ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATIONS IN *O-SPACES

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(Received July 27, 1961)

The purpose of the paper is to generalize some of recent results of S. Tachibana and S. Ishihara [9][1] concerning with infinitesimal holomorphically projective transformation in Kählerian spaces to the case of *O-spaces. In almost complex spaces, such transformations were defined in the case when the affine connection under consideration is a φ-connection.[2] However an *O-space is not endowed with a φ-connection but a symmetric affine connection with respect to which the almost complex structure $\varphi_j^{\ i}$ satisfies $\nabla_r\varphi_j^{\ r} = 0$. We shall study the infinitesimal holomorphically projective transformation (briefly an HP-transformation) of such a connection.

In §1, we shall give the notion of an *O-space, an H-space and a K-space and other preliminary facts. After introducing an analytic HP-transformation in §2, we shall define in §3 the HP-curvature tensor which is an invariant under such a transformation. In §4, we shall deal with an *O-space of constant holomorphic sectional curvature and prove some theorems on the HP-curvature tensor in this space. In §5 and §6, so called decomposition theorem for an analytic HP-transformation in an Einstein *O-space and conformally flat K-space will be given. In the last §7 we shall see that an analytic HP-transformation is necessarily an isometry in a compact *O-space of constant curvature.

Throughout the paper ∇_j denotes the operator of covariant differentiation with respect to the Riemannian connection.

The author wishes to express his sincere thanks to Professor S. Tachibana for his valuable suggestions and advises.

1. Preliminaries. In a $2n$-dimensional real differentiable space of class C^∞ with local coordinates $\{x^i\}$, a field $\varphi_j^{\ h}$ such that

$$(1.1) \qquad \varphi_j^{\ r}\varphi_r^{\ i} = -\delta_j^i$$

is called an *almost complex structure* and the space with such a structure is

1) The number in bracket refers to the Bibliography at the end of the paper.
2) Tashiro, Y. [5].

called an *almost complex space*, and a positive definite Riemannian metric tensor field such that

$$(1.2)\qquad g_{ji} = \varphi_j{}^l \varphi_i{}^m g_{lm}$$

can be always introduced in an almost complex space.[3] The space with a pair $(\varphi_j{}^i, g_{ji})$ satisfying (1.1) and (1.2) is called an almost Hermitian space.

If the structure tensor field $\varphi_j{}^i$ satisfies

$$(1.3)\qquad \nabla_j \varphi_i{}^h + \varphi_j{}^l \varphi_i{}^m \nabla_l \varphi_m{}^h = 0,$$

then the almost Hermitian space is called an $^*O$-space.
If $\varphi_{ji} = g_{ir}\varphi_j{}^r$ satisfies

$$(1.4)\qquad \nabla_j \varphi_{ih} + \nabla_i \varphi_{hj} + \nabla_h \varphi_{ji} = 0,$$

then the almost Hermitian space is called an *H-space* or an *almost Kählerian space*. If φ_{ji} satisfies

$$(1.5)\qquad \nabla_j \varphi_{ih} + \nabla_i \varphi_{jh} = 0,$$

then the almost Hermitian space is called a K-space or an *almost Tachibana space*. We see that an $^*O$-space, an H-space and a K-space satisfy

$$(1.6)\qquad \nabla_r \varphi_j{}^r = 0.$$

It is verified that an H-space or a K-space is an $^*O$-space respectively.[4] In an almost Hermitian space, we shall define the following operator

$$(1.7)\qquad O_{ji}^{lm} \equiv \frac{1}{2}(\delta_j{}^l \delta_i{}^m - \varphi_j{}^l \varphi_i{}^m),$$

$$(1.8)\qquad {}^*O_{ji}^{lm} \equiv \frac{1}{2}(\delta_j{}^l \delta_i{}^m + \varphi_j{}^l \varphi_i{}^m).$$

A tensor is called *pure* (*hybrid*) in two indices if the tensor vanishes by transvection of $^*O(O)$ on these indices.

Since (1.2) and (1.3) can be written in the form

$$(1.9)\qquad O_{ji}^{lm} g_{lm} = 0,$$

$$(1.10)\qquad {}^*O_{ji}^{lm} \nabla_l \varphi_m{}^k = 0$$

respectively, the metric tensor g_{ji} is hybrid in j and i, and $\nabla_j \varphi_i{}^h$ is pure in j and i in an $^*O$-space.

For the two operators with the same indices, we have

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3) Frölicher, A. [1].
4) Kotō, S. [2].

$$(1.11)\qquad \begin{cases} OO = O, & O^*O = {}^*OO = 0, \\ {}^*O^*O = {}^*O, & {}^*O + O = O + {}^*O = E, \end{cases}$$

where E denotes an identity operator.

For simplicity we denote ${}^*O_{ji}^{lm}\ T_{lm} \equiv {}^*OT_{ji}$, $O_{ji}^{lm}\ T_{lm} = OT_{ji}$ for a tensor of order 2, for example (1.9) is replaced by $Og_{ji} = 0$.

