ALMOST HERMITIAN STRUCTURE ON S^6

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A differentiable manifold M of even dimension is called to have an *almost Hermitian structure*, if there exist on M a metric tensor field g and a tensor field φ of type (1, 1) such that

$$\varphi^i_a\varphi^a_j = -\delta^i_j, \qquad g_{ab}\varphi^a_i\varphi^b_j = g_{ij}.$$

It is well known that an almost Hermitian structure can be defined on a sphere S^6 of dimension 6 by making use of the algebra C of Cayley numbers [1]. The group G of all automorphisms of the algebra C acts on S^6 transitively as a group of isometries [3]. We shall show that almost Hermitian structure on S^6 is invariant under the group G, i.e. the tensor fields g and φ are invariant under G. Moreover there exists one and only one affine connection Λ invariant under G with respect to which g and φ are covariantly constant. The unique connection Λ is actually obtained by making use of the fields g and φ, and its torsion and curvature tensor fields are covariantly constant.

1. Almost Hermitian structure on S^6. Let C be the algebra of Cayley numbers. Any element ξ of C may be written in the form

$$\xi = XI + X^0e_0 + X^1e_1 + \dots + X^6e_6 = XI + x,$$

where $X, X^0, X^1, \dots, X^6$ are real numbers and $I, e_0, e_1, \dots, e_6$ form a natural base of the algebra C, I being the unit element of C. ξ is called pure imaginary if $X = 0$. All pure imaginary Cayley numbers form a 7-dimensional subspace E^7 of C. The multiplication table is given by the following:

$$\begin{array}{llll} e_i^2 = -I & & (i = 0, 1, \dots, 6), & \\ e_i\cdot e_j = -e_j\cdot e_i & & (i \neq j;\ i, j = 0, 1, \dots, 6), & \\ e_0\cdot e_1 = e_2, & e_0\cdot e_3 = e_4, & e_0\cdot e_5 = e_6, & \\ e_1\cdot e_4 = e_6, & e_1\cdot e_3 = -e_5, & e_2\cdot e_3 = e_6, & e_2\cdot e_4 = e_5, \end{array}$$

and the other $e_i\cdot e_j$ are given by cyclic permutations of indices.

If $\eta = YI + \sum^6_{=0} Y^ie_i = YI + y$ is an element of C, then we have

$$\xi\cdot\eta = XYI - \sum_{i=0}^{6} X^iY^iI + Xy + Yx + \sum_{i,k=0}^{6}{}_{i\neq k}\, X^iY^ke_i\cdot e_k.$$

For any two pure imaginary Cayley numbers x, y we have

$$x\cdot y = -\sum_{i=0}^{6} X^iY^iI + \sum_{i,k=0}^{6}{}_{i\neq k}\, X^iY^ke_i\cdot e_k,$$

where $\sum^6_{i=0} X^iY^i$ is the *scalar product* of two vectors x and y of E^7 and is denoted by (x, y). The pure imaginary Cayley number $\sum_{i\neq k} X^iY^ke_ie_k$ is called the *vector product* of x and y and is denoted by $x \times y$. Then we have

$$x \cdot y = -(x, y)I + x \times y.$$

The operation of vector product is bilinear, and $x \times y$ is orthogonal to both x and y. We have moreover $x \times y = -y \times x$.

The set of all vectors x of E^7 such that $(x, x) = 1$ is the unit sphere S^6. The scalar product in E^7 induces the natural metric tensor field g on S^6. The tangent space T_x of S^6 at $x \in S^6$ can naturally be identified with the subspace of E^7 orthogonal to x, and we use this identification throughout this paper A linear mapping φ_x of T_x onto itself is defined as follows:

$$\varphi_x(y) = x \times y \quad \text{for} \quad y \in T_x.$$

The correspondence $x \to \varphi_x$ defines a tensor field φ, which together with g gives an almost Hermitian structure on S^6 [1].

If G is the group of all automorphisms of the algebra C of Cayley numbers, then it is a compact simple Lie group with the structure of Cartan's exceptional simple Lie algebra of type (G) [3]. Every element of G leaves the subspace E^7 invariant and is an orthogonal transformation on E^7. In this way G acts on S^6 transitively. Let H be the isotropic subgroup of G leaving e_0 fixed. Then it is easily seen that H is isomorphic to the natural real representation of the unimodular unitary group in three complex variables. Thus H is connected. Hence G is connected, because G/H is a sphere.

