}$ be a bundle with a compact space X as its bundle space, and let X' be a completely regular space. For a given map $f_0:X'\to\mathfrak{B}$ and given uniform homotopy $\bar{f}:X'\times I\to X$ of the map $pf_0=\bar{f}_0$ there exists a uniform homotopy $f:X'\times I\to B$ of f_0 which covers $\bar{f}$ and f is stationary with $\bar{f}$.*

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