NOTE ON DIRICHLET SERIES (XIV) ON THE SINGULARITIES OF DIRICHLET SERIES (VI)

CHUJI TANAKA

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1. Introduction. Let us put

$$(1.1)\qquad F(s)=\sum_{n=1}^{\infty}a_n\exp(-\lambda_n s)\quad (s=\sigma+it,\ 0\leqq\lambda_1<\lambda_2<\ldots<\lambda_n\to+\infty).$$

We shall begin with

DEFINITION. ([1, p. 605]) *If* (1. 1) *is simply convergent for* $\sigma>0$, *we call the point* s_0 *on* $\sigma=0$ *Picard's point, provided that* (1. 1) *assumes every value, except perhaps two* (∞ *included*), *infintely many times, in the half-circle*: $|s-s_0|<\varepsilon$, $\sigma>0$, *where* ε *is an arbitrary positive constant.*

In this note, we shall study the relation between Picard's points and Dirichlet-coefficients $\{a_n\}$ of (1. 1) in the most general cases. In the special cases, we have already established some results in the previous note [1, p. 610, p. 612].

The main theorem reads as follows:

MAIN THEOREM. *Let* (1. 1) *have the simple convergence-abscissa* $\sigma_s=0$. *Then* $s=0$ *is Picard's point for* (1. 1), *provided that there exist two sequences* $\{x_k\}$ $(0<x_k\uparrow\infty)$, $\{\gamma_k\}$ (γ_k : *real*) *such that*

$$(1.2)\ \text{(a)}\ \overline{\lim_{k\to\infty}}\,1/x_k\cdot\log|\Delta_k|=0,\quad \overline{\lim_{k\to\infty}}\,1/\log x_k\cdot\log^+\log^+|\Delta_k|=1/2+\alpha$$

$(0<\alpha\leqq1/2)$, *where* $\Delta_k=\sum_{[x_k]\leqq\lambda_n<x_k}b_{n,k}$, $b_{n,k}=\Re(a_n\exp(-i\gamma_k))$, $[x]$: *Gauss's symbol.*

(b) $\underline{\lim}_{k\to\infty}\log\sigma_k/\log[x_k]<\alpha$, *where* σ_k : *the number of sign-changes of* $b_{n,k}$, $\lambda_n\in I_k\,[[x_k](1-w),\ [x_k](1+w)]$ $(0<w<1)$.

(c) *the sequence* $\{b_{n,k}\}$ $(\lambda_n\in\{I_k\})$ *has the normal sign-change in* $\{I_k\}$ $(k=1,2,\ldots)$ [2, p. 285. Definition III].

2. Lemmas. To establish our main theorem, we need some lemmas.

LEMMA 1 [1, p. 610. corollary 1]. *Let* (1. 1) *have the simple convergence-abscissa* $\sigma_s=0$. *Then* $s=0$ *is Picard's pioint for* (1. 1), *provided that*

$$(2.1)\qquad \begin{aligned}&\text{(a)}\ \overline{\lim_{m\to\infty}}\,|O_m|^{\frac{1}{m}}\geqq1,\\ &\text{(b)}\ \overline{\lim_{m\to\infty}}\,1/\log m\cdot\log^+\log^+|O_m|>1/2,\end{aligned}$$

where $O_m=(e/m)^m\sum_{m(1-w)\leqq\lambda_n\leqq m(1+w)}a_n\lambda_n^m\exp(-\lambda_n)\qquad(0<w<1).$

LEMMA 2. *Under the same assumptions as in lemma* 1, *the following* *conditions* (2.2) *are equivalent to* (2.1):

$$(2.2)\qquad \begin{aligned} &(a)\ \overline{\lim_{m\to\infty}}\, |F^{(m)}(1)/m!|^{\frac{1}{m}} = 1,\\ &(b)\ \overline{\lim_{m\to\infty}}\, 1/\log m\cdot \log^+\log^+|F^{(m)}(1)/m!| > 1/2. \end{aligned}$$

PROOF. By A. Ostrowski's theorem ([3, 4; pp. 12–16], [2, p. 288]) (2.1) (*a*) is the necessary-sufficient condition for $s=0$ to be the singular point of (1.1). Hence (2.1) (*a*) is equivalent to (2.2) (*a*).

Since $s=1$ is a regular point of (1.1), (1.1) is developable in Taylor series:

$$F(s) = \sum_{m=1}^{\infty} F^{(m)}(1)/m!\cdot(s-1)^m,$$

where $F^{(m)}(1) = (-1)^m \sum_{n=1}^{\infty} a_n \lambda_n^m \exp(-\lambda_n) = (-1)^m\{R_m + H_m + S_m\}$,

$$(2.3)\quad R_m = \sum_{\lambda_1\leqq\lambda_n<m(1-w)}, \qquad H_m = \sum_{m(1-w)\leqq\lambda_n\leqq m(1+w)}, \qquad S_m = \sum_{\lambda_n > m(1+w)};$$

$(0<w<1)$.