The tensor field φ defined above is invariant under G. In fact, for any s of G, x of S^6 and y of T_x we have

$$s(\varphi_x(y)) = s(x \times y) = s(x \cdot y) = s(x) \cdot s(y) = s(x) \times s(y) = \varphi_{s(x)}(s(y))^{1)}.$$

Conversely, let f be an orthogonal transformation of E^7 leaving invariant the tensor field φ on S^6. There corresponds to f a vector-space automorphism f' of the algebra C such that $f'(I) = I$ and the restriction of f' to E^7 coincides with f. It is not hard to see that $f'(x \cdot y) = f'(x) \cdot f'(y)$ for x, y of E^7, and hence $f'(\xi \cdot \eta) = f'(\xi) \cdot f'(\eta)$ for any two Cayley numbers ξ and η, i. e. f' is an automorphism of the algebra C. We assume throughout this paper that the sphere S^6 has almost Hermitian structure defined above. Thus we have

THEOREM 1. *The almost Hermitian structure on S^6 is invariant under the group G of all automorphisms of Cayley numbers. Conversely, the group of all isometric transformations leaving invariant the almost Hermitian structure on S^6 is isomorphic to G.*

2. An invariant affine-connection Λ on S^6. We shall define on S^6 an invariant affine connection with required properties. Let $x = \sum_{i=0}^{6} X^i e_i$ and $y = \sum_{i=0}^{6} Y^i e_i$ be two pure imaginary Cayley numbers. Then we have

$$x \times y = \sum_{j,k=0}^{6} A_k^j Y^k e_j,$$

1) For any $s \in G$ we denote by the same letter s the differential mapping of the transformation $s: S^6 \to S^6$.

where the matrix (A_k^j) has the form [1]:

$$\begin{pmatrix} A_0^0 \dots A_6^0 \\ \dots\dots\dots \\ A_0^6 \dots A_6^6 \end{pmatrix} = \begin{pmatrix} 0 & -X^2 & X^1 & -X^4 & X^3 & -X^6 & X^5 \\ X^2 & 0 & -X^0 & X^5 & -X^6 & -X^3 & X^4 \\ -X^1 & X^0 & 0 & -X^6 & -X^5 & X^4 & X^3 \\ X^4 & -X^5 & X^6 & 0 & -X^0 & X^1 & -X^2 \\ -X^3 & X^6 & X^5 & X^0 & 0 & -X^2 & -X^1 \\ X^6 & X^3 & -X^4 & -X^1 & X^2 & 0 & -X^0 \\ -X^5 & X^4 & -X^3 & X^2 & X^1 & X^0 & 0 \end{pmatrix}.$$

As a local coordinate system $(x^1, \dots, x^6)$ of S^6 in a neighborhood of the point $P_0(=e_0) = (X^0, X^1, \dots, X^6) = (1, 0, \dots, 0)$, we set $x^i = X^i$ $(i = 1, 2, \dots, 6)$. If we denote by φ_j^i the components of the tensor field φ, then we have

$$\varphi_k^j = A_k^j - \frac{A_0^j X^k}{X^0} \qquad (j, k = 1, 2, \dots, 6),$$

where

$$X^k = x^k, \qquad X^0 = \left\{1 - \sum_{i=1}^{6} (x^i)^2\right\}^{\frac{1}{2}}.$$

At the point P_0 we obtain

$$(\varphi_k^j)_{P_0} = A_k^j, \qquad \left(\frac{\partial \varphi_k^j}{\partial x^l}\right)_{P_0} = \frac{\partial A_k^j}{\partial X^l}.$$

The metric tensor field g has the components

$$g_{ij} = \delta_{ij} + \frac{x^i x^j}{(X^0)^2}, \qquad g^{ij} = \delta_{ij} - x^i x^j,$$

where $g^{ia} g_{aj} = \delta_j^i$. Hence the Christoffel symbol Γ_{jk}^i of g_{ij} is given by

$$\Gamma_{jk}^i = x^i \left\{ \delta_{jk} + \frac{x^j x^k}{(X^0)^2} \right\}.$$

We shall define an affine connection Λ_{jk}^i on S^6 by the formula:

$$\Lambda_{jk}^i = \Gamma_{jk}^i - \frac{1}{2} \varphi_{k;a}^i \varphi_j^a, \tag{1}$$

where $\varphi_{j;k}^i$ is the covariant derivative of φ_j^i with respect to the Riemannian connection Γ_{jk}^i. On the other hand we have

$$\varphi_{j;k}^i + \varphi_{k;j}^i = 0, \tag{2}$$

because the group G is transitive on S^6 and

$$(\Gamma_{jk}^i)_{P_0} = 0,$$

$$\left(\frac{\partial \varphi_j^i}{\partial x^k}\right)_{P_0} + \left(\frac{\partial \varphi_k^i}{\partial x^j}\right)_{P_0} = \frac{\partial A_j^i}{\partial X^k} + \frac{\partial A_k^i}{\partial X^j} = 0.$$