Let $K_{kji}{}^h$ be the Riemannian curvature tensor and put

$$(1.12)\qquad \begin{cases} K_{ji} = g^{kh}K_{kjih}, & \widetilde{K}_{ji} = \varphi_j{}^r K_{ri},\ K = g^{ji}K_{ji} \\ H_{ji} = \varphi^{kh}K_{kjih}, & K^*{}_{ji} = -\ \varphi_j{}^r H_{ri},\ K^* = \varphi^{ji}H_{ji}. \end{cases}$$

We see that $H_{ji} = -\dfrac{1}{2}\varphi^{lm}K_{lmji}$ by the Bianchi's identity.

Let $\underset{v}{\pounds}$ be the Lie derivative with respect to a vector v^i, then v^i is called contravariant almost analytic or analytic if it satisfies

$$(1.13)\qquad \underset{v}{\pounds}\varphi_j{}^i \equiv \nabla^r\nabla_r\varphi_j{}^i - \varphi_j{}^r\nabla_r v^i + \varphi_r{}^i\nabla_j v^r = 0.$$

The following identities are valid.[5)]

$$(1.14)\qquad \underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \nabla_j\nabla_i v^h + K_{rji}{}^h v^r.$$

$$(1.15)\qquad \underset{v}{\pounds}\,\nabla_j\varphi_i{}^h - \nabla_j\underset{v}{\pounds}\varphi_i{}^h = \varphi_i{}^r\,\underset{v}{\pounds}\left\{ {h \atop jr} \right\} - \varphi_r{}^h\,\underset{v}{\pounds}\left\{ {r \atop ji} \right\}.$$

$$(1.16)\qquad \underset{v}{\pounds}K_{kji}{}^h = \nabla_k\underset{v}{\pounds}\left\{ {h \atop ji} \right\} - \nabla_j\underset{v}{\pounds}\left\{ {h \atop ki} \right\}.$$

2. Infinitesimal holomorphically projective transformations. A vector field v^i is called an infinitesimal holomorphically projective transformation or briefly an *HP*-transformation if it satisfies

$$(2.1)\qquad \underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \delta_j{}^h\rho_i + \delta_i{}^h\rho_j - \varphi_j{}^h\widetilde{\rho}_i - \varphi_i{}^h\widetilde{\rho}_j$$

where ρ_i is a vector and $\widetilde{\rho}_i = \varphi_i{}^r\rho_r$. We shall call ρ_i the *associated vector* of the *HP*-transformation.

Contracting (2.1) with respect to i and h, we get $\rho_j = \dfrac{1}{2(n+1)}\nabla_j\nabla_r v^r$, and therefore ρ_i is gradient.

Next we shall introduce the curve satisfying the differential equations

$$(2.2)\qquad \frac{d^2x^h}{dt^2} + \left\{ {h \atop ji} \right\}\frac{dx^j}{dt}\frac{dx^i}{dt} = \alpha(t)\frac{dx^h}{dt} + \beta(t)\varphi_i{}^h\frac{dx^i}{dt}.$$

5) Yano, Y. [11].

Such a curve is called a *holomorphically planer curve.*[6]

Let v^i be an infinitesimal transformation and we assume that an infinitesimal point transformation $'x^i = x^i + \varepsilon v^i$ transforms any homomorphically planer curve into such a curve.

A necessary and sufficient condition v^i be such a transformation is that

$$\dot{x}^j \underset{v}{\pounds} \varphi_j{}^i = a\dot{x}^i + b\varphi_j{}^i \dot{x}^j, \tag{2.3}$$

$$\dot{x}^j \dot{x}^i \underset{v}{\pounds} \left\{ {h \atop ji} \right\} = c\dot{x}^h + d\varphi_j{}^h \dot{x}^j \tag{2.4}$$

hold for any direction $\dot{x}^i$, where a,b,c and d are some functions of x^i and $\dot{x}^i$.

Now let v^i be such a transformation, then taking account of Lemma 1 in [9, Appendix I] and (2.3) we have

$$\underset{v}{\pounds} \varphi_j{}^i = 0. \tag{2.5}$$

Next, from (2.4) and Lemma 3 in [9, Appendix I] we have

$$\underset{v}{\pounds} \left\{ {h \atop ji} \right\} = \delta_j{}^h \rho_i + \delta_i{}^h \rho_j + \varphi_j{}^h \sigma_i + \varphi_i{}^h \sigma_j. \tag{2.6}$$

If we substitute (2.5) into (1.15), then we get

$$\underset{v}{\pounds} \nabla_j \varphi_i{}^h = \varphi_i{}^r \underset{v}{\pounds} \left\{ {h \atop jr} \right\} - \varphi_r{}^h \underset{v}{\pounds} \left\{ {r \atop ji} \right\}. \tag{2.7}$$

