ON CONVEX MAPPING

YOSIO MUTÔ

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1. Introduction. It is well-known that the total curvature K of an ovaloid in Euclidean 3-space E^3 satisfies the inequality $K \geqq 0$ everywhere, if one assumes sufficient differentiability of the surface. It will be natural to present the converse problem whether a closed surface in E^3 with $K \geqq 0$ is an ovaloid or not, or at least the problem whether it is homeomorphic to a sphere.

When $K > 0$ the problem is easily solved by using the spherical representation. The second fundamental form is then positive definite and the surface is so to say locally strictly convex. Similarly a locally strictly convex hypersurface in E^{n+1} is homeomorphic to an n-sphere if one can assume sufficient differentiability. But if one assumes only less differentiability and moreover less strict convexity, that is, that the rank of the second fundamental form may become less than n, the problem becomes more difficult. The following study might be the first attack to such a problem.

The present article contains another result, for the mapping f considered is not assumed to be a homeomorphism, but it is proved to be a homeomorphism in the sequel.

2. Definition and the Theorem. Consider a regular mapping f of class C^2 of an n-dimensional compact manifold M^n of class C^2 into an $n+1$-dimensional Riemannian manifold V^{n+1} of class C^α $(\alpha \geqq 3)$. This means that for any point P in M^n we can find out a neighborhood $U(P)$ and a coordinate system such that any point in $U(P)$ is given by the coordinates $u^1, \ldots, u^n$, while in V^{n+1} we can find out a neighborhood of $f(P)$ and a coordinate system $x^1, \ldots, x^{n+1}$ so that the image $f(U(P))$ of $U(P)$ is given by

$$x^\lambda = f^\lambda(u^1, \ldots, u^n) \qquad (\lambda = 1, \ldots, n+1),$$

where the functions f^λ are differentiable of class C^2 and the rank of the matrix $\partial f^\lambda / \partial u^i$ is n at P. A mapping f will be called a convex mapping if $f(U(P))$ in V^{n+1} is convex (or concave) at $f(P)$ for any point P in M^n.

We might take some other definition of "convex", but in the present paper "convex" means

$$\begin{aligned} &\sum \Omega_{jk} v^j v^k \leqq 0 && \text{for any } v^j \ (j, k = 1, \ldots, n) \\ &\sum \Omega_{jk} v^j v^k < 0 && \text{for some } v^j, \end{aligned}$$

where Ω_{jk} is the second fundamental tensor of $f(U(P))$ for a suitably chosen direction of the normal.

The purpose of the present paper is to prove the

THEOREM. *If an n-dimensional compact manifold M^n of class C^2 is mapped into an $(n+1)$-dimensional euclidean space E^{n+1} by a convex mapping f and $n \geqq 2$, then f is a homeomorphism, that is, M^n and $f(M^n)$ are homeomorphic, and hence $f(M^n)$ has no singular (or multiple) point, and M^n is homeomorphic to an n-sphere S^n.*

3. **Proof**. The proof is given in nine steps.

(i) We prove that M^n is orientable. We fix an orientation in E^{n+1} and take a neighborhood $U(a)$ for each point a in M^n [1]. Its image $f(U(a))$ has a normal at $f(a)$ directed to the convex side[2], and this normal fixes an orientation in $f(U(a))$. Then the orientation in $f(U(a))$ fixes an orientation in $U(a)$.

(ii) As M^n is orientable and has integral cycles, we can speak of the order of a point in $E^{n+1}-f(M^n)$ with respect to $f(M^n)$. The orientation of M^n and $f(M^n)$ being given by the convexity of $f(M^n)$, we can determine the order of a point in such a way that it increases when the point crosses $f(M^n)$ from the convex side to the concave side. Then we can find out a point P whose order is one or more[3]. This fact is proved as follows.