By a lemma ([1, p. 606 lemma 2], [5, p. 32 (1.13)]), we have easily

$$(2.4)\qquad \lim_{m\to\infty} 1/m!\cdot|R_m + S_m| = 0.$$

From the assumption (2.1) (*b*), we get

$$(2.5)\qquad 1/2 < \overline{\lim_{m\to\infty}}\, 1/\log m\cdot\log^+\log^+|O_m| = \overline{\lim_{m\to\infty}}\, 1/\log m\cdot \log\log|O_m|.$$

Since, by Stirling's formula, we can put

$$H_m/m! = O_m(1+o(1))/(2\pi m)^{1/2},$$

by (2.5) we can easily prove

$$(2.6)\qquad \overline{\lim_{m\to\infty}}\, 1/\log m\cdot \log^+\log^+|H_m/m!| \geqq \overline{\lim_{m\to\infty}}\, 1/\log m\cdot\log\log|O_m| > 1/2.$$

Since we can put

$$|F^{(m)}(1)/m!| = |1/m!\cdot(R_m+H_m+S_m)| \geqq |H_m/m!| - |1/m!\cdot(R_m+S_m)|,$$

by (2.4), (2.6) we get

$$(2.7)\qquad \overline{\lim_{m\to\infty}}\, 1/\log m\cdot\log^+\log^+|F^{(m)}(1)/m!| > 1/2.$$

By the slight modification of the above arguments, we can prove that (2.5) follows from (2.7). q. e. d.

LEMMA 3 [1, p. 606. lemma 3]. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *have* $|z|<1$ *as its convergence-circle. Then*

$$\overline{\lim_{n\to\infty}}\, 1/\log n\cdot \log^+\log^+|a_n| = \rho/(1+\rho),$$

where
$$\begin{aligned} &(\mathrm{i})\ \ \rho = \overline{\lim_{r\to 1}}\, [\log(1/(1-r))]^{-1}\log^+\log^+ M(r),\\ &(\mathrm{ii})\ M(r) = \operatorname*{Max}_{|z|=r}|f(z)|. \end{aligned}$$

LEMMA 4. *Let $\phi(z)$ be the integral function such that, for any given ε (>0)*

$$|\phi(z)| < \exp(\varepsilon |z|) \qquad \textit{for} \quad |z| > R(\varepsilon). \tag{2.8}$$

Put

$$F_\phi(s) = \sum_{n=1}^{\infty} a_n \phi(\lambda_n) \exp(-\lambda_n s).$$

Then (2.10) *follows, provided that $F_\phi(s)$ satisfies* (2.9):

$$\text{(2.9)} \quad \begin{cases} \text{(a)} & \overline{\lim_{m\to\infty}} |F_\phi^{(m)}(1)/m!|^{\frac{1}{m}} = 1, \\ \text{(b)} & \overline{\lim_{m\to\infty}} 1/\log m \cdot \log^+ \log^+ |F_\phi^{(m)}(1)/m!| > 1/2, \end{cases}$$

$$\text{(2.10)} \quad \begin{cases} \text{(a)} & \overline{\lim_{m\to\infty}} |F^{(m)}(1)/m!|^{\frac{1}{m}} = 1, \\ \text{(b)} & \overline{\lim_{m\to\infty}} 1/\log m \cdot \log^+ \log^+ |F^{(m)}(1)/m!| > 1/2. \end{cases}$$

REMARK. By a lemma [2, p. 287 lemma 3, (2.11)], $F_\phi(s)$ is simply convergent at least for $\sigma > 0$. Hence, taking account ot (2.9) (a), the simple convergence-abscissa of $F_\phi(s)$ is exactly equal to $\sigma = 0$.

PROOF. Let us put

$$\phi(z) = \sum_{n=0}^{\infty} c_n/n! \cdot z^n, \qquad \phi^*(z) = \sum_{n=0}^{\infty} (-1)^n c_n / z^{n+1}.$$

Since

$$\overline{\lim_{n\to\infty}} |c_n|^{1/n} = \overline{\lim_{r\to\infty}} \log M(r)/r \qquad (M(r) = \max_{|z|=r} |\phi(z)|)$$

[6, p. 62], (2.8) yields us

$$\overline{\lim_{n\to\infty}} |c_n|^{1/n} = 0. \tag{2.11}$$

Then, by Cramer-Ostrowski's theorem [4, pp. 49–52], for any given $\varepsilon\,(>0)$, we have

$$F_\phi(s) = 1/2\pi i \cdot \int_{|z|=\epsilon} F(s+z)\phi^*(z)\,dz \tag{2.12}$$

On account of (2.12), the regularity of $F_\phi(s)$ at $s = s_0$ follows from the regularity of $F(s)$ at $s = s_0$. Hence, (2.10) (a) follows from (2.9) (a). Thus, to establish our lemma, it suffices to prove that (2.14) follows from (2.13):

$$\text{(2.13)} \quad \begin{cases} \text{(a)} & \overline{\lim_{m\to\infty}} |F^{(m)}(1)/m!|^{\frac{1}{m}} = 1, \\ \text{(b)} & \overline{\lim_{m\to\infty}} 1/\log m \cdot \log^+ \log^+ |F^{(m)}(1)/m!| \leqq 1/2. \end{cases}$$

$$\text{(2.14)} \quad \begin{cases} \text{(a)} & \overline{\lim_{m\to\infty}} |F_\phi^{(m)}(1)/m!|^{\frac{1}{m}} = 1, \\ \text{(b)} & \overline{\lim_{m\to\infty}} 1/\log m \cdot \log^+ \log^+ |F_\phi^{(m)}(1)/m!| \leqq 1/2. \end{cases}$$