By virtue of the equation (2), the connection Λ_{jk}^i defined by (1) exactly coincides with the one introduced by A. Frölicher [1]. The affine connection Λ_{jk}^i is obviously invariant under G. It is well known [1] that the tensor field φ is a covariantly constant with respect to the connection Λ_{jk}^i and the torsion tensor field of Λ_{jk}^i is nothing but the Nijenhuis tensor field, i. e. the torsion tensor field of the almost complex structure φ on S^6. A. Frölicher [1] has shown that the torsion tensor field does not vanish at the point P_0. Hence it does not vanish at every point of S^6.

The covariant derivative $D_k g_{ij}$ with respect to Λ^i_{jk} is zero. In fact,

$$\begin{aligned} 2D_k g_{ij} &= \varphi^b_{k;a}\varphi^a_i g_{bj} + \varphi^b_{k;a}\varphi^a_j g_{ib} = -\varphi^b_{a;k}\varphi^a_i g_{bj} - \varphi^b_{a;k}\varphi^a_j g_{ib} \\ &= \varphi^b_{j;k}\varphi^a_i g_{ba} - \varphi^a_{b;k}\varphi^b_j g_{ia} = -\varphi^b_{j;k}\varphi^a_b g_{ia} + \varphi^a_b\varphi^b_{j;k} g_{ia} = 0, \end{aligned}$$

since

$$\varphi^i_{j;k} + \varphi^i_{k;j} = 0, \quad g_{ia}\varphi^a_{j;k} + g_{ja}\varphi^a_{i;k} = 0, \quad \varphi^i_{a;k}\varphi^a_j + \varphi^i_a\varphi^a_{j;k} = 0.$$

For simplicity, an affine connection Λ on an almost Hermitian space M is called a *C-connection*, if the metric tensor field g and the tensor field φ defining the almost Hermitian structure on M are covariantly constant with respect to Λ.

It has been proved in this section that the affine connection on S^6 defined by (1) is a C-connection and invariant under G.

3. Uniqueness of the invariant C-connection on S^6. We shall prove in this section that there exists uniquely on S^6 an invariant C-connection. For this purpose we shall give some preliminary remarks.

Let G and H be as before, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{h}$ the subalgebra of $\mathfrak{g}$ corresponding to H. Since H is compact, there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathrm{ad}(h)\cdot\mathfrak{m} \subset \mathfrak{m}, \quad \mathfrak{g} = \mathfrak{m} + \mathfrak{h} \qquad \text{(direct sum as vector space)}$$

for any $h \in H$, where $\mathrm{ad}(h)$ denotes the adjoint transformation of h in $\mathfrak{g}$; that is, the homogeneous space G/H is reductive [4]. We may identify the subspace $\mathfrak{m}$ with the tangent space T at the point $P_0 = H$ of $G/H = S^6$. According to this identification, since g and φ are invariant under G, there exist a symmetric bilinear form B on $\mathfrak{m}\times\mathfrak{m}$ corresponding to the metric tensor field g on S^6 and a linear mapping $J: \mathfrak{m}\to\mathfrak{m}$ ($-J^2 =$ identity) corresponding to the tensor field φ on S^6 such that

$$\begin{aligned} &B(\mathrm{ad}(h)\cdot X,\ \mathrm{ad}(h)\cdot Y) = B(X, Y), \\ &J\circ\mathrm{ad}(h) = \mathrm{ad}(h)\circ J, \qquad B(J\cdot X, J\cdot Y) = B(X, Y) \end{aligned}$$

for any $X, Y \in \mathfrak{m}$ and $h \in H$.