In E^{n+1} a hyperplane $E^n(c)$ is given by $x^{n+1}=c$. If c is sufficiently large, $f(M^n)$ and $E^n(c)$ have no common point. We can find out a number c_0 such that for $c>c_0$, $E^n(c)$ has no common point with $f(M^n)$, while for $c=c_0$, $E^n(c)$ has at least one common point with $f(M^n)$. Let $f(a)$ be one of such common points. Then $E^n(c_0)$ is the tangent of $f(U(a))$ at $f(a)$ and the normal of $f(U(a))$ is directed to the positive direction of x^{n+1} axis. If we extend this normal to the concave side of $f(U(a))$ and take a point near $f(a)$ on this extension as the point P, then P is the point demanded.

(iii) Consider a mapping φ obtained from f by

$$\varphi^\lambda(u^1,\dots,u^n)=f^\lambda(u^1,\dots,u^n)+Kl^\lambda(u^1,\dots,u^n),$$

where K is a positive constant and $l^\lambda(u^1,\dots,u^n)$ is the unit normal vector of the image by f of neighborhoods of the point $(u^1,\dots,u^n)$. As f is convex and K is positive, φ is regular. This fact is proved as follows.

If we take a suitable cartesian coordinate system with $f(a)$ as the origin, $f(U(a))$ is given by

$$x^{n+1}=g(x^1,\dots,x^n),$$

where g is a differentiable function of class C^2 satisfying

$$(1)\qquad g=0,\quad \frac{\partial g}{\partial x^i}=0,\quad \frac{\partial^2 g}{\partial x^j\partial x^k}=-\Omega_{jk}(a)$$

at the origin $x^\lambda=0$. As we can take $x^1,\dots,x^n$ as the coordinates $u^1,\dots,$

1) Neighborhoods are always taken sufficiently small.
2) In this paper a normal is a half straight line directed to the convex side.
3) This is not trivial, for, as we do not assume that f be a homeomorphism, we can take the Klein bottle and its image in E^3, which has double points, as a counter example, if we omit the assumption that f be convex.

u^n in $U(a)$, we get

$$f^i = u^i, \quad f^{n+1} = g(u^1, \ldots, u^n), \tag{2}$$

$$l^i = \frac{\dfrac{\partial g}{\partial u^i}}{G}, \quad l^{n+1} = \frac{-1}{G}, \quad G = \left[1 + \sum \left(\frac{\partial g}{\partial u^i}\right)^2\right]^{1/2}, \tag{3}$$

$$\frac{\partial l^i}{\partial u^k} = \frac{\dfrac{\partial^2 g}{\partial u^i \partial u^k}}{G} - \frac{\partial g}{\partial u^i}\frac{\partial l^{n+1}}{\partial u^k}, \quad \frac{\partial l^{n+1}}{\partial u^k} = \frac{\sum \dfrac{\partial g}{\partial u^i}\dfrac{\partial^2 g}{\partial u^i \partial u^k}}{G^3}. \tag{4}$$

We can calculate the expressions

$$\frac{\partial \varphi^j}{\partial u^k} = \delta_{jk} + K\frac{\partial l^j}{\partial u^k} \tag{5}$$

and find that the determinant $|\partial\varphi^j/\partial u^k|$ does not vanish at the origin for positive K. Hence this determinant does not vanish in a sufficiently small $V(a)$ in $U(a)$, for $\partial\varphi^j/\partial u^k$ are continuous functions in $U(a)$.

(iv) As M^n is compact, $f(M^n)$ is contained in a sphere with P as the center and with sufficiently large radius R. If we take $K > 2R$, then $\varphi(M^n)$ has no tangent line from P, for $\varphi(M^n)$ has no point in the sphere with P as the center and of radius $K - R$, while a normal of $\varphi(M^n)$ must join a point of $\varphi(M^n)$ and a point of $f(M^n)$. Hence if we take a unit sphere Σ^n with P as the center and consider a projection π caused by half straight lines from P, we can map $\varphi(M^n)$ into Σ^n, and the mapping is locally homeomorphic. This means that we can map M^n into Σ^n, and the mapping $\pi\varphi$ is locally homeomorphic. M^n being compact, this mapping is onto.

