ON AUTOMORPHISMS OF W^*-ALGEBRAS LEAVING THE CENTER ELEMENTWISE INVARIANT

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The present paper deals mainly with automorphisms of W^*-algebras leaving the center elementwise invariant. By an automorphism of a W^*-algebra M, we shall always understand a ring-automorphism $a \to a^\alpha$ of M such that $a^{*\alpha} = a^{\alpha *}$ for all $a \in M$. As one easily see, an inner automorphism (by a unitary element) of a W^*-algebra leaves the center elementwise invariant, and carries each projection on an equivalent one. We shall discuss on the question whether an automorphism of a W^*-algebra leaving the center elementwise invariant carries each projection on an equivalent one or not. Even for factors, we have, it seems, very few knowledge on this question in the literature. In § 1, it turns out that if M is finite, each projection carries by the automorphism considered on an equivalent one, and that if M is properly infinite, each infinite projection behaves likely, but not so for finite projections, as it is shown by an example due to Y. Misonou. In the course of the proof, we employ certain results in my preceding paper *on the invariants of W^*-algebras* [3]. Next, in § 2, § 3, we shall find the condition that any automorphism of W^*-algebra leaving the center elementwise invariant be inner. We see that one of our results is a generalization of the theorem of Kaplansky for a W^*-algebra of type I.

Throughout this paper, we shall refer the same definitions and notations as in [3]. A W^*-algebra M means a weakly closed self-adjoint operator algebra with the identity on a Hilbert space H. We denote by $M^\natural$ the center of M.

1. In this section, we will answer the question mentioned in the introduction, and in connection with this, it is shown that there exists an automorphism of a factor of type II which is not inner.

Our theorem is stated as follows:

THEOREM 1. *Let M be a W^*-algebra and α an automorphism of M, leaving the center elementwise invariant.*

(A) *If M is finite, then each projection e of M is equivalent to e^α.*

(B) *If M is properly infinite, then each infinite projection e of M is equivalent to e^α.*

PROOF OF (A). We first notice that $a^{\natural\alpha} = a^{\alpha\natural}$ for all $a \in M$. Since α leaves $M^\natural$ elementwise invariant, we have $e^\natural = e^{\natural\alpha} = e^{\alpha\natural}$. Applying the comparability theorem to e and e^α, there is a projection $h \in M^\natural$ such that $eh \precsim e^\alpha h$ and $e(1-h) \succsim e^\alpha(1-h)$. Hence there is a projection $p \in M$ such that $eh \sim p \leqq e^\alpha h$. Then.

$$(e^\alpha h - p)^\natural = e^{\alpha\natural} h - p^\natural = e^{\alpha\natural} h - e^\natural h = 0.$$

Thus

$$e^\alpha h - p = 0, \ e^\alpha h = p, \quad \text{or } eh \sim e^\alpha h.$$

Similarly, $e(1-h) \sim e^\alpha(1-h)$. Therefore $e \sim e^\alpha$.

To prove (B), we need the following:

LEMMA 1. *Let M be a properly infinite W^*-algebra without purely infinite part, whose center $M^\natural$ is countably decomposable, and let $\{e_i\}_{i\in I}$, $\{e'_j\}_{j\in J}$ be infinite families of orthogonal finite projections in M. If*

$$1 = \sum_{i\in I} e_i = \sum_{j\in J} e'_j,$$

then the Cardinal of I equals to the Cardinal of J.

PROOF. Since $M^\natural$ is countably decomposable, we may assume that the invariant of $M_{e_i H} \geqq 1$ for all $i \in I$. By Lemma 1 in [3] $M_{e_i H}$ has a separating vector φ_i. Then the proof is completed as in the proof of Lemma 2 [3].

PROOF OF (B). At first, we assume that M is properly infinite without purely infinite part. Since α leaves the center elementwise invariant, we may assume as follow;

(i) $e = \sum_{i\in I} e_i$, where $\{e_i\}_{i\in I}$ is an infinite family of equivalent, orthogonal finite projections in M.

