ON Π-STRUCTURES OF FINITE GROUPS

NOBORU ITÔ

(Received June 24, 1952)

Introduction. In his recent papers S. Čunihin skilfully introduced the notion of Π-solubility in the theory of finite groups, where Π is a certain set of prime numbers, and obtained some interesting results of types of L. Sylow and P. Hall on Π-soluble groups. These results of S. Čunihin can, in my opinion, be in a more precise way derived even in a little more generalized form from the following two propositions——one is due to I. Schur and the other is due to H. Zassenhaus (H. Zassenhaus [1]).

THEOREM (S). *Let N be a normal subgroup of a group G with index prime to its order. Then there exists in G at least one complemented subgroup S of N.*

HYPOTHESIS (Z). Under the same assumption of Theorem (S), all the complemented subgroups of N are conjugate one another.

This hypothesis (Z) is probably true (Cf. H. Zassenhaus [1]) and seems to be considered as a generalization of the D. Hilbert-A. Speiser theorem on crossed product theory and has been verified affirmatively, by Zassenhans, in case either N or G/N is soluble. So we shall expose our results on assuming the validity of the hypothesis, and in case a result is free of it we shall remark that.

The purposes of this paper are

(1) to reproduce the theory of S. Čunihin with a suitable modification from this point of view (§1),

(2) to generalize the theory of P. Hall on soluble groups to the case of Π-soluble groups of a certain type (§2), and

(3) to generalize some results in the theory of O. Grün on p-Sylow groups to the case of Π-Sylow groups under a certain condition (§3).

1. On the theory of S. Čhunihin (S. Čunihin [1], [2], [3], [4], [5] and [6]).

First we prove the following

INCLUSION THEOREM. *Under the assumption in Theorem (S), let T be a subgroup of G with order dividing the index of N in G. Then T is contained in a suitable conjugate subgroup of S.*

REMARK. If T is soluble, the result is free from the hypothesis.

PROOF. From $S(NT) = (NT)S = G$, we clearly have $[G:S] = [NT:(NT) \cap S]$, whence $(NT) \cap S^A = T$ with a suitable element A of N by hypothesis (Z). This proves $S^A \supseteqq T$.

Let Π be a certain set of prime numbers. We say that an integer n belongs to Π, if all the prime factors of n belong to Π, and that n is prime to Π, if none of the prime factors of n belongs to Π.

A group is a Π-group, if its order belongs to Π. A Π-subgroup is a Π-Sylow subgroup, if the index is prime to Π. A subgroup with order prime to Π is a Π-Sylow complement, if the index belongs to Π.

DEFINITION. *A group G is Π-soluble, if the order of each factor group of a normal series of G either belongs to Π or is prime to Π.*

PROPOSITION 1. (1) *If G is Π-soluble, there exist in G at least one Π-Sylow complement and also at least one Π-Sylow subgroup.*

(2). *If G is Π-soluble, then all the Π-Sylow complements are conjugate one another and all the Π-Sylow subgroups are conjugate one another.*

(3). *If G is Π-soluble, then any subgroup of order prime to Π is contained in some Π-Sylow complement and also any Π-subgroup is contained in some Π-Sylow subgroup.*

REMARK. If either a Π-Sylow complement or a Π-Sylow subgroup is soluble, then (2) and (3) are free from the hypothesis.

PROOF. By symmetry, we have only to prove these for Π-complement. Now we prove these by induction. If G is simple, then all the assertions trivially hold good. Let N be a minimal normal subgroup of G. If N is of order prime to Π, then all the assertions can readily be proved. Let thus N be a Π-group. Unless G/N is of order prime to Π, all the assertions can easily be proved by induction. So let G/N be of order prime to Π. Then all the assertions can be proved easily by Theorem (S), Hypothesis (Z) and Inclusion Theorem.

DEFINITION. A group G is Π-faithful, if there exists in G at least one Π-Sylow subgroup and all the Π-Sylow subgroups are conjugate one another.

PROPOSITION 2. *Let N be a normal subgroup of a group G. If N and G/N are Π-faithful, then G is also Π-faithful.*

PROOF. Existence. Let K and H/N be Π-Sylow subgroups of N and G/N respectively. Then we have clearly $N_H(K)N = H$ and $H/N \cong N_H(K)/N \cap N_H(K)$. The latter is a Π-group. Now $N \cap N_H(K)/K$ is of order prime to Π and is normal in $N_H(K)/K$. Therefore we have the assertion by Theorem (S).