According to Monodromiesatz a locally homeomorphic mapping of M^n onto an n-sphere S^n is a homeomorphic mapping and hence M^n is homeomorphic to S^n, for $n \geqq 2$. As $\pi\varphi$ is homeomorphic, a half straight line from P has only one common point with $\varphi(M^n)$, and the order of P with respect to $\varphi(M^n)$ is ± 1.

(v) Let us denote the mapping φ in which K is replaced by tK by f_t. Then $f_0 = f$ and $f_1 = \varphi$. f_t shows a continuous deformation of f to φ. For any point a in M^n and its neighborhood $U(a)$, $f_t(U(a))$ moves toward the convex side for increasing t[4]. According to (ii) the order of P with respect to $f(M^n)$ never exceeds that with respect to $\varphi(M^n)$, for the order of a point with respect to $f_t(M^n)$, which is not defined when it belongs to $f_t(M^n)$, is a non-decreasing function of t. As the order of P with respect to $\varphi(M^n)$ is ± 1 and that with respect to $f(M^n)$ is one at least, the former and the latter are the same and we have $d = 1$.

(vi) If P belongs to $f_t(M^n)$ for some value of $t, t = t_1, 0 < t_1 < 1$, the order of P with respect to $f_t(M^n)$ is greater for $t = t_1 + \varepsilon$ than for $t = t_1 - \varepsilon$. Then the order of P with respect to $\varphi(M^n)$ becomes more than one, contrary to (iv). Hence the point P never belongs to $f_t(M^n)$ for $0 \leqq t \leqq 1$. If a normal of $f(M^n)$, that is, the normal of $f(U(a))$ at $f(a)$ for some point a of M^n passes

4) Though we can not say that f_t is differentiable of class C^2, we can understand the sides of $f_t(M^n)$, for f_t means a continuous deformation.

P, P belongs to $f_t(M^n)$ for some t such that $tK < R$, hence no normal of $f(M^n)$ passes P.

(vii) Take a point c and its neighborhood $U(c)$ in M^n. The angle between the normal of $f(U(c))$ at $f(c)$ and the vector $\overrightarrow{f(c)P}$ is denoted by $\theta(c)$. As f is convex, if $\theta(c)$ is an acute angle, we can find out a point c' in $U(c)$ such that $\theta(c') < \theta(c)$, except the case $\theta(c) = 0$, as shown in the following:

If we take the point c instead of a in (iii), a point c' in $U(c)$ has the coordinates $u^1, \dots, u^n$, satisfying $|u^i| < \varepsilon$ for some $\varepsilon > 0$. Let the coordinates of the point P be $p^1, \dots, p^{n+1}$. Then we have

$$\cos\theta(c') = \frac{\sum l^j(p^j - u^j) + l^{n+1}(p^{n+1} - g)}{[(p^1 - u^1)^2 + \dots + (p^n - u^n)^2 + (p^{n+1} - g)^2]^{1/2}}, \tag{6}$$

and this is a function of class C^1 with respect to $u^1, \dots, u^n$. Differentiating partially with respect to u^k and putting $u^i = 0$, we get

$$-\left(\frac{\partial\theta}{\partial u^k}\right)_c \sin\theta(c) = \frac{-\sum\Omega_{jk}(c)p^j}{|p|} + \frac{-p^{n+1}p^k}{|p|^3} \tag{7}$$

where $|p| = [(p^1)^2 + \dots + (p^{n+1})^2]^{1/2}$. If $\theta(c)$ is an acute angle, then $p^{n+1} < 0$, and we have

$$\frac{-\sum\Omega_{jk}(c)p^jp^k}{|p|} + \frac{-p^{n+1}\sum(p^k)^2}{|p|^3} > 0 \tag{8}$$

for any p^i except $p^i = 0$, in which case we have $\theta(c) = 0$. If we consider a curve $x^i = u^i = tp^i$, $x^{n+1} = g(tp^1, \dots, tp^n)$ in $f(U(c))$, and denote the angle $\theta(c')$ at the points on this curve by $\alpha(t)$, this function is differentiable of class C^1 with respect to t, satisfying

$$\frac{d\alpha}{dt} = -\frac{1}{\sin\alpha}\left[\frac{-\sum\Omega_{jk}(c)p^jp^k}{|p|} + \frac{-p^{n+1}\sum(p^k)^2}{|p|^3}\right] < 0$$

at $t = 0$, because of (7) and (8), except the case $\sin\theta(c) = 0$. Hence we get $\alpha(t) < \theta(c)$ for some t, that is, $\theta(c') < \theta(c)$ for some point c' in $U(c)$ as long as we have $0 < \theta(c) < \pi/2$.