(ii) e is countably decomposable for $M^\natural$.

Then, an infinite family $\{e_i^\alpha\}$ consists of equivalent, orthogonal finite projections in M and $e^\alpha = \sum_{i\in I} e_i^\alpha$. Hence e^α is also infinite and countably decomposable for $M^\natural$. For a fixed $i_0 \in I$, there exists a projection $f_{i_0} < e^\alpha$ such that $e_{i_0} \sim f_{i_0}$. Let $\{f_j\}_{j\in J}$ be a maximal family of equivalent orthogonal finite projections $< e^\alpha$ in M containing f_{i_0}. Then the Cardinal of I equals to the Cardinal of J. In fact, if $e^\alpha = \sum_{j\in J} f_j$, applying Lemma 1 to $M_{e^\alpha H}$, this fact is immediately obtained. If $e^\alpha \neq \sum_{j\in J} f_j$, $e_\alpha - \sum_{j\in J} f_j$ is a finite projection. Hence, by Lemma 1, we see also it. Then, since $e_i \sim e_{i_0} \sim f_{i_0} \sim f_i$ for each $i \in I$, $e \precsim e^\alpha$. By symmetry, $e \succsim e^\alpha$. Therefore $e \sim e^\alpha$.

Next, we assume that M is a purely infinite W^*-algebra. Let $\{e_i\}_{i\in I}$ be an infinite family of orthogonal cyclic projections in M such that $e = \sum_{i\in I} e_i$. Since each $M_{e_i H}$ has a separating vector, $M_{e_i H}$ is countably decomposable, and also $M_{e_i H}$ has a generating vector since it is purely infinite. Then $\{e_i^\alpha\}$ is an infinite family of orthogonal cyclic projections in M such that $e^\alpha = \sum_{i\in I} e_i^\alpha$ and each $M_{e_i^\alpha H}$ has a separating vector and a generating vector. Since the central support of e and e^α is identical, by Proposition 3 in [1], $e_i \sim e_i^\alpha$ for each $i \in I$. Therefore $e \sim e^\alpha$.

We must notice that Theorem 1 does not hold in general for a finite projection in a properly infinite W^*-algebra. This is shown by the following example due to Y. Misonou. Let A be an approximately finite factor and

N a factor of type I_∞. Then $M = A \otimes N$ is a factor of type II_∞. Let e be a finite projection in M and f be a finite projection in M which properly contained in e. M is isomorphic to $M_{eH} \otimes N_1$ and $M_{fH} \otimes N_2$, where N_1, N_2 are factors of type I_∞, and A is isomorphic to M_{eH} and M_{fH}. This means that M_{eH}, M_{fH} also are approximately finite factors. Therefore there exists an isomorphism α_1 between M_{eH} and M_{fH}. We can represent as $e \simeq 1_e \otimes p$, $f \simeq 1_f \otimes q$, where 1_e, 1_f are identities in M_{eH}, M_{fH} respectively, and p, q are minimal projections in N_1, N_2 respectively. If α_2 is an isomorphism between N_1 and N_2 which carries p on q, then $\alpha = \alpha_1 \otimes \alpha_2$ define an automorphism of M, which carries e on f. From our construction, α is an automorphism of M as desired. Moreover, by this example, we see that there exists a factor of type II_∞ which admits a non-inner automorphism.

2. We shall research, in this section, the necessary and sufficient condition under which an automorphism of a W^*-algebra leaving the center elementwise invariant is inner. The following lemma is fundamental.