Contracting (2.7) with j and h, and using (1.6) and (2.6), we have $\sigma_j = -\tilde{\rho}_j$. Hence we obtain (2.1). Therefore v^i is analytic and at the same time an *HP*-transformation. The converse is evident. Thus we have the following

THEOREM 2.1. *In an *O-space, an infinitesimal transformation preserves the holomorphically planer curve, if and only if it is an analytic HP-transformation.*

For *HP*-transformations it holds that

$$\nabla_j \nabla_i v^h + K_{rji}{}^h v^r = \delta_j{}^h \rho_i + \delta_i{}^h \rho_j - \varphi_j \tilde{\rho}_i{}^h - \varphi_i{}^h \tilde{\rho}_j.$$

Contracting with respect to j and i, we get

$$\nabla^r \nabla_r v^h + K_r{}^h v^r = 0.$$

This is a necessary and sufficient condition in order that a vector v^i be analytic in compact H-spaces.[7]

THEOREM 2.2. *In a compact H-space an HP-transformation is analytic.*

6) Ōtsuki, T. and Tashiro, Y. [4].
7) Tachibana, S. [6].

3. The *HP*-curvature tensor. Now, in an $^*O$-space, let v^i be an analytic *HP*-transformation. If we substitute (2. 1) into (1. 16), we have

$$\underset{v}{\pounds} K_{kji}{}^h = \delta_j^h \nabla_k \rho_i - \delta_k^h \nabla_j \rho_i - \varphi_j{}^h \nabla_k \tilde{\rho}_i + \varphi_k{}^h \nabla_j \tilde{\rho}_i - \varphi_i{}^h (\nabla_k \tilde{\rho}_j - \nabla_j \tilde{\rho}_k) - \tilde{\rho}_j \nabla_k \varphi_i{}^h + \tilde{\rho}_k \nabla_j \varphi_i{}^h + \tilde{\rho}_i (\nabla_j \varphi_k{}^h - \nabla_k \varphi_j{}^h). \tag{3.1}$$

Transvecting (3. 1) with $\varphi_h{}^k$ and making use of (1. 6) and $\underset{v}{\pounds}\varphi_j{}^i = 0$, we have

$$\underset{v}{\pounds} H_{ji} = - 2\varphi_j{}^r \nabla_r \rho_i + 2n\varphi_i{}^r \nabla_r \rho_j + (2n+1)(\nabla_j \varphi_i{}^r)\rho_r - (\nabla_i \varphi_j{}^r)\rho_r.$$

Taking the alternating and symmetric part, we have

$$\underset{v}{\pounds} H_{ji} = - (n+1)[(\varphi_j{}^r \nabla_r \rho_i - \varphi_i{}^r \nabla_r \rho_j) - (\nabla_j \varphi_i{}^r - \nabla_i \varphi_j{}^r)\rho_r] \tag{3.2}$$

and

$$(n-1)(\varphi_j{}^r \nabla_r \rho_i + \varphi_i{}^r \nabla_r \rho_j) + n(\nabla_j \varphi_i{}^r + \nabla_i \varphi_j{}^r)\rho_r = 0.$$

The last equation turns to

$$2(n-1)O(\nabla_j \rho_i) + n(\nabla_j \varphi_i{}^r + \nabla_i \varphi_j{}^r)\tilde{\rho}_r = 0. \tag{3.3}$$

On the other hand, if we contract (3. 1) with respect to h and k, and take account of (3.3), then we find

$$\underset{v}{\pounds} K_{ji} = (n+1)[(\nabla_j \varphi_i{}^r + \nabla_i \varphi_j{}^r \tilde{\rho}_r) - 2^*O(\nabla_j \rho_i)]. \tag{3.4}$$

Operating $^*O$ to (3. 4) we have

$$\underset{v}{\pounds}(^*OK_{ji}) = - 2(n+1)^*O(\nabla_j \rho_i) \tag{3.5}$$

because of (1. 10) and $^*O\underset{v}{\pounds} = \underset{v}{\pounds}{}^*O$.

It is easily verified that the following identities hold good.

$$^*O_{kj}^{lm} O_{it}^{rh} (\delta_m{}^t \xi_{lr}) = \frac{1}{2} (\delta_j{}^{h*} O\xi_{ki} + \varphi_j{}^h \varphi_k{}^{l*} O\xi_{li}). \tag{3.6}$$

$$^*O_{jk}^{lm} O_{it}^{rh} (\varphi_m{}^t \xi_{lr}) = \frac{1}{2} (\varphi_j{}^{h*} O\xi_{ki} - \delta_j{}^h \varphi_k{}^{l*} O\xi_{li}),$$

where ξ_{ji} is a tensor.