By (2.11), for any given $\varepsilon\,(>0)$, there exists a constant $K(\varepsilon)$ such that

$$|c_n| < K(\varepsilon/2)^n \qquad (n = 0, 1, 2, \ldots).$$

Therefore

$$(2.15)\qquad \max_{|z|=\varepsilon}|\phi^*(z)| \leqq \sum_{n=0}^{\infty}|c_n|/\varepsilon^{n+1} < 2K/\varepsilon.$$

Combining (2.12) with (2.15), we get

$$M(r, F_\phi) = \max_{|s-1|=r}|F_\phi(s)| = |F_\phi(s_0)| \qquad (|s_0-1| = r)$$

$$\leqq M(R,F)\frac{1}{2\pi}\int_{|z|=(1-r)/2}|\phi^*(z)|\,|dz| \leqq 2K\cdot M(R,F),$$

Where $M(R,F) = \max_{|s-1|=R}|F(s)|$, $R = r + (1-r)/2$, $0 < r < 1$.

Hence,

$$(2.16)\qquad \rho_\phi = \overline{\lim_{r\to1}}\, 1/\log(1/1-r)\cdot\log^+\log^+ M(r,F_\phi)$$

$$\leqq \overline{\lim_{R\to1}}\, 1/\log(1/1-R)\cdot\log^+\log^+ M(R,F) = \rho.$$

Therefore, by (2.13) and lemma 3,

$$\rho_\phi \leqq \rho \leqq 1,$$

so that, again by this lemma, we have (2.14)(b), which proves our lemma.

LEMMA 5. *Under the same assumptions as in the main theorem, we have*

$$\text{(i)}\quad \underline{\lim_{\nu\to\infty}}(r_{\nu+1}-r_\nu) > 0, \qquad \underline{\lim_{\nu,n\to\infty}}|r_\nu - \lambda_n| > 0,$$

$$\text{(ii)}\quad \overline{\lim_{\nu\to\infty}}\log\nu/\log r_\nu < \alpha\ (\leqq 1/2),$$

provided that $\{r_\nu\}$ *is the sequence arranged in the order of magnitude of* $\{1/2\cdot(\lambda_n+\lambda_{n-1})\}$, *where the sign-change occurs between* $b_{n,k}$ *and* $b_{n-1,k}$ (λ_n, $\lambda_{n-1} \in I_k$; $k = 1, 2, \ldots$).

PROOF. On account of (c) of the main theorem, (i) is evident. Taking suitable subsequence, if necessary, we can suppose that

$$[x_{k+1}] > 2[x_k]\cdot(1+w)/(1-w), \qquad (k = 1, 2, \ldots).$$

Hence,

$$(2.17)\qquad \begin{cases} 1/2\cdot[x_{k+1}] > [x_k] \\ [x_{k+1}](1-w) > [x_k](1+w), \end{cases} \quad \text{so that } I_i\cdot I_j = 0,\ i \neq j.$$

Putting $\overline{\lim_{k\to\infty}}\log\sigma_k/\log[x_k] = \beta < \alpha$, by virtue of ($b$), for any given $\varepsilon\,(>0)$, there exists $k(\varepsilon)$ such that

$$(2.18)\qquad \sigma_{k+i} < [x_{k+i}]^{\beta+\varepsilon} \qquad (\beta+\varepsilon<\alpha,\ i = 0, 1, 2, \ldots).$$

Therefore, by (2.17) and (2.18)

$$(2.19)\qquad \sum_{i=0}^{r}\sigma_{k+i} < \sum_{i=0}^{r}[x_{k+i}]^{\beta+\varepsilon} < [x_{k+r}]^{\beta+\varepsilon}\cdot\sum_{i=0}^{r}\left(\frac{1}{2}\right)^{(\beta+\varepsilon)i}$$

$$< [x_{k+r}]^{\beta+\varepsilon}\left\{1-\left(\frac{1}{2}\right)^{\beta+\varepsilon}\right\}^{-1}$$

For sufficiently large $r \geqq r(\varepsilon)$, we have easily

$$(2.20) \qquad \sum_{i=1}^{k-1} \sigma_i < [x_{k+r}]^{\beta+\epsilon} \qquad \text{for } r \geqq r(\varepsilon).$$

By (2.19) and (2.20)

$$(2.21) \qquad \sum_{i=1}^{m} \sigma_i < [x_m]^{\beta+\epsilon} \cdot O(1) \qquad \text{for } m \geqq k(\varepsilon) + r(\varepsilon).$$

If $r_\nu \in I_k$, $\nu = \sigma_1 + \sigma_2 + \dots + \sigma_{k-1} + \sigma'_k$, $\sigma'_k \leqq \sigma_k$. Hence, by (2.21)

$$\log \nu < \log\left(\sum_{i=1}^{k} \sigma_i\right) < (\beta + \varepsilon) \log[x_k] + O(1),$$

so that

$$\lim_{\nu\to\infty} \log \nu / \log r_\nu \leqq (\beta + \varepsilon) \overline{\lim_{k\to\infty}} \log[x_k] / \log\{[x_k]\cdot(1 - w)\} = (\beta + \varepsilon).$$

Letting $\varepsilon \to 0$,

$$\overline{\lim_{\nu\to\infty}} \log \nu / \log r_\nu \leqq \beta < \alpha,$$

which proves our lemma.