LEMMA 2. *Let M be a W^*-algebra and let $\{e_i\}_{i\in I}$ a family of equivalent orthogonal projections in M such that $1 = \sum_{i\in I} e_i$. If an automorphism α of M leaves the subalgebra $e_{i_0}Me_{i_0}$ (for a fixed $i_0 \in I$) elementwise invariant, then it is inner.*

PROOF. Let w_i be a partially isometric operator such that $w_i^* w_i = e_{i_0}$ and $w_i w_i^* = e_i$. Define $u = \sum_{i\in I} w_i^\alpha w_i^*$, it is easy to see that u is a well defined unitary operator in M, we shall complete the proof by showing $u^* a^\alpha u = a$ for all $a \in M$:

$$\begin{aligned} u^* a^\alpha u &= \sum_{i,j\in I} w_i w_i^{*\alpha} a^\alpha w_j^\alpha w_j^* = \sum_{i,j\in I} w_i (w_i^* a w_j)^\alpha w_j^* \\ &= \sum_{i,j\in I} w_i w_i^* a w_j w_j^* \quad (\text{since } w_i^* a w_j \in e_{i_0} M e_{i_0}) \\ &= \sum_{i,j\in I} e_i a e_j = \left(\sum_{i\in I} e_i\right) a \left(\sum_{j\in J} e_j\right) = a. \end{aligned}$$

REMARK. In the above lemma, it is not difficult to show $M = e_{i_0}Me_{i_0} \otimes N$ where N is a factor of type I. Using this and the theorem of Kaplansky [2], which states that an automorphism of a factor of type I is inner, Lemma 2 can be easily verified. But we will derive from our lemma a generalization of the theorem of Kaplansky [2].

Let M be a finite (resp. a properly infinite) W^*-algebra, and let α an automorphism of M leaving the center elementwise invariant. Then Theorem 1 shows that each (resp. each infinite) projection $e \in M$ is equivalent to e^α, say by w: $w^* w = e$, $ww^* = e^\alpha$. If e satisfies the following property:

(P) $(eae)^\alpha = weaew^*$ for all $a \in M$,

we shall call that e satisfies (P).

THEOREM 2. *Let M be a finite (resp. a properly infinite) W^*-algebra, and let α an automorphism of M leaving the center elementwise invariant. If there exists a (resp. an infinite) propection $e \in M$ with central support* 1 *and satisfying* (*P*), *then α is inner.*

PROOF. We first notice that if e satisfies (P), $eh (h \in M^\natural)$, a projection g equivalent to e and a finite (resp. an infinite) projection bounded by e also satisfy (P). If M is finite, there exists a projection $g \leqq e$ such that $h = \sum_{i \in I} h_i \in M^\natural$; where $\{h_i\}_{i \in I}$ is a family of equivalent orthogonal projections in M containing g. If M is properly infinite, let $\{e_i\}_{i \in I}$ be a maximal family of equivalent orthogonal projections in M containing e, then there exists a projection $h \in M^\natural$ such that $h = \sum_{i \in I} h_i$, where $\{h_i\}_{i \in I}$ is an infinite family of equivalent orthogonal projections in M containing he. Therefore we can take up the maximal family of orthogonal projections $\{p_i\}_{i \in I}$ in $M^\natural$ having the following properties: $p_i = \sum_{j \in J_i} p_{ij}$, where $\{p_{ij}\}_{j \in J_i}$ is a family of equivalent orthogonal projections in M containing p_{ij_0} satisfying (P).

Then $1 = \sum_{i \in I} p_i$, in fact, if $1 - \sum_{i \in I} p_i = p \neq 0$, it is impossible that $e \precsim p$. As the central support of e is 1, there is a non-zero projection $g \in M^\natural$; $pg \leqq eg$. Since eg satisfies (P), there exists a projection f eatisfying (P) bounded by p. This contradicts to the maximality of $\{p_i\}_{i \in I}$.