If we operate $^*O_{kj}^{lm} O_{jt}^{rh}$ to (3. 1) and make use of (1.10), $O_{im}^{lh} \nabla_j \varphi_l{}^m = 0$ and (3. 6), then we obtain

$$\underset{v}{\pounds}{}^*O_{kj}^{lm} O_{it}^{rh} K_{lmr}{}^t = \delta_j{}^{h*}O(\nabla_k \rho_i) - \delta_k{}^{h*}O(\nabla_j \rho_i) + \varphi_j{}^h \varphi_k{}^{l*}O(\nabla_l \rho_i) - \varphi_k{}^h \varphi_j{}^{l*}O(\nabla_l \rho_i) + 2\varphi_i{}^h \varphi_k{}^{l*}O(\nabla_l \rho_j).$$

Substituting (3. 5) into the last equation we have

$$\underset{v}{\pounds}{}^*O_{kj}^{lm} O_{it}^{rh} K_{lmr}{}^t = - \frac{1}{2(n+1)} \underset{v}{\pounds}(\delta_j{}^{h*}OK_{ki} - \delta_k{}^{h*}OK_{ji} + \varphi_j{}^{h*}O\tilde{K}_{ki} - \varphi_k{}^{h*}O\tilde{K}_{ji}$$

$$+ 2\varphi_i^{\ h} {}^*O\widetilde{K}_{kj})$$

$$= -\frac{1}{n+1} \underset{v}{\pounds} {}^*O_{kj}^{lm} O_{it}^{rh} (\delta_m^{\ t} K_{lr} - \delta_l^{\ t} K_{mr} + \varphi_r^{\ t} \widetilde{K}_{lm})$$

by virtue of $\underset{v}{\pounds}\delta_j^{\ i} = 0$ and (3. 6).

If we put

$$P_{kji}^{\ \ \ h} \equiv {}^*O_{kj}^{lm} O_{it}^{rh} A_{lmr}^{\ \ \ \ t}, \tag{3. 7}$$

$$A_{lmr}^{\ \ \ \ t} = K_{lmr}^{\ \ \ \ t} + \frac{1}{n+1} (\delta_m^{\ t} K_{lr} - \delta_l^{\ t} K_{mr} + \varphi_r^{\ t} \widetilde{K}_{lm}),$$

then we have

$$\pounds P_{kji}^{\ \ \ h} = 0. \tag{3. 8}$$

Hence $P_{kji}^{\ \ \ h}$ is an invariant under the analytic *HP*-transformation v^i. We shall call $P_{kji}^{\ \ \ h}$ in (3. 7) the *holomorphically projective curvature tensor* or briefly the *HP-curvature tensor* in an $^*O$-space. Equation (3. 7) can be also written as follows

$$P_{kji}^{\ \ \ h} \equiv {}^*O_{kj}^{lm} {}^*O_{ih}^{rt} K_{lmrt} + \frac{1}{2(n+1)} (g_{jh} {}^*OK_{ki} - g_{kh} {}^*OK_{ji} + \varphi_{jh} {}^*O\widetilde{K}_{ki} - \varphi_{kh} {}^*O\widetilde{K}_{ji} + 2\varphi_{ih} {}^*O\widetilde{K}_{kj}). \tag{3. 9}$$

It can be verified that the following identities hold good

$$P_{kjr}^{\ \ \ r} = 0,\ P_{(kj)i}^{\ \ \ \ \ h} = 0,\ O_{kj}^{lm} P_{lmi}^{\ \ \ \ h} = 0,\ {}^*O_{it}^{rh} P_{kjr}^{\ \ \ t} = 0,$$

$$P_{rji}^{\ \ \ r} = -\frac{1}{2} {}^*O(K_{ji} - K^*_{\ ji}),\ g^{ji} P_{rji}^{\ \ \ r} = -\frac{1}{2} (K - K^*).$$

We shall put

$$P_{ji} \equiv {}^*O(K_{ji} - K^*_{\ ji}),\ P \equiv K - K^*. \tag{3. 10}$$

Then we see that $P_{ji} = P_{ij}$, $OP_{ji} = 0$.
From (3. 8) we get

$$\underset{v}{\pounds} P_{ji} = 0 \tag{3. 11}$$

and therefore from (3. 5)

$$\underset{v}{\pounds}({}^*OK^*_{\ ji}) = -2(n+1) {}^*O(\nabla_j \rho_i). \tag{3. 12}$$

From (3. 4) and (3. 2) we find

$$\underset{v}{\pounds}(OK_{ji}) = (n+1)(\nabla_j \varphi_i^{\ r} + \nabla_i \varphi_j^{\ r}) \widetilde{\rho}_r, \tag{3. 13}$$

$$\underset{v}{\pounds}(OH_{ji}) = (n+1)(\nabla_j \varphi_i^{\ r} - \nabla_i \varphi_j^{\ r}) \rho_r, \tag{3. 14}$$

respectively.