LEMMA 6. *Under the same assumptions as above, there exists the integral function* $\phi(z)$ ($\phi(\lambda_n)$: *real*) *such that*

(i) $\{b_{n,k} \cdot \phi(\lambda_n)\}$ ($\lambda_n \in I_k$ $k = 1, 2, \dots$) *has the same sign.*

(ii) *for any given* $\varepsilon\,(> 0)$,

$$(2.22) \qquad \begin{cases} \text{(a)} \quad |\phi(z)| < \exp(|z|^{\gamma+\epsilon}) & \textit{for } |z| > R(\varepsilon), \\ \text{(b)} \quad \exp(-\lambda_n^{\gamma+\epsilon}) < |\phi(\lambda_n)| < \exp(\lambda_n^{\gamma+\epsilon}) & \textit{for } \lambda_n > R(\varepsilon), \end{cases}$$

where $\gamma = \overline{\lim_{\nu\to\infty}} \log \nu / \log r_\nu$ $\quad (< \alpha \leqq 1/2)$.

(iii) $F_\phi(s) = \sum_{n=1}^{\infty} a_n\, \phi(\lambda_n) \exp(-\lambda_n s)$ has the simple convergence-abscissa $\sigma = 0$.

PROOF. The convergence-exponent of $\{r_\nu\}$ is $\gamma\,(< \alpha \leqq 1/2)$ (lemma 5 (ii)). Let $\phi(z)$ be the canonical product of $\{r_\nu\}$

$$\phi(z) = \prod_{\nu=1}^{\infty} E(z/r_\nu, 0) = \prod_{\nu=1}^{\infty} (1 - z/r_\nu).$$

Then, (i) is evident. Since the order of the canonical product of $\{r_\nu\}$ is equal to the convergence-exponent of $\{r_\nu\}$, the order of $\phi(z)$ is γ. Hence, (2.22) (*a*) is obvious. By the well-known property of the canonical product, we have

$$|\phi(z)| > \exp(-|z|^{\gamma+\epsilon}) \text{ for } |z| > R(\varepsilon),\ |z - r_\nu| > r_\nu^{-(\gamma+\epsilon)} \ (\nu = 1, 2, \dots).$$

Therefore, by lemma 5 (i), (2.22) (*b*) follows immediately.

By (2.22) and a lemma [2, p. 287 lemma 3], $F_\phi(s)$ has the simple convergence-abscissa $\sigma = 0$. q. e. d.

3. Proof of the main Theorem. Let us put

$$O_m(F_\phi) = (e/m)^m \sum_{m(1-w)\leqq\lambda_n\leqq m(1+w)} a_n\phi(\lambda_n)\lambda_n^m \exp(-\lambda_n) \qquad (0<w<1),$$

where $\phi(z)$ is the integral function defined in lemma 6.
Then by lemma 6(i),

$$\begin{aligned} (3.1) \qquad |O_{[x_k]}(F_\phi)| &= |\exp(-i\gamma_k)\cdot O_{[x_k]}(F_\phi)| \\ &\geqq \Big|\sum_{\lambda_n\in I_k} b_{n,k}\,\phi(\lambda_n)(\lambda_n e/[x_k])^{[x_k]}\exp(-\lambda_n)\Big| \\ &\geqq \Big|\sum_{[x_k]\leqq\lambda_n<x_k} b_{n,k}\,\phi(\lambda_n)(\lambda_n e/[x_k])^{[x_k]}\exp(-\lambda_n)\Big| \\ &> \Big|\sum_{[x_k]\leqq\lambda_n<x_k} b_{n,k}\,\phi(\lambda_n)\Big|\cdot e^{-1}. \end{aligned}$$

On the other hand, by (2.23) (b)

$$\begin{aligned} |\Delta_k| &= \Big|\sum_{[x_k]\leqq\lambda_n<x_k} b_{n,k}\Big| = \Big|\sum_{[x_k]\leqq\lambda_n<x_k} b_{n,k}\,\phi(\lambda_n)\cdot 1/\phi(\lambda_n)\Big| \\ &\leqq \Big|\sum_{[x_k]\leqq\lambda_n<x_k} b_{n,k}\cdot\phi(\lambda_n)\Big|\cdot\exp(x_k^{\gamma+\varepsilon}) \qquad \text{for } x_k > R(\varepsilon), \end{aligned}$$

so that

$$(3.2) \qquad \Big|\sum_{[x_k]\leqq\lambda_n<x_k} b_{n,k}\,\phi(\lambda_n)\Big| \geqq \exp(-x_k^{\gamma+\varepsilon})\cdot|\Delta_k| \qquad \text{for } x_k > R(\varepsilon).$$