REMARK. This is free from the hypothesis.

Conjugacy. Let P_1 and P_2 be any two Π-Sylow subgroups of G. Considering in G/N, we have clearly $P_1N \supseteq P_2^A$ with a suitable element A of G. So we have only to show that all the Π-Sylow subgroups of P_1N are conjugate one another. Let P_2 and P_3 be any two Π-Sylow subgroups of P_1N. Now $P_2 \cap N$ and $P_3 \cap N$ are Π-Sylow subgroups of N and therefore they are conjugate one another. Further $N_{P_1N}(P_2 \cap N) \supseteq P_2$ and $N_{P_1N}(P_3 \cap N) \supseteq P_3$. Therefore $N_{P_1N}(P_2 \cap N) \supseteq P_3$. Therefore $N_{P_1N}(P_2 \cap N) \supseteq P_3^B$ with a suitable element B of P_1N. So we have only to show that all the Π-Sylow subgroups of $N_{P_1N}(P_2 \cap N)$ are conjugate one another. Now since $N/P_2 \cap N$ is normal in $N_{P_1N}(P_2 \cap N)/P_2 \cap N$, we have the result by Hypothesis (Z).

REMARK. We need not Hypothesis (Z), if a Π-Sylow subgroup of G is soluble.

PROPOSITIN 3. *Let q be a prime not belonging to Π. If G is Π-soluble, then G is $\{\Pi, q\}$-faithful. Similarly let q be a prime belonging to Π. If G is Π-soluble, then G is $\{\Pi^C, q\}$-faithful.*

PROOF. We can easily prove this by an induction argument.

REMARK. If either a Π-Sylow complement or a Π-Sylow subgroup is nilpotent then the result is free from the hypothesis. (N. Itô [1]).

2. On the theory of P. Hall (P. Hall [1], [2], [3], [4] and [5]).

DEFINITION. A Π-soluble group G is a Π-soluble group of type S, if at least one Π-Sylow complement of G is soluble.

Now the theory of P. Hall on soluble groups can be naturally extended to Π-soluble groups of type S. We omit the detailed proofs and formulations and only list the necessary definitions, since any proof and formulation of the soluble case can be easily modified in the case of Π-soluble groups of type S. Naturally the result is free from the hypothesis.

Let q be a prime not belonging to Π. There exists in G at least one q-Sylow complement $C_q(G)$. All the q-Sylow complements are conjugate with one another. Any subgroup whose order divides the order of $C_q(G)$ is contained in some conjugate subgroup of $C_q(G)$.

DEFINITION. We call any system $\{C_q(G)\}$, where q runs all the prime factors of the order of G not belonging to Π, a Π-Sylow system. A Π-system normalizer is the meet $\bigcap_q N(C_q(G))$.

DEFINITION. A commutator $[X, Y] = X^{-1}Y^{-1}XY$ is called a Π-commutator, if X and Y possess orders prime to Π. We denote by $D(\Pi, G)$ the characteristic subgroup generated by all the Π-commutators and call it the Π-commutator subgroup. Then the Π-commutator series can be naturally defined as follows: $D^{(1)}(\Pi, G) = D(\Pi, G)$, and $D^{(n+1)}(\Pi, G) = D(\Pi, D^{(n)}(\Pi, G))$ for $n \geqq 1$.

DEFINITION. A group G is a Π-soluble group of type A, if $D(\Pi, G) = E$.

A Π-soluble group of type A is clearly a Π-soluble group of type S. A group G is a Π-soluble group of type A, if and only if $G = S(\Pi, G) \cdot C(\Pi, G)$, with abelian normal $C(\Pi, G)$, where $S(\Pi, G)$ and $C(\Pi, G)$ denote Π-sylow subgroup and Π-Sylow complement of G. A group G is a Π-soluble group of type S, if and only if $D^{(n+1)}(\Pi, G) = E$ for some $n \geqq 0$.