4. Holomorphic sectional curvatures in an $^*O$-space. In the first place we shall assume that $P_{kji}{}^h = 0$ in (3. 9), then we have

$$0 = g^{ji}P_{kjih} = \frac{1}{2(n+1)}[(n-1)^*OK_{kh} + (n+1)\,{}^*OK^*{}_{kh} - g_{kh}K].$$

From which we find

$$(4.1) \qquad {}^*OK_{kh} = \frac{K}{2n}\, g_{kh}$$

because $P_{kh} = 0$. Substituting (4. 1) into (3. 9) we have

$$(4.2) \qquad {}^*O_{kj}^{lm}\,{}^*O_{ih}^{rt}K_{lmrt} = \frac{k}{4}(g_{ji}g_{kh} - g_{jh}g_{ki} - \varphi_{ji}\varphi_{kh} - \varphi_{jh}\varphi_{ki} - 2\varphi_{ih}\varphi_{kj})$$

where

$$k = \frac{K}{n(n+1)}.$$

It can be verified that

$$(4.3) \qquad {}^*O_{kj}^{lm}\,{}^*O_{ih}^{rt}K_{lmrt} + {}^*O_{ij}^{lm}\,{}^*O_{kh}^{rt}K_{lmrt} - \varphi_k{}^a\varphi_h{}^b\,{}^*O_{bj}^{lm}\,{}^*O_{ia}^{pt}K_{lmrt}$$
$$= -k(g_{ki}g_{hj} + \varphi_{kh}\varphi_{ij} + \varphi_{kj}\varphi_{ih})^{8)}$$

by virtue of (4. 2).
Transvecting (4. 3) with $\varphi_q{}^k\varphi_p{}^i$ we find

$$\varphi_{(q}{}^m\varphi_p{}^lK_{|m|j|l|h)} + kg_{(qp}g_{jh)} = 0.$$

Hence for any vector u^i the following equation identically holds good.

$$(\varphi_q{}^m\varphi_p{}^lK_{mjlh} + kg_{qp}g_{jh})u^qu^pu^ju^h = 0,$$

and from which we get

$$k = \frac{-K_{mjlh}\varphi_q{}^mu^qu^j\varphi_p{}^lu^pu^h}{g_{qj}u^qu^jg_{ph}u^pu^h}.$$

This is the holomorphic sectional curvature with respect to the vector u^i which determined by two orthogonal vectors u^i and $\varphi_q{}^iu^q$.
Thus we have the following

THEOREM 4. 1. *In an $^*O$-space if the HP-curvature tensor vanishes, then the holomorphic sectional curvature k is independent on direction.*

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8) Mizusawa, H. and Koto, S. [3]

Especially in a K-space $OK_{ji}=0$ or $^*OK_{ji}=K_{ji}$ holds good.[9] In this case we have from (4.1)

$$K_{kh}=\frac{K}{2n}g_{kh}.$$

Hence K is an absolute constant and so is k by virtue of $k=\frac{K}{n(n+1)}$. Thus we find that a K-space with $P_{kjih}=0$ is of constant holomorphic sectional curvature.

But we can prove that this fact is also true for an H-space, that is,

THEOREM 4.2. *In an H-space or a K-space if $P_{kjih}=0$, then the space is of constant holomorphic curvature.*

In the first place we shall give the following lemma.

LEMMA. *If an H-space satisfies $P_{ji}=0$ in* (3.10), *then it is Kählerian.*

PROOF. In an $^*O$-space, from (1.3) we have

$$\varphi_j^{\ r}\nabla_r\varphi_i^{\ h}-\varphi_i^{\ r}\nabla_j\varphi_r^{\ h}=0.$$

Operating ∇_h to the last equation and using $\nabla_r\varphi_j^{\ r}=0$, we find

$$(4.4)\qquad (\nabla^r\varphi_i^{\ h})(\nabla_h\varphi_{jr}+\nabla_j\varphi_{rh})+2^*O(K_{ji}-K^*_{\ ji})=0.$$

In an H-space, by (1.4), it turns to

$$(4.5)\qquad (\nabla^r\varphi_i^{\ h})\nabla_r\varphi_{jh}=-P_{ji}$$

and in a K-space, by (1.5), (4.4) becomes

$$(4.6)\qquad (\nabla^r\varphi_i^{\ h})\nabla_r\varphi_{jh}=P_{ji}.$$

If $P_{ji}=0$, then from (4.5) or (4.6) we have $\nabla_j\varphi_i^{\ h}=0$ respectively and therefore it is Kählerian.

PROOF OF THEOREM.

Now, if an H-space or a K-space has $P_{kjih}=0$, then by the Lemma it is Kählerian. On the other hand in a Kählerian space

$$O_{kj}^{lm}K_{lmih}=0,\ OK_{ji}=0,\ OH_{ji}=0,\ H_{ji}=\widetilde{K}_{ji}$$

hold good.[10]
Making use of these identities, (3.9) and (4.1) turn to

$$(4.6)\qquad P_{kjih}=K_{kjih}+\frac{1}{2(n+1)}(g_{jh}K_{ki}-g_{kh}K_{ji}+\varphi_{jh}H_{ki}-\varphi_{kh}H_{ji}+2\varphi_{ih}H_{kj})$$

9) Tachibana, S. [7].
10) Yano, K. [11]

and

$$K_{kjih} = k(g_{kh}g_{ji} - g_{jh}g_{ki} + \varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{ih}\varphi_{kj}) \tag{4.7}$$

respectively.
In Kählerian spaces a necessary and sufficient condition fo $P_{kjih} = 0$ in (4.6) is that (4.7) holds good[11]. Thus Theorem 4.2 was proved.