Hence, by (3.1) and (3.2)

$$(3.3) \qquad |O_{[x_k]}(F_\phi^\gamma)|\cdot\exp(1+x_k^{\gamma+\varepsilon}) \geqq \Delta_k \qquad \text{for } x_k > R(\varepsilon)$$

Since $\gamma+\varepsilon<\alpha\leqq 1/2$, by (3.3) and (1.2)

$$\overline{\lim_{k\to\infty}}\, |O_{[x_k]}(F_\phi)|^{1/[x_k]} \geqq \overline{\lim_{k\to\infty}}\, |\Delta_k|^{1/[x_k]} = 1,$$

$$\begin{aligned} \overline{\lim_{k\to\infty}}\, 1/\log[x_k]\cdot\log^+\log^+|O_{[x_k]}(F_\phi)| &\geqq \overline{\lim_{k\to\infty}}\, 1/\log x_k\cdot\log^+\log^+|\Delta_k| \\ &\quad -(\gamma+\varepsilon) = 1/2 + (\alpha-\gamma-\varepsilon) > 1/2, \end{aligned}$$

so that

$$(3.4) \qquad \begin{aligned} &\lim_{m\to\infty} |O_m(F_\phi)|^{\frac{1}{m}} \geqq 1, \\ &\overline{\lim_{m\to\infty}}\, 1/\log m\cdot\log^+\log^+|O_m(F_\phi)| > 1/2. \end{aligned}$$

Since $F_\phi(s)$ has the simple convergence-abscissa $\sigma=0$ by lemma 6(iii), by (3.4) and lemma 2,

$$(3.5) \qquad \begin{cases} \overline{\lim_{m\to\infty}}\, |F_\phi^{(m)}(1)/m!|^{\frac{1}{m}} = 1, \\ \overline{\lim_{m\to\infty}}\, 1/\log m\cdot\log^+\log^+|F_\phi^{(m)}(1)/m!| > 1/2. \end{cases}$$

Hence, by (2.22) (a) and lemma 4,

$$\begin{cases} \overline{\lim_{m\to\infty}}\, |F^{(m)}(1)/m!|^{\frac{1}{m}} = 1, \\ \overline{\lim_{m\to\infty}}\, 1/\log m\cdot\log^+\log^+|F^{(m)}(1)/m!| > 1/2. \end{cases}$$

Therefore, by lemma 2 and lemma 1, $s=0$ is Picard's point for (1.1), which proves our main theorem.

4. Theorems. We can deduce series of theorems from the main theorem.

THEOREM 1. *Let* (1.1) *have the simple convergence-abscissa* $\sigma_s = 0$. *Then* $s = 0$ *is Picard's point for* (1.1), *provided that there exist two sequences* $\{\lambda_{n_k}\}$, $\{\gamma_k\}$ (γ_k: real) *such that*

(4.1) (a) $\overline{\lim_{k\to\infty}}\, 1/\lambda_{n_k}\cdot \log|b_k| = 0,\ \overline{\lim_{k\to\infty}}\, 1/\log\lambda_{n_k}\cdot\log^+\log^+|b_k| = 1/2+\alpha,$ *where* $0<\alpha\leqq 1/2,\ b_k = \Re\,(a_{n_k}\exp(-i\gamma_k))$.

(b) $\overline{\lim_{k\to\infty}}\, \log\sigma_k/\log[\lambda_{n_k}] < \alpha$, *where* σ_k: *the number of sign-changes of* $b_{n,k} = \Re(a_n\exp(-i\gamma_k))$, $\lambda_n\in I_k[[\lambda_{n_k}](1-w),\ [\lambda_{n_k}](1+w)]$ $(0<w<1)$.

(c) *the sequence* $b_{n,k}\,(\lambda_n\in I_k)$ *has the normal sign-change in* $\{I_k\}$ $(k = 1,2,\ldots)$.

PROOF. Taking account of the main theorem, it suffices to prove the existence of a sequence $\{x_k\}$ such that

$$(4.2)\qquad \begin{cases} \text{(i)}\ \ [x_k] = [\lambda_{n_k}] & (k=1,2,\ldots) \\ \text{(ii)}\ \ \overline{\lim\limits_{k\to\infty}}\, 1/x_k\cdot\log|\Delta_k| = 0, & \\ \qquad \overline{\lim\limits_{k\to\infty}}\, 1/\log x_k\cdot\log^+\log^+|\Delta_k| = 1/2+\alpha' & (0<\alpha\leqq\alpha'\leqq 1/2) \end{cases}$$

where $\Delta_k = \sum\limits_{[x_k]\leqq\lambda_n<x_k} b_{n,k}$.