DEFINITION. An element X with order prime to Π is called a Π-central element, if X is commutative with all elements with order prime to Π. We denote by $Z(\Pi, G)$ the characteristic subgroup generated by all the Π-central elements and call it the Π-centre of G. Then the Π-upper central series can be naturally defined as follows: $Z^{(1)}(\Pi, G) = Z(\Pi, G)$, and $Z^{(n+1)}(\Pi, G)/Z^{(n)}(\Pi, G) = Z(\Pi, G/Z^{(n)}(\Pi, G))$ for $n \geqq 1$.

$Z(\Pi, G)$ is clearly a Π-soluble group of type A.

DEFINITION. A group G is a Π-soluble group of type N, if $G = Z^{(n+1)}(\Pi, G)$ for some $n \geqq 0$.

A group G is a Π-soluble group of type N, if and only if $G = S(\Pi, G) \cdot C(\Pi, G)$ with nilpotent normal $C(\Pi, G)$.

Last, the Π-lower central series can be naturally defined as follows: $H^{(1)}(\Pi, G) = G$, and $H^{(n+1)}(\Pi, G)$ is the characteristic subgroup generated by all the Π-commutator $[X, Y]$ such that X belongs to $H^{(n)}(\Pi, G)$.

A group G is a Π-soluble group of type N, if and only if $H^{(n+1)}(\Pi, G) = E$ for some $n \geqq 0$. The lengths of the Π-upper central series and the Π-lower central series of a group are the same with each other.

After O. Ore we say that a maximal subgroup M of a group G belongs to a normal subgroup N of G, if N is the largest among normal subgroups of G contained in M. The following proposition is a generalization of Theorem IV-14 of O. Ore [1].

PROPOSITION 4. *Let G be a Π-soluble group of type S and let N be a normal subgroup of G. If any two maximal subgroups, which have indices prime to Π, belong to N, then they are conjugate to each other.*

PROOF. We have only to prove this for $N = E$, as we see by an induction argument. Now, there exists in G the least normal subgroup P of order a power of a prime p, where p is prime to Π. And for any maximal subgroup M belonging to E, we have clearly $G = M \cdot P$ and $M \cap P = E$. Let L/P be a miniml normal subgroup of G/P. If L is a p-group, then the centre $Z(L)$ of L is different from E and is normal in G. Further we clearly have $Z(L) \supseteqq P$, whence we have also that $M \cap L$ is normal in G. This is a contradiction. Thus the order of L/P must be prime to p. Therefore there exists in L a complemented subgroup Q of P by Theorem (S). Further all the Q's are conjugate one another by Theorem (Z). Now then $M \cap L = Q^X$ for some element $X \in L$ and the normalizer of Q^X in G is M.

3. **On the theory of O. Grün** (O. Grün [1] and [2]).

In this section we assume that a group G and its all subgroups are Π-faithful. First we state some results of O. Grün in a formally generalized form without proofs. Let P be a Π-Sylow subgroup of G and let $Z(P)$ be the centre of P. After O. Grün we denote by $V(P)$ the weak closure of $Z(P)$ in P itself, i.e., the join of all the conjugate subgroups of $Z(P)$ which are contained in P.

DEFINITION. Let $Z(P), Z(P^{S_1}), \cdots, Z(P^{S_n})$ be all the conjugate subgroups of $Z(P)$ which are contained in P. We call the complex $N(Z(P))(E + S_1 + \cdots + S_n)(E + S_1^{-1} + \cdots + S_n^{-1})N(Z(P))$ the conjugator complex, after O. Grün, and denote it by $\mathfrak{R}(P)$.

Any two elements of P which are conjugate one another in G are transformable with a suitable element of $\mathfrak{R}(P)$.

Let U be any subgroup of G containing $\mathfrak{K}(P)$. Then we have $G/G'(\Pi) \supseteqq U/U'(\Pi)$, where $G'(\Pi)$ and $U'(\Pi)$ are Π-factor commutator subgroups of G and U respectively.

PROPOSITION 5 (N. Itô[2]). *If G is a Π-soluble group, then $V(P)$ is abelian.*

PROOF. In case there exists in G a normal subgroup with order prime to Π, the assertion can be proved by induction. So let us assume that there exists in G no normal subgroup with order prime to Π. Let M be the largest normal Π-subgroup of G. Then M contains the centralizer $K(M)$ of M. In fact, if $M \not\supseteqq K(M)$, let N be the largest normal Π-subgroup of $K(M)$, and let H/N be a minimal normal subgroup of G/N, which is contained in $K(M)/N$. Then by Theorem (S) there exists in H a complemented subgroup K of N. Since $M \supseteqq N$ and $K(M) \supseteqq K$, we clearly have $KN = K \times N$. This is a contradiction. Thus M contains all the conjugate subgroups of $Z(P)$. Therefore $V(P)$ is normal in G. In particular, $V(P)$ is abelian.