5. Einstein $^*O$**-spaces.** In this section we shall consider an Einstein $^*O$-space with $K \neq 0$, then it holds that

$$K_{ji} = \frac{K}{2n} g_{ji} \tag{5.1}$$

from which we have $OK_{ji} = 0$, and therefore from (3.13) we find

$$(\nabla_j \varphi_i^{\ r} + \nabla_i \varphi_j^{\ r}) \tilde{\rho}_r = 0 \tag{5.2}$$

and from (3.3) we have

$$O(\nabla_j \rho_i) = 0. \tag{5.3}$$

Substituting (5.2) and (5.3) into (3.4), we have

$$\underset{v}{\pounds} K_{ji} = - 2(n + 1)\nabla_j \rho_i. \tag{5.4}$$

(5.2) is equivalent to

$$(\nabla_j \varphi_i^{\ r} + \nabla_i \varphi_j^{\ r})\rho_r = 0. \tag{5.5}$$

Making use of (5.5), (5.3) and $\nabla_j \rho_i = \nabla_i \rho_j$, we have

$$\nabla_j \tilde{\rho}_i + \nabla_i \tilde{\rho}_j = (\nabla_j \varphi_i^{\ r} + \nabla_i \varphi_j^{\ r})\rho_r + (\varphi_i^{\ r} \nabla_j \rho_i + \varphi_j^{\ r} \nabla_i \rho_r) = 0, \tag{5.6}$$

which shows $\tilde{\rho}_i$ is a Killing vector.

If we operate $\underset{v}{\pounds}$ to (5.1) and take account of (5.4) and $\underset{v}{\pounds} g_{ji} = \nabla_j v_i + \nabla_i v_j$, then it holds that

$$\nabla_j \left(v_i - \frac{1}{k} \rho_i \right) + \nabla_i \left(v_j - \frac{1}{k} \rho_j \right) = 0,$$

where $k = - \dfrac{K}{2n(n + 1)}$. If we put $p_i = v_i - \dfrac{1}{k} \rho_i$ the p_i is a Killing vector. Next, if we put $q_i = \dfrac{1}{k} \tilde{\rho}_i$, then q_i is also a Killing vector by virtue of (5.6). Thus we obtain the following

11) Tashiro, Y. [5].

THEOREM 5. 1[12]. *In an Einstein *O-space with* $K \neq 0$, *an analytic HP-transformation* v^i *is uniquely decomposed into the form*

$$(5.7) \qquad v^i = p^i + \varphi_r{}^i q^r$$

where p^i *and* q^i *are both Killing vectors and* $\varphi_i{}^r q_r$ *is gradient.*

From (5. 7), we have $v_i = p_i - \tilde{q}_i = p_i + \frac{1}{k}\rho_i$. Hence it holds that

$$\underset{v}{\pounds}\left\{ {h \atop ji} \right\} = \underset{p}{\pounds}\left\{ {h \atop ji} \right\} - \underset{\tilde{q}}{\pounds}\left\{ {h \atop ji} \right\} = \frac{1}{k}\underset{\rho}{\pounds}\left\{ {h \atop ji} \right\},$$

because p_i is a Killing vector. If we substitute (2. 1) into the last equation, we find

$$(5.8) \qquad \underset{\rho}{\pounds}\left\{ {h \atop ji} \right\} = \nabla_j\nabla_i\rho^h + K_{rji}{}^h q_i = k(\delta_j^h\rho_i + \delta_i^h\rho_j - \varphi_j{}^h\tilde{\rho}_i - \varphi_i{}^h\tilde{\rho}_j).$$

Thus it follows that

THEOREM 5. 2. *In an Einstein *O-space with* $K \neq 0$, *the associated vector of an analytic HP-transformation is an HP-transformation.*

From (5. 8) we have

$$\nabla_j\nabla_i\rho_h + K_{rjih}\rho^r = k(g_{jh}\rho_i + g_{ih}\rho_j - \varphi_{jh}\tilde{\rho}_i - \varphi_{ih}\tilde{\rho}_j).$$

Taking the symmetric part of the last equation with respect to i and h, we have

$$(5.9) \qquad K_{rjih}\rho^r = \frac{k}{2}(g_{jh}\rho_i - g_{ji}\rho_h - \varphi_{jh}\tilde{\rho}_i + \varphi_{ji}\tilde{\rho}_h - 2\varphi_{ih}\tilde{\rho}_j).$$

The equation (5. 9) can be written in the following form

$$K_{rjih}\rho^r = \frac{k}{2}(g_{ri}g_{jh} - g_{rh}g_{ji} + \varphi_{ri}\varphi_{jh} - \varphi_{rh}\varphi_{ji} + 2\varphi_{rj}\varphi_{ih})\rho^r.$$

Hence applying Lemma in [9, Appendix II] we have

THEOREM 5. 3. *If an Einstein *O-space with* $K \neq 0$, *admits an analytic non-affine HP-transformation, then the restricted homogeneous holonomy group contains the full unitary group* $U(n)$.

