INTEGRABILITY OF NONNEGATIVE TRIGONOMETRIC SERIES

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1. Introduction. It is known that if a trigonometric series converges everywhere to a nonnegative sum $f(x)$ then f is integrable and the series is a Fourier series [5, p.328], whereas when a trigonometric series converges to a nonnegative sum only in an interval (a, b), its sum is integrable over interior intervals, but is integrable over (a, b) if and only if the integrated series converges at the endpoints of the interval [5, p.372, no.14]. However, the sum belongs to $L^{1-\delta}$, for every positive δ, over the whole interval (a, b) [5, p.371, no.13]. I shall give a simple proof of this last result by showing first that $(x-a)^{\alpha}(b-x)^{\alpha}f(x)$ is integrable over (a, b) for every positive α.

There is another natural sense in which a nonnegative function f can be associated with a trigonometric series, namely that the coefficients in the series are the Fourier coefficients of f in a generalized sense. If we consider the case when f is integrable except in the neighborhood of one point, which we may take to be 0, we can obtain necessary and sufficient conditions for the integrability of $x^{\alpha}f(x)$ for certain nonnegative values of α. These may be considered as analogues of the known results that connect integrability of $x^{\alpha}f(x)$ with the convergence of $\Sigma c_n n^{-\alpha-1}$ when $\alpha < 0$ (see, for example, [1], [2], [4], where further references are given).

2. Convergent trigonometric series.

THEOREM 1. *If a trigonometric series $\Sigma c_n e^{inx}$ converges in some $(0, \delta)$ to sum $f(x)$ and $f(x) \geqq 0$ in this neighborhood then $x^{\alpha}f(x) \in L$ in a right-hand neighborhood of 0 for every positive α.*

PROOF. Since the series converges in an interval, $c_n \to 0$. We know that f is integrable on every (a, b), $0 < a < b < \delta$. (Cf. [5], pp.328 and 371, no.13.) Since the Fourier series of the function equal to $f(x)$ on (a,b) and to 0 elsewhere is equiconvergent with $\Sigma c_n e^{inx}$ over any closed subinterval of (a, b) ([5], p.330), we can integrate $\Sigma c_n e^{inx}$ formally over (x, ε), where $0 < x < \varepsilon < \delta$, and obtain an integral of f. Since the series $\Sigma c_n e^{inx}/(in)$ is a Fourier series, and indeed the Fourier series of a function that belongs to every $L^p (p < \infty)$, by the Hausdorff-Young theorem, $\int_x^{\varepsilon} f(t)dt \in L^p$ for every p. Then by Hölder's inequality

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$$\alpha^{-1}\int_0^\varepsilon t^\alpha f(t)dt = \int_0^\varepsilon f(t)dt\int_0^t x^{\alpha-1}dx = \int_0^\varepsilon x^{\alpha-1}dx\int_x^\varepsilon f(t)dt$$
$$\leqq \left\{\int_0^\varepsilon dx\left[\int_x^\varepsilon f(t)dt\right]^p\right\}^{1/p}\left\{\int_0^\varepsilon x^{(\alpha-1)p'}dx\right\}^{1/p'} < \infty$$

provided that $p' = p/(p-1) > 1/\alpha$.

THEOREM 2. *With the hypotheses of Theorem* 1, $f \in L^{1-\eta}$ *in a right-hand neighborhood of* 0, *for every positive* η.

We have

$$\int_0^\varepsilon f(t)^{1-\eta}dt = \int_0^\varepsilon f(t)^{1-\eta}t^\lambda t^{-\lambda}\,dt$$
$$\leqq \left\{\int_0^\varepsilon f(t)t^{\lambda/(1-\eta)}dt\right\}^{1-\eta}\left\{\int_0^\varepsilon t^{-\lambda/\eta}dt\right\}^{\eta} < \infty$$

by Hölder's inequality, provided that $0 < \lambda < \eta$.

3. Generalized sine series. We now consider generalized Fourier series of nonnegative functions. We discuss sine and cosine series separately.

THEOREM 3. *If* $0 < \alpha < 1$, $x^\alpha f(x) \in L$, *and*

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\,\sin nx dx, \tag{1}$$

then

$$\sum n^{-\alpha-1}b_n \tag{2}$$

converges.