Let us put

$$(4.3)\qquad \begin{cases} \Delta_1 = \overline{\lim\limits_{\substack{x\to\infty\\ x\in\{I_k\}}}}\, 1/x\cdot\log\left|\sum\limits_{[x]\leqq\lambda_n<x} b_{n,k}\right|, \\ \Delta_2 = \overline{\lim\limits_{\substack{x\to\infty\\ x\in\{I_k\}}}}\, 1/\log x\cdot\log^+\log^+\left|\sum\limits_{[x]\leqq\lambda_n<x} b_{n,k}\right| = 1/2+\alpha', \end{cases}$$

where I_k: $[\lambda_{n_k}]\leqq x<[\lambda_{n_k}]+1$ $(k=1,2,\ldots)$. Then, by the entirely similar arguments as in the previous note [2, p. 290], we have

$$(4.4)\qquad \Delta_1 = 0.$$

Hence, we can easily prove that

$$(4.5)\qquad 0\leqq\Delta_2\leqq 1,\ \text{i.e.}\ -1/2\leqq\alpha'\leqq 1/2.$$

On account of (4.3), for any given $\varepsilon\,(>0)$, there exists $X(\varepsilon)$ such that

$$(4.6)\qquad \left|\sum_{\substack{[x]\leqq\lambda_n<x\\ x\in\{I_k\}}} b_{n,k}\right| < \exp\{x^{1/2+\alpha'+\varepsilon}\} \qquad \text{for } [x]>X(\varepsilon).$$

Now we have easily

$$b_k = \sum_{[\lambda_{n_k}]\leqq\lambda_n\leqq\lambda_{n_k}} b_{n,k} - \sum_{[\lambda_{n_k}]\leqq\lambda_n<\lambda_{n_k-1}} b_{n,k},\quad \text{if } [\lambda_{n_k}]\leqq\lambda_{n_k-1}<\lambda_{n_k}.$$

$$= \sum_{[\lambda_{n_k}] \leqq \lambda_n \leqq \lambda_{n_k}} b_{n,k}, \qquad \text{if } \lambda_{n_k-1} < [\lambda_{n_k}] < \lambda_{n_k}.$$

Therefore, by (4.6)

$$|b_k| < 2\exp\{x^{1/2+\alpha'+\epsilon}\} < 2\exp\{([\lambda_{n_k}]+1)^{1/2+\alpha'+\epsilon}\} \qquad \text{for } [\lambda_{n_k}] > X(\varepsilon),$$

so that

$$1/2 + \alpha = \overline{\lim_{k\to\infty}}\, 1/\log \lambda_{n_k} \cdot \log^+ \log^+ |b_k| \leqq 1/2 + \alpha' + \varepsilon.$$

Letting $\varepsilon \to 0$, on account of (4.1) and (4.5),

$$0 < \alpha \leqq \alpha' \leqq 1/2. \tag{4.7}$$

Taking account of (4.3), (4.4) and (4.7), there exist two sequences $\{x_k'\}$, $\{x_k''\}$ such that

$$\begin{cases} \Delta_1 = \overline{\lim_{k\to\infty}}\, 1/x_k' \cdot \log \left| \sum_{[x_k'] \leqq \lambda_n < x_k'} b_{n,k} \right| = 0 & ([x_k'] = [\lambda_{n_k}]), \\ \Delta_2 = \overline{\lim_{k\to\infty}}\, 1/\log x_k'' \cdot \log^+ \log^+ \left| \sum_{[x_k''] \leqq \lambda_n < x_k''} b_{n,k} \right| = 1/2 + \alpha' & ([x_k''] = [\lambda_{n_k}], \end{cases} \tag{4.8}$$

$$(0 < \alpha \leqq \alpha' < 1/2).$$

Let us define $\{x_k\}$ as follows:

$$\operatorname{Max}\left\{ \left| \sum_{[x_k'] \leqq \lambda_n < x_k'} b_{n,k} \right|, \left| \sum_{[x_k''] \leqq \lambda_n < x_k''} b_{n,k} \right| \right\} = \left| \sum_{[x_k] \leqq \lambda_n < x_k} b_{n,k} \right|,$$

$$(x_k = x_k' \text{ or } x_k'', \ [x_k] = [x_k'] = [x_k''] = [\lambda_{n_k}]).$$

Then, by (4.3) and (4.8), we can easily establish (4.2), which proves our theorem.

THEOREM 2. *Let* (1.1) *have the simple convergence-abscissa* $\sigma_s = 0$. *Then* $s = 0$ *is Picard's point for* (1.1), *provided that there exists a sequence* $\{x_k\}$ $(0 < x_k \uparrow \infty)$ *such that*

(4.9) (a) $\overline{\lim_{k\to\infty}}\, 1/x_k \cdot \log |\Delta_k| = 0, \quad \overline{\lim_{k\to\infty}}\, 1/\log x_k \cdot \log^+ \log^+ |\Delta_k| > 1/2,$

where $\Delta_k = \sum_{[x_k] \leqq \lambda_n < x_k} \Re(a_n)$.

(b) $\Re(a_n) \geqq 0$; *for* $[x_k](1-w) \leqq \lambda_n \leqq [x_k](1+w)$, $(k = 1, 2, \ldots, \ 0 < w < 1)$.