Transvecting (5. 8) with g^{ji}, we find

$$\nabla^r\nabla_r\rho^h + K_r{}^h\rho^r = 0.$$

Hence we have the following

12) For a K-space, see Tachibana, S. [10], and for a Kählerian space, see Tachibana, S. and Ishihara, S. [11].

THEOREM 5. 4. *In a compact Einstein H-space with* $K > 0$[13] *the associated vector of an analytic HP-transformation is also an analytic HP-transformation.*

In the next place we shall consider a K-space which is not necessarily an Einstein one. Then $OH_{ji} = 0$ holds good. Consequently from (3. 14) we get

$$(\nabla_j \varphi_i^{\ r} - \nabla_i \varphi_j^{\ r})\rho_r = 0,$$

and by (1. 5) it becomes

$$2\rho^r \nabla_r \varphi_{ji} = 0. \tag{5. 10}$$

On the other hand, from (1. 5) and (3. 3) we get $O(\nabla_j \rho_i) = 0$.
Now, the definition (1. 13) of an analytic vector v^i can be written as

$$v^r \nabla_r \varphi_{ji} - 2\varphi_j^{\ r} O(\nabla_r v_i) = 0.$$

If we put $v^i = \rho^i$, then we find that ρ^i is analytic. Hence we have

LEMMA. *In a K-space the associated vector* ρ_i *of an analytic HP-transformation is also analytic.*

Taking account of Theorem 5. 2 and the above lemma, we have the following

THEOREM 5. 5[14]. *In an Einstein K-space with* $K \neq 0$, *the associated vector of analytic HP-transformation is also an analytic HP-transformation.*

By an automorphism v^i, we shall mean an analytic vector v^i which is an infinitesimal isometry, i.e. a Killing vector.

THEOREM 5. 6. *In a K-space, if* ρ_i *is the associated vector of an analytic HP-transformation, then* $\widetilde{\rho}_i$ *is an automorphism.*

PROOF. We notice that (5. 6) is valid in a K-space which is not necessarily Einstein one, in fact, in order to obtain (5. 6) we did not use (5. 1) but only $OK_{ji} = 0$. Therefore $\widetilde{\rho}_i$ is an infinitesimal isometry.

Next, we shall prove that $\widetilde{\rho}_i$ is analytic, that is, $\underset{\widetilde{\rho}}{\pounds} \varphi_j^{\ i} = 0$, which is equivalent to

$$\rho^r \nabla_r \varphi_{ji} - 2\varphi_j^{\ r} O(\nabla_j \widetilde{\rho}_i) = 0.$$

Transvecting the last equation with $\varphi_h^{\ j}$ and using (1. 3) and (5. 3), we get

$$\rho^r \nabla_r \varphi_{hi} + 2\rho^r \nabla_h \varphi_{ir} = 0.$$

13) In a compact Einstein H-space with $K < 0$, there does not exist a non-trivial analytic vector. Tachibana, S. [6].

14) For compact case, see Tachibana, S. [8].

It turns to $3\rho^r\nabla_r\varphi_{hi} = 0$ by virtue of (1.5), and therefore from (5.10) we find that $\tilde{\rho}_i$ is analytic.

THEOREM 5.7. *In an Einstein H-space with* $K \neq 0$, *if* $\tilde{\rho}_i$ *is the associated vector of an analytic HP-transformation, then* $\tilde{\rho}_i$ *is an automorphism.*

In fact (5.6) is valid in an Einstein H-space. Next, by making use of (5.10) and (1.4), $\underset{\tilde{\rho}}{\pounds}\varphi_j{}^i = 0$ turns to

$$\rho^r(\nabla_h\varphi_{ir} + \nabla_i\varphi_{hr}) = 0.$$

But it is valid in an Einstein $^*O$-space because of (3.13), and hence $\tilde{\rho}_i$ is analytic.

In the compact case this theorem is trivially contained in the well known theorem that in an compact H-space a Killing vector is analytic.[15]

6. Conformally flat K-spaces. In this section we shall consider an HP-transformations in conformally flat K-spaces. In the first place, the following fact was proved.