From the above argument, we may assume that there exists a family of equivalent orthogonal projections $\{e_i\}_{i \in I}$ in M containing e_{i_0} satisfying (P) such that $1 = \sum_{i \in I} e_i$. Then e_i and e_i^α are equivalent, say by $w_i : w_i^* w_i = e_i$, $w_i w_i^* = e_i^\alpha$. Define $u_0 = \sum_{i \in I} w_i^*$, it is easy to see that u_0 is a unitary operator in M and $e_i = u_0 e_i^\alpha u_0^*$. Put $a^\beta = u_0 a u_0^*$ for all $a \in M$. $\gamma = \alpha \cdot \beta$ is an automorphism of M which leaves each e_i invariant. Set $v_0 = u_0 w_{i_0}$, $v_0 v_0^* = u_0 w_{i_0} w_{i_0}^* u_0^* = u_0 e_{i_0}^\alpha u_0^* = e_{i_0}$ and $v_0^* v_0 = w_{i_0}^* u_0^* u_0 w_{i_0} = w_{i_0}^* w_{i_0} = e_{i_0}$. This shows that $a \to v_0 a v_0^*$ $(a \in e_{i_0} M e_{i_0})$ is an inner automorphism of the subalgebra $e_{i_0} M e_{i_0}$. And, γ and $a \to v_0 a v_0^*$ $(a \in e_{i_0} M e_{i_0})$ coincide on $e_{i_0} M e_{i_0}$, in fact, $(e_{i_0} a e_{i_0})^\gamma = u_0 (e_{i_0} a e_{i_0})^\alpha u_0^* = u_0 w_{i_0} e_{i_0} a e_{i_0} w_{i_0}^* u_0^* = v_0 e_{i_0} a e_{i_0} v_0^*$ for all $a \in M$.

Define $v = v_0 + \sum_{i \in I - i_0} e_i$, then

$$
\begin{aligned}
vv^* &= \left(v_0 + \sum_{i \in I - i_0} e_i\right) \cdot \left(v_0^* + \sum_{i \in I - i_0} e_i\right) \\
&= v_0 v_0^* + v_0 \left(\sum_{i \in I - i_0} e_i\right) + \left(\sum_{i \in I - i_0} e_i\right) v_0^* + \left(\sum_{i \in I - i_0} e_i\right)\left(\sum_{i \in I - i_0} e_i\right) \\
&= e_{i_0} + \sum_{i \in I - i_0} e_i = 1.
\end{aligned}
$$

Similarly $v^* v = 1$. It follows that put $a^\delta = vav^*$ for all $a \in M$. δ is an inner automorphism of M.

Now we shall show that δ and γ coincide on $e_{i_0} M e_{i_0}$. For all $a \in e_{i_0} M e_{i_0}$,

$$
a^\delta = vav^* = \left(v_0 + \sum_{i \in I - i_0} e_i\right) a \left(v_0^* + \sum_{i \in I - i_0} e_i\right)
$$

$$= v_0 a v_0^* + v_0 a \left(\sum_{i \in I - i_0} e_i\right) + \left(\sum_{i \in I - i_0} e_i\right) a v_0^* + \left(\sum_{i \in I - i_0} e_i\right) a \left(\sum_{i \in I - i_0} e_i\right)$$

$$= v_0 a v_0^*.$$

Put $\sigma = \delta^{-1} \cdot \gamma$, σ leaves the subalgebra $e_{i_0} M e_{i_0}$ elementwise invariant. By Lemma 1, σ is inner. Therefore γ is inner since δ is inner, since β is inner, we can conclude that α is inner.

3. Applying the result obtained in the preceeding section, we intend to investigate an automorphism of a W^*-algebra of type I, type II and type III.

For an automorphism of a W^*-algebra of type I, we show that the following theorem easily follows from our Theorem 2.

THEOREM 3 (Kaplansky) *Let M be a W^*-algebra of type* I. *If an automorphism of M leaves the center elementwise invariant, it is inner.*

PROOF. Let e be an abelian projection not annihilated by any center projections. Then since e^α is so, e and e^α are equivalent from Lemma 19 in [2], say by w; $w^*w = e$ and $ww^* = e^\alpha$. Since $eMe = M^\natural e$, assume that $eae = a^\# e$ for all $a \in M$; $a^\# \in M^\natural$, then

$$(eae)^\alpha = a^\# wew^* = wa^\# ew^* = weaew^* \text{ for all } a \in M.$$

Therefore it is easily known that we may consider e to be a projection in Theorem 2. Hence α is inner.