This is a well-known elementary fact for $\alpha = 0$. For $\alpha = 1$, see Theorem 6. We have

$$\frac{1}{2}\pi\sum_m^M n^{-\alpha-1}b_n = \sum_m^M n^{-\alpha-1}\int_0^\pi f(x)\,\sin nx dx$$
$$= \int_0^\pi f(x)\sum_m^M n^{-\alpha-1}\,\sin nx dx = \int_0^\varepsilon + \int_\varepsilon^\pi = I_1 + I_2.$$

In I_1, we have

$$\left|\sum_m^M \frac{\sin\,nx}{n^{\alpha+1}}\right| \leqq \sum_m^{[1/x]}\left|\frac{\sin nx}{nx}\,\frac{x}{n^\alpha}\right| + \sum_{[1/x]}^M\left|\frac{\sin nx}{n^{\alpha+1}}\right|$$
$$\leqq x\sum_m^{[1/x]} n^{-\alpha} + \sum_{[1,x]}^M n^{-\alpha-1} \leqq Ax^\alpha,$$

where A is independent of m and M. Hence

$$|I_1| \leqq A \int_0^{\varepsilon} |f(x)| x^{\alpha} dx.$$

In I_2, we have

$$\left| \sum_m^M n^{-\alpha-1} \sin nx \right| \leqq \sum_m^M n^{-\alpha-1} \leqq A m^{-\alpha}.$$

We obtain

$$|I_2| \leqq A m^{-\alpha} \int_{\varepsilon}^{\pi} |f(x)| dx.$$

If we take ε small and then m large we can therefore make $I_1 + I_2$ arbitrarily small, and so (2) converges.

Theorem 3 assumes nothing about the sign of $f(x)$. When $f(x) \geqq 0$, it has a converse.

THEOREM 4. *If* $0 < \alpha < 1$, $f(x) \geqq 0$ *on* $(0, \pi)$, $xf(x) \in L$, b_n *are defined by* (1), *and* $\Sigma n^{-\alpha-1} b_n$ *converges, then* $f(x) x^{\alpha} \in L$.

We have

$$\begin{aligned} \frac{1}{2} \pi \sum_1^M n^{-\alpha-1} b_n &= \sum_1^M n^{-\alpha-1} \int_0^{\pi} f(x) \sin nx dx \\ &= \int_0^{\pi} f(x) \sum_1^M n^{-\alpha-1} \sin nx dx. \end{aligned}$$

Since $\sum n^{-1} \sin nx$ has nonnegative partial sums, partial summation shows that $\Sigma n^{-\alpha-1} \sin nx$ also has nonnegative partial sums. Hence by Fatou's lemma

$$\int_0^{\pi} f(x) \sum_1^{\infty} n^{-\alpha-1} \sin nx dx \leqq \liminf_{M\to\infty} \frac{1}{2} \pi \sum_1^M n^{-\alpha-1} b_n. \tag{3}$$

Now

$$\sum_1^{\infty} n^{-\alpha} \cos nt \sim A t^{\alpha-1} \qquad (t \to 0)$$

and so

$$\sum_1^{\infty} n^{-\alpha-1} \sin nx = \int_0^x \sum_1^{\infty} n^{-\alpha} \cos nt dt \sim A x^{\alpha} \qquad (x \to 0).$$

Hence (3) implies that $\int_0^{\pi} f(x) x^{\alpha} dx < \infty$. We have not used the full force of the hypothesis that $\Sigma n^{-\alpha-1}b_n$ converges; it would be enough for this series to have a sequence of bounded partial sums.

Theorem 4 is still true when $\alpha = 0$ but the proof is slightly different.

THEOREM 5. *If* $f(x) \geqq 0$ *on* $(0, \pi)$, b_n *are defined by* (2), *and* $\Sigma b_n/n$ *converges then* $f(x) \in L$.

The reasoning leading to (3) is unchanged when $\alpha = 0$, and the series on the left is now equal to $(\pi - x)/2$. Hence

$$\int_0^\pi f(x)(\pi - x)dx \leqq \liminf_{M\to\infty} \pi \sum_1^M n^{-1}b_n. \tag{4}$$

Since $xf(x) \in L$, (4) shows that $f(x) \in L$.

When $\alpha = 1$, Theorem 4 is vacuous and Theorem 3 fails; as an example we may take an odd function equal to $x^{-2}(\log x)^{-2}$ in a right-hand neighborhood of 0. We have the following substitute.

THEOREM 6. *If* $xf(x)\log(1/x) \in L$ *and* b_n *is defined by* (1) *then* $\Sigma n^{-2}b_n$ *converges; if* $\Sigma n^{-2}b_n$ *converges and* $f(x) \geqq 0$ *then* $xf(x)\log(1/x) \in L$.