LEMMA[15]. *In a K-space,* $P = K - K^*$ *is an absolute constant.*

Now the curvature tensor takes the following form

$$2(n-1)K_{kjih} = g_{kh}K_{ji} - g_{ki}K_{jh} + g_{ji}K_{kh} - g_{jh}K_{ik} - \frac{K}{2n-1}(g_{ji}g_{kh} - g_{ki}g_{jh}). \tag{6.1}$$

Transvecting (6.1) with $\varphi_t{}^j\varphi^{kh}$ and using $OK_{kj} = 0$, we get

$$2(n-1)K^*{}_{ti} = 2K_{ti} - \frac{K}{2n-1}g_{ti}. \tag{6.2}$$

Transvecting (6.2) with g^{ti}, we have

$$(2n-1)(K - K^*) = 2(n-1)K.$$

Hence from the above lemma, in a conformally flat K-space, K is also an absolute constant.

On the other hand, in section 5 we have seen that in a K-space ρ_i is analytic and $\tilde{\rho}_i$ is a Killing vector and

$$\underset{v}{\pounds}K^*{}_{ji} = \underset{v}{\pounds}K_{ji} = -2(n+1)\nabla_j\rho_i \tag{6.3}$$

hold good.

Operating $\underset{v}{\pounds}$ to (6.2) we have

15) Tachibana, S. [10].

$$2(n-2)\underset{v}{\pounds}K_{ji} = -\frac{K}{2n-1}\underset{v}{\pounds}g_{ji}.$$

Taking account of (6.3) and $\underset{v}{\pounds}g_{ji} = \nabla_j v_i + \nabla_i v_j$, we obtain

$$\nabla_j\left(v_i - \frac{1}{k}\rho_i\right) + \nabla_i\left(v_j - \frac{1}{k}\rho_j\right) = 0,$$

where $$k = \frac{K}{2(n+1)(n-2)(2n-1)} \qquad (n > 2).$$

If we put $p_i = v_i - \frac{1}{k}\rho_i$ and $q_i = \frac{1}{k}\tilde{\rho}_i$, then p_i and q_i are both Killing vectors. Thus we have

THEOREM 6.1. *In a $2n$ $(n > 2)$ dimensional conformally flat K-space with $K \neq 0$, an analytic HP-transformation v^i is uniquely decomposed into the form*

$$v^j = p^i + \varphi_r{}^i q^r$$

where p^i and q^i are both Killing vectors and $\varphi_i{}^r q_r$ is gradient.

We shall prove the uniqueness of this decomposition.
If we have

$$v^i = p^i + \varphi_r{}^i q_r, \; v^i = {}'p^i + \varphi_r{}^i \; {}'q_r,$$

then $$p^i - {}'p^i = \varphi_r{}^i({}'q^r - q^r).$$

If we put $\xi^i = p^i - {}'p^i$, then ξ^i is a Killing vector and at the same time gradient, and therefore it holds that

$$\nabla_j \xi^h = 0.$$

By the Ricci's identity, we have

$$\nabla_j\nabla_i\xi^h - \nabla_i\nabla_j\xi^h = K_{jir}{}^h\xi^r = 0$$

from which we get $K_{ir}\xi^r = 0$, $K^*{}_{ir}\xi^r = 0$. Substituting (6.2) into the last equation, we obtain

$$\frac{K}{2(n-1)(2n-1)}\xi_j = 0.$$

Thus we have $\xi^i = p^i - {}'p^i = 0$, and $q^i - {}'q^i = 0$. In the same way as in section 5 we have the following

THEOREM 6.2. *In a $2n (n > 2)$ dimensional conformally flat K-space with $K \neq 0$, the associated vector of an analytic HP-transformation is also an* *analytic HP-transformation.*

THEOREM 6. 3. *If a $2n(n>2)$ dimensional conformally flat K-space with $K \neq 0$ admits an analytic non-affine HP-transformation, then the restricted homogeneous holonomy group contains the full unitary group $U(n)$.*

7. *O-spaces of constant curvature. Let us consider an *O-space of of constant curvature with $K \neq 0$, then the curvature tensor takes the form

$$K_{kjih} = \frac{K}{2n(2n-1)}(g_{ji}g_{kh} - g_{ki}g_{jh}),$$

from which we have $K_{ji} = \frac{K}{2n} g_{ji}$ and $H_{ji} = \frac{K}{2n(2n-1)} \varphi_{ji}$.
Now, let v^i be an analytic HP-transformation and ρ_i its associated vector, then Theorem 5.2 is valid. If we transvect (5. 9) with φ^{ji}, then we get

$$H_{rh}\rho^r = k(n+1)\tilde{\rho}_h = -\frac{K}{2n}\tilde{\rho}_h.$$

From the last two equations we have $\frac{(n-1)K}{n(2n-1)}\tilde{\rho}_h = 0$ and $\rho_h = 0$. Therefore we have the following

THEOREM 7. 1[16]. *In an *O-space of constant curvature, an analytic HP-transformation is necessarily affine.*

COROLLARY. *In a compact *O-space of constant curvature, an analytic HP-transformation is necessarily an isometry.*

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