For its proof, we need only to put $\gamma_k = 0$ $(k = 1, 2, \ldots)$ in the main theorem.

THEOREM 3. *Let* (1.1) *have the simple convergence-abscissa* $\sigma_s = 0$. *Then* $s = 0$ *is Picard's point for* (1.1), *provided that there exists a sequence* $\{x_k\}$ *such that*

(a) $\overline{\lim_{k\to\infty}}\, 1/x \cdot \log |\Delta_k^*| = 0, \quad \overline{\lim_{k\to\infty}}\, 1/\log x_k \cdot \log^+ \log^+ |\Delta_k^*| = 1/2 + \alpha$ $(\alpha > 0)$,

where $\Delta_k^* = \sum_{[x_k] \leqq \lambda_n < x_k} a_n$

(4. 10) (b) $\Re(a_n) \geqq 0$ *for* $\lambda_n \in [[x_k](1-w),\ [x_k](1+w)]$ $(k=1,2,\dots\ 0<w<1)$,

(c) (i) $\lim\limits_{\substack{n\to\infty \\ \lambda_n \in \{I_k\}}} 1/\lambda_n \cdot \log(\cos\theta_n) = 0$,

(ii) $\overline{\lim\limits_{\substack{n\to\infty \\ \lambda_n \in \{I_k\}}}}\ 1/\log\lambda_n \cdot \log^+ \log^+\{(\cos\theta_n)^{-1}\} < \alpha$,

where $\theta_n = \arg(a_n)$, $I_k : [x_k] \leqq x < x_k$ $(k=1,2,\dots)$.

PROOF. On account of theorem 2, it is sufficient to prove (4. 9) (*a*). By the entirely similar arguments as in the previous note [7, p. 293], and (4. 10) (*a*), (*c*),

$$\overline{\lim_{k\to\infty}}\ 1/x_k \cdot \log|\Delta_k| = 0.$$

Hence we need only establish

$$\overline{\lim_{k\to\infty}}\ 1/\log x_k \cdot \log^+ \log^+ |\Delta_k| > 1/2.$$

On the other hand,

$$|\Delta_k| = \Big|\sum_{\lambda_n \in I_k} \Re(a_n)\Big| = \sum_{\lambda_n \in I_k} |a_n| \cos\theta_n$$
$$\geqq \cos(\theta_{n_k}) \sum_{\lambda_n \in I_k} |a_n| \geqq \cos(\theta_{n_k}) \Big| \sum_{\lambda_n \in I_k} a_n \Big|,$$

where $\cos(\theta_{n_k}) = \operatorname*{Min}\limits_{\lambda_n \in I_k} \{\cos(\theta_n)\}$. Therefore

$$\overline{\lim_{k\to\infty}}\ \log\lambda_{n_k}/\log x_n \cdot 1/\log\lambda_{n_k} \cdot \log^+ \log^+ \{(\cos\theta_{n_k})^{-1}\}$$
$$+ \overline{\lim_{k\to\infty}}\ 1/\log x_k \cdot \log^+ \log^+ |\Delta_k| \geqq \overline{\lim_{k\to\infty}}\ 1/\log x_k \cdot \log^+ \log^+ |\Delta_k^*|$$
$$= 1/2 + \alpha.$$

so that, by (c) (ii),

$$\overline{\lim_{k\to\infty}}\ 1/\log x_k \cdot \log^+ \log^+ |\Delta_k| \geqq 1/2 + \alpha - \overline{\lim_{\substack{n\to\infty \\ \lambda_n \in \{I_k\}}}}\ 1/\log\lambda_n \cdot \log^+ \log^+\{(\cos\theta_n)^{-1}\}$$
$$> 1/2,$$

which is to be proved.

As its immediate corollary, we get

COROLLARY 1 [1, p. 612]. *Let* (1. 1) *with* $|\arg(a_n)| \leqq \theta < \pi/2$ *have the simple convergence-abscissa* $\sigma_s = 0$. *Then* $s = 0$ *is Picard's point provided that there exists a sequence* $\{x_k\}$ $(0 < x_k \uparrow \infty)$ *such that*

$$\overline{\lim_{k\to\infty}}\ 1/x_k \cdot \log|\Delta_k^*| = 0, \qquad \overline{\lim_{k\to\infty}}\ 1/\log x_k \cdot \log^+ \log^+ |\Delta_k^*| > 1/2,$$

where $\Delta_k^* = \sum\limits_{[x_k] \leqq \lambda_n < x_k} a_n$.

THEOREM 4. *Let* (1. 1) *have the simple convergence-abscissa* $\sigma_s = 0$. *Then every point on* $\sigma = 0$ *is Picard's point for* (1. 1), *provided that there exists a sequence* $\{\lambda_{n_k}\}$ *such that*

(4.11) (a) $\overline{\lim_{k\to\infty}} 1/\lambda_{n_k} \cdot \log|a_{n_k}| = 0, \quad \overline{\lim_{k\to\infty}} 1/\log\lambda_{n_k} \cdot \log^+ \log^+ |a_{n_k}| = 1/2+\alpha, \quad (0<\alpha\leqq 1/2).$

(b) $\overline{\lim_{k\to\infty}} \log s_k/\log[\lambda_{n_k}] < \alpha$, *where* s_k: *the number of* $a_n \neq 0$, $\lambda_n \in I_k[[\lambda_{n_k}](1-w),\ [\lambda_{n_k}](1+w)]$ $(k=1,2,\dots,\ 0<w<1)$.