(viii) As the angle $\theta(c)$ is a continuous function of c and M^n is compact, $\theta(c)$ has a least value θ_0. According to (vii) we know that θ_0 can not satisfy $0 < \theta_0 < \pi/2$, while according to (vi) θ_0 can not be zero. Hence we get $\theta_0 \geqq \pi/2$. This fact is true even if we let the point P move in a sufficiently small neighborhood $V(P)$ of P such that $V(P) \cap f(M^n) = \phi$, for the order of the point P' in $V(P)$ with respect to $f(M^n)$ is the same and all previous discussions are valid for P' too.

Now suppose that a half straight line PX from P is tangent to $f(M^n)$, that is, PX is tangent to $f(U(a))$ for some $U(a)$. We take the point a in such a way that $f(a)$ is the point of contact. Then we can find out a point b in $U(a)$ and a point P' in $V(P)$ such that the normal of $f(U(a))$ at $f(b)$ and the vector $\overrightarrow{f(b)P'}$ hold an acute angle. This fact is proved as follows.

As $\theta(c) = \pi/2$, we have $p^{n+1} = 0$ in the formulas considered in (vii), hence

$$-\frac{\partial\theta}{\partial u^k}\sin\theta = -\frac{\partial\theta}{\partial u^k} = \frac{-\sum\Omega_{jk}(c)p^j}{|p|}$$

at $f(c)$. As in (vii) we can find out a point c' for which $\theta(c') < \theta(c)$ as long as p^i satisfy $\Sigma\,\Omega_{jk}(c)p^j \neq 0$, for $\Sigma\,\Omega_{jk}(c)p^jp^k$ vanishes only when $\Sigma\,\Omega_{jk}(c)p^j = 0$ because of the definition of "convex". Then $\theta(c')$ is an acute angle and we can take P for P'. If $\Sigma\,\Omega_{jk}(c)p^j = 0$, we take a point P' in $V(P)$ and denote the coordinates of P' by p'^λ. Let $p'^{n+1} = 0$, so that $P'f(c)$ is a tangent of $f(U(c))$ at $f(c)$. If we have $\Sigma\,\Omega_{jk}(c)p'^jp'^k < 0$ for some p'^i, we can obtain $\theta(c') < \pi/2$ for this point P' and the proof is completed. The only difficulty occurs only when $\Sigma\,\Omega_{jk}(c)p'^jp'^k = 0$ for every p'^j satisfying $|p'^j - p^j| < \varepsilon$ for some $\varepsilon > 0$. But this means $\Omega_{jk}(c) = 0$, which contradicts our definition of "convex".

Thus, we see, for some point P' in $V(P)$ at least, we get $\theta(c') < \pi/2$ contrary to $\theta_0 \geqq \pi/2$. Hence we know that PX is not tangent to $f(M^n)$.

(ix) As there is no half straight line PX which is tangent to $f(M^n)$, we can consider a projection π as considered in (iv) such that πf becomes a local homeomorphism of M^n onto Σ^n. By Monodromiesatz we conclude that πf is a homeomorphism, hence f is also a homeomorphism and the theorem is proved.

According to the Jordan-Brouwer's theorem we get the

COROLLARY. *Under the same assumption as before, $f(M^n)$ divides E^{n+1} into two parts, one of which is bounded and has $f(M^n)$ as the common boundary.*

MATHEMATICAL INSTITUTE OF THE YOKOHAMA NATIONAL UNIVERSITY.