PROPOSITION 6 (N. Itô [2]). *If $V(P)$ is abelian, then $\mathfrak{K}(P) = N(Z(P))N(V(P)) \cdot N(Z(P))$.*

PROOF. We have only to prove that $\mathfrak{K}(P) \subseteqq N(Z(P))N(V)P))N(Z(P))$. Since $V(P)$ is abelian, we have clearly $N(Z(P)) \supseteqq V(P_{s^{-1}}P)$, whence $V(P_{s_i^{-1}}) = V(P^T)$, where T is a suitable element of $N(Z(P))$. This proves our proposition.

PPOPOSITION 7. *If G is a Π-soluble group, then $G/G'(\Pi) \cong N(V(P)/N(V(P)'(\Pi)$.*

PROOF. We saw in our proof of Proposition 5 that if there exists in G no normal subgroup with order prime to Π, then $V(P)$ is normal in G. Let N be the largest normal subgroup of G with ordr prime to Π. Then we have $G = N(V(P))N$. In fact, let us assume that $Z(P^A) \subseteqq PN$, where A is a suitable element of G. Then clearly $Z(P^A) \subseteqq V(P^B)$, where B is a suitable element of N. Therefore $Z(P^A) \subseteqq V(P)N$. This shows that $V(P)N$ is normal in G, whence we have easily $G = N(V(P)) \cdot N$. On the other hand, we clearly have $G'(\Pi) = N(V(P))'(\Pi)N$ and $N(V(P))'(\Pi)N \cap N(V(P)) = N(V(P))'(\Pi)$. The assertion is now evident by the second isomorphism theorem.

BIBLIOGRAPHY

S. UNIHIN [1] Über Gruppen mit vorgegebenen Untergruppen, Mat. sb., 4 (46), 1938.
[2] O p-razložimyh gruppah, Mat. sb., 15(57), 1944, 325-338.
[3] O Π-svoĭstvah konečnyh grupp, Mat. sb., 25 (67), 1949, 321-346.
The following four papers are not yet communicated in Japan. We see their existence only in Mat. Rev.,
[4] DAN SSSR, 66 (1949), 165-168. Cf. Mat. Rev. 10, 678.
[5] DAN SSSR, 69 (1949), 735-737. Cf. Mat. Rev. 11, 321.

[6] DAN SSSR, 73 (1950), 29-32. Cf. Mat. Rev. 12, 156.
[7] DAN SSSR, 77 (1951), 973-975. Cf. Mat. Rev. 12, 800.
O. GRÜN [1] Beiträge zur Gruppentheorie I, Crelle, 174 (1935), 1-14.
[2] Beiträge zur Gruppentheorie III, Math. Nachr. Berlin, 1 (1948), 1-24.
P. HALL [1] A note on soluble groups, Journal London Math. Soc., 3 (1928), 98-105.
[2] A characteristic property of soluble groups, Journal London Math. Soc., 12 (1937), 198-200.
[3] On the Sylow systems of a soluble group, Proc. London Math. Soc. (2), 43 (1937), 316-323.
[4] On the system normalizers of a soluble group, Proc. London Math. Soc. (2), 43 (1937), 507-528.
[5] The construction of soluble groups, Crelle, 182 (1940).
N. ITÔ [1] Remarks on factorizable groups, Acta. Sci. Math., 14 (1951), 83-84.
[2] Remarks on O. Grün's paper "Beiträge zur Gruppentheorie III", Math. Nachr. Berlin, 6 (1952), 319-325.
O. ORE [1] Contributions to the theory of groups of finite order, Duke Math. J., 5 (1939), 431-460.
H. ZASSENHAUS [1] Lehrbuch der Gruppentheorie, 1937, Leipzig-Berlin.

MATHEMATICAL INSTITUTE, NAGOYA UNIVERSITY, NAGOYA.