If Λ is an invariant affine connection on G/H, then there exists in a unique manner a bilinear mapping α: $\mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ which is invariant by the adjoint representation of H on $\mathfrak{m}$, where the mapping α is the so-called connection function of Λ [4]. If we introduce a linear mapping $\tilde{\alpha}$ of $\mathfrak{m}$ into the Lie algebra $E(\mathfrak{m})$ of all linear endomorphisms of $\mathfrak{m}$ such that

$$\tilde{\alpha}(Z)\cdot Y = \alpha(Z, Y)$$

for any $Y, Z \in \mathfrak{m}$, then by virtue of invariance of the connection function α we have

$$\begin{aligned} \tilde{\alpha}(\mathrm{ad}(U)\cdot Z) &= \mathrm{ad}(U)\circ\tilde{\alpha}(Z) - \tilde{\alpha}(Z)\circ\mathrm{ad}(U) \\ &= [\mathrm{ad}(U), \tilde{\alpha}(Z)] \end{aligned} \tag{3}$$

for any $Z \in \mathfrak{m}$ and $U \in \mathfrak{h}$, where $\mathrm{ad}(U)$ is the endomorphism of $\mathfrak{m}$ nduced by the adjoint transformation of $U(\in \mathfrak{h})$ in $\mathfrak{g}$. Let W_α be the image of $\mathfrak{m}$ by $\widetilde{\alpha}$, and $\mathrm{ad}(\mathfrak{h})$ the subalgebra of $E(\mathfrak{m})$ composed of all endomorphisms induced by the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$. Then W_α is invariant under $\mathrm{ad}(\mathfrak{h})$ because of the relation (3).

Assume moreover that the invariant tensor fields g and φ on S^6 are covarianly constant with respect to the connection Λ. Then the following relations hold true:

$$B(\widetilde{\alpha}(Z)\cdot X, Y) + B(X, \widetilde{\alpha}(Z)\cdot Y) = 0, \quad J\circ\widetilde{\alpha}(Z) = \widetilde{\alpha}(Z)\circ J$$

for any $X, Y, Z \in \mathfrak{m}$. Hence the subspace W_α is contained in $\mathfrak{U}(3)$ which is the subalgebra of $E(\mathfrak{m})$ corresponding to the real representation of the unitary group in three complex variables. On the other hand, the subalgebra $\mathrm{ad}(\mathfrak{h})$ of $E(\mathfrak{m})$ coincides with the subalgebra $\mathfrak{B}(3)$ corresponding to the real representation of the unimodular unitary group in three complex variables. Thus, if we denote by $[\mathfrak{h}, \mathfrak{m}]$ the subspace of $\mathfrak{m}$ spanned by all elements of the form $[U, X] = \mathrm{ad}(U)\cdot X$, $U \in \mathfrak{h}$, $X \in \mathfrak{m}$, then we have $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$. From the fact just obtained and the relation (3), it follows easily that W_α is contained in the derived subalgebra of $\mathfrak{U}(3)$, that is, in $\mathfrak{B}(3)$. Consequently, W_α is an ideal of $\mathfrak{B}(3)$.

Since W_α is an ideal of the simple Lie algebra $\mathfrak{B}(3)$ and $\dim W_\alpha \leqq 6$, we have $W_\alpha = \{0\}$. Hence $\widetilde{\alpha} = 0$, i. e. the connection function α corresponding to the given invariant affine connection Λ is equal to zero. That is to say, there exists one and only one invariant C-connection S^6 and it is the canonical connection of the second kind of the given reductive homogeneous space G/H [4]. Then the covariant derivatives of its torsion and curvature tensor fields are zero.

Summing up the results obtained in the present and previous sections, we have the following theorem:

THEOREM 2. *On the homogeneous almost Hermitian space $S^6 = G/H$ there exists one and only one invariant C-connection Λ, which is given by* (1) *in* §2. *The covariant derivatives of its torsion and curvature tensor fields are both zero, but its torsion tensor field istelf does not vanish at every point of S^6.*

Recently, K. Yano [6] has given three remarkable C-connections in almost Hermitian spaces, and J. A. Schouten and K. Yano [5] have found a C-connection distinct from these three ones. By virtue of (2), these fonr C-connections on S^6 coincide with the one given by (1). This fact is, however a direct consequence of Theorem 2, because the four C-connections are invariant under G.

Let G/H be a homogeneous almost Hermitian space of dimension $2n$, and $\dim G = n^2 + 2n - 1$ $(n > 1)$. If G/H is not flat as a homogeneous Riemannian space, then $n = 3$ and moreover, if the homogeneous space G/H is simply connected, it is isomorphic with the homogeneous almost Hermitian space S^6 considered in this paper[2]. Analogously, we have for general cases that there exists on G/H a unique invariant C-connection Λ.

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