(c) $\lim_{\substack{n\to\infty \\ \lambda_n,\lambda_{n+1}\in I_k}} (\lambda_{n+1}-\lambda_n) > 0.$

PROOF. Putting $\gamma_k = \arg(a_{n_k})$ $(k=1,2,\dots)$ in theorem 1, all the assumptions of theorem 1 is evidently satisfied. Hence $s=0$ is Picard's point. By the transformation $s=s'+it$ and the arguments as above, $s=it$ is Picard's point for (1.1), which proves our theorem.

As its corollary we obtain

COROLLARY 2. *Let* (1.1) *with* $\underline{\lim}_{n\to\infty}(\lambda_{n+1}-\lambda_n)>0$ *have the simple convergence-abscissa* $\sigma_s=0$. *If* $\overline{\lim_{n\to\infty}} 1/\log\lambda_n \cdot \log^+\log^+|a_n| = 1/2+\alpha$ $(\alpha>0)$, *and* $\overline{\lim_{n\to\infty}} \log n/\log\lambda_n < \alpha$, *then every point on* $\sigma=0$ *is Picard's point.*

PROOF. Since evidently $\lim_{n\to\infty}\log n/\lambda_n = 0$, by G. Valiron's theorem [4, p. 4] the simple convergence-abscissa of (1.1) is determined by

(4.12) $$\overline{\lim_{n\to\infty}} 1/\lambda_n \cdot \log|a_n| = 0.$$

Hence we can easily prove that

$$\overline{\lim_{n\to\infty}} 1/\log\lambda_n \cdot \log^+\log^+|a_n| = 1/2+\alpha,\ 0<\alpha\leqq 1/2.$$

Therefore, for any given $\varepsilon(>0)$, there exists a sequence $\{\lambda_{n_k}\}$ such that

(4.13) $$\overline{\lim_{k\to\infty}} 1/\log\lambda_{n_k} \cdot \log^+\log^+|a_{n_k}| = 1/2+\alpha,\ 0<\alpha\leqq 1/2,$$

$$|a_{n_k}| > \exp\{\lambda_{n_k}^{\frac{1}{2}+\alpha-\epsilon}\} \qquad (k=1,2,\dots).$$

so that, by (4.12)

$$0 = \overline{\lim_{n\to\infty}} 1/\lambda_n \cdot \log^+|a_n| \geqq \overline{\lim_{k\to\infty}} 1/\lambda_{n_k} \cdot \log^+|a_{n_k}| \geqq 0, \text{ i.e.}$$

(4.14) $$\overline{\lim_{k\to\infty}} 1/\lambda_{n_k} \cdot \log^+|a_{n_k}| = 0.$$

Denoting by $N(r)$ the number of λ_n's contained in $[0,r]$, by $\overline{\lim_{n\to\infty}} \log n/\log\lambda_n = \beta < \alpha$, for any given $\varepsilon(>0)$,

$$N(r) < \lambda_{N(r)}^{\beta+\epsilon} < r^{\beta+\epsilon} \qquad \text{for } r \geqq R(\varepsilon),\ \beta+\varepsilon<\alpha.$$

Hence $\quad 0 \leqq s_k \leqq N([\lambda_{n_k}](1+w)) < \{[\lambda_{n_k}](1+w)\}^{\beta+\epsilon}$,

where s_k: the number of $a_n \neq 0$, $\lambda_n \in I_k[[\lambda_{n_k}](1-w),\ [\lambda_{n_k}](1+w)]$, so that

$$\overline{\lim_{k\to\infty}} \log s_k/\log[\lambda_{n_k}] \leqq \beta+\varepsilon<\alpha.$$

Letting $\varepsilon\to 0$,

(4.15) $$\overline{\lim_{k\to\infty}} \log s_k/\log[\lambda_{n_k}] \leqq \beta<\alpha.$$

By (4.14), (4.13), (4.15) and $\varliminf_{n\to\infty}(\lambda_{n+1}-\lambda_n)>0$, all the assumptions of theorem 4 are satisfied, which proves our corollary.

Since $\lim_{n\to\infty}\log n/\log\lambda_n = 0$, follows from $\varliminf_{n\to\infty}\lambda_{n+1}/\lambda_n>1$, we get

COROLLARY 3 [1, p. 610]. *Let* (1.1) *with* $\varliminf_{n\to\infty}\lambda_{n+1}/\lambda_n>1$ *have the simple convergence-abscissa* $\sigma_s=0$. *If* $\varlimsup_{n\to\infty} 1/\log\lambda_n\cdot\log^+\log^+|a_n|>1/2$, *then every point on* $\sigma=0$ *is Picard's point for* (1.1).

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MATHEMATICAL INSTITUTE, WASEDA UNIVERSITY.