Let M be a W^*-algebra and let $\{e_i\}_{i \in I}$ be a family of equivalent orthogonal projections in M such that $1 = \sum_{i \in I} e_i$. Let w_i be a partially isometric operator such that $w_{i1}^* w_{i1} = e_1$ and $w_{i1} w_{i1}^* = e_i$. Define $w_{11} = e_1$ $w_{1i} = w_{i1}^*$, $w_{ij} = w_{i1} w_{1j}$. We refer to $\{w_{ij}\}_{i,j \in I}$ as a system of matrix units. Let D be the set of all elements of M commuting with all w_{ij}. Clearly, D is a W^*-subalgebra of M. Then we will refer to the following result of I. Kaplansky [2; Lemma 14, 15].

LEMMA 3. (1) $M^\natural = D^\natural$, (2) $e_i M e_j = D w_{ij}$, *and the representation in this form of an element $e_i M e_j$ is unique.*

Now let M be a finite W^*-algebra of type II (resp. a properly infinite W^*-algebra). Then, there exist families of equivalent, orthogonal projections $\{e_i\}_{i \in I}$ in M such that $1 = \sum_{i \in I} e_i$(resp. each $e_i \sim 1$ and $1 = \sum_{i \in I} e_i$). Let D_f (resp. D_i) be a subalgebra defined in above.

THEOREM 4. *Let M be a finite W^*-algebra of type* II (*resp. a properly infinite W^*-algebra*). *If an automorphism α of M leaves any subalgebra D_f* (*resp. D_i*) *elementwise invariant, then it is inner.*

PROOF. Let $\{e_i\}_{i \in I}$ be a family of equivalent orthogonal projections in M such that $1 = \sum_{i \in I} e_i$(resp. each $e_i \sim 1$ and $1 = \sum_{i \in I} e_i$) and let $\{w_{ij}\}_{i,j \in I}$ be a system of matrix units. Let D_f(resp. D_i) in Theorem be the subalgebra of M commuting with all w_{ij}.

PROOF of the case (D_f). Put $N_1 = D_f' \cap M$, α leaves a W^*-algebra N_1 invariant, in fact, if a lies in N_1, $ab = ba$ for all $b \in D_f$, and hence $a^\alpha b = ba^\alpha$ for all $b \in D_f$, or $a^\alpha \in N_1$. Next we show that $N_1^\natural = M^\natural$. In order to show this, it is sufficient to prove $e_i(ab - ba)e_j = 0$ for all $b \in M$, where a is any fixed operator in $N_1^\natural$. By Lemma 3(2), $e_i b e_j = b^\# w_{ij}$, $b^\# \in D_f$. Since e_i and w_{ij} lie in N_1, $e_i a b e_j = a e_i b e_j = a b^\# w_{ij} = b^\# w_{ij} a = e_i b e_j a = e_i b a e_j$. Thus $e_i(ab - ba)e_j = e_i a b e_j - e_i b a e_j = 0$.

Let E be a subalgebra of N_1 commuting with w_{ij}. Then, from the above fact and Lemma 3, $E = D^\natural = M^\natural = N_1^\natural$, and hence $e_i N_1 e_i = N_1^\natural e_i$ for all $i \in I$. This shows that each e_i is an abelian projection in N_1. Therefore we obtain that $e_i \sim e_i^\alpha \pmod{N_1}$, say by w_i in N_1; $w_i^* w_i = e_i$, $w_i w_i^* = e_i^\alpha$. By Lemma 3, for all $a \in M$, there is $a^\# \in D_f$ such that $e_i a e_i = a^\# e_i$. Thus $(e_i a e_i)^\alpha = a^\# e_i^\alpha = a^\# w_i e_i w_i^* = w_i a^\# e_i w_i^* = w_i e_i a e_i w_i^*$. Theorem 2 shows that α is inner.

PROOF of the case (D_i). Put $N_2 = D_i' \cap M$, N_2 is also properly infinite and each e_i is an abelian projection in N_2. Hence the part remained of the proof is similar to that in the case (D_f).

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