If $xf(x)\log(1/x) \in L$ we have

$$\frac{1}{2}\pi \sum_m^M n^{-2}b_n = \sum_m^M n^{-2} \int_0^\pi f(x) \sin nx dx$$
$$= \int_0^\pi f(x) \sum_m^M n^{-2} \sin nx dx = \int_0^\varepsilon + \int_\varepsilon^\pi = I_1 + I_2.$$

By the same reasoning as in Theorem 3 we see that $\sum_m^M n^{-2}\sin nx$ is $O(x\log(1/x))$ uniformly in m and M as $x \to 0$, and $O(1/m)$ for $x > \varepsilon$ as $m \to \infty$. The conclusion follows.

Conversely,

$$\frac{1}{2}\pi \sum_1^M n^{-2}b_n = \int_0^\pi f(x) \sum_1^M n^{-2} \sin nx dx,$$

and if $\Sigma n^{-2}b_n$ converges we have

$$\int_0^\pi f(x) \sum_1^\infty n^{-2} \sin nx dx \leqq \frac{1}{2}\pi \sum_1^\infty n^{-2}b_n.$$

Now $\sum_1^\infty n^{-1}\cos nx = -\log(2\sin x/2)$ and hence $\sum_1^\infty n^{-2}\sin nx \sim x\log(1/x)$ as $x \to 0$. The conclusion follows.

4. Generalized cosine series. For cosine series the situation is somewhat different. If we assume $f(x)\geqq 0$ then the existence of the cosine coefficient a_0 automatically makes $f\in L$. We shall therefore suppose that $a_0=0$, and require $f(x)\geqq 0$ only in a neighborhood of 0. We can then work with a wider range of α than in § 3.

THEOREM 7. *If* $0<\alpha<2$, $x^\alpha f(x)\in L$, *and*

$$a_n=\frac{2}{\pi}\int_{\to 0}^{\pi} f(x)\cos nx\,dx, \qquad n=1,2,\cdots, \tag{5}$$

then

$$\Sigma n^{-\alpha-1}a_n \tag{6}$$

converges.

We have

$$\frac{1}{2}\pi\sum_m^M n^{-\alpha-1}a_n=\int_0^\pi f(x)\sum_m^M\frac{\cos nx}{n^{\alpha+1}}\,dx=-\int_0^\pi f(x)\sum_m^M\frac{1-\cos nx}{n^{\alpha+1}}\,dx$$

since $\int_0^\pi f(x)dx=0$. Thus

$$\frac{1}{2}\pi\sum_m^M n^{-\alpha-1}a_n=-\left(\int_0^\varepsilon+\int_\varepsilon^\pi\right)f(x)\sum_m^M\frac{2\sin^2 nx/2}{n^{\alpha+1}}\,dx=I_1+I_2.$$

In I_1 we have

$$\sum_m^M\frac{\sin^2 nx/2}{n^{\alpha+1}}\leqq\sum_m^{[1/x]}+\sum_{[1/x]}^M\leqq\frac{1}{4}x^2\sum_m^{[1/x]}n^{1-\alpha}+\sum_{[1/x]}^M n^{-\alpha-1}<Ax^\alpha.$$

In I_2,

$$\sum_m^M n^{-\alpha-1}\sin^2 nx/2\leqq Am^{-\alpha}$$

as in the proof of Theorem 3. The convergence of (6) then follows.

THEOREM 8. *If* $0<\alpha<2$, $f\in L$ *in every* (ε,π), $\varepsilon>0$, $f(x)\geqq 0$ *in a right-hand neighborhood of* 0, a_n *are defined by* (5) *with* $a_0=0$, *and* $\Sigma n^{-\alpha-1}a_n$ *converges, then* $f(x)x^\alpha\in L$.

Thus $\alpha=1$ is not an exceptional case for cosine series.

We have

$$\frac{1}{2}\pi\sum_1^M n^{-\alpha-1}a_n=\int_0^\pi f(x)\sum_1^M n^{-\alpha-1}\cos nx\,dx.$$

Now $\sum_{1}^{\infty} n^{-\beta}$ $\cos nx$ has its partial sums uniformly bounded below for β sufficiently near 1; hence, by partial summation, so does $\Sigma n^{-\alpha-1}\cos nx$, $0 < \alpha < 1$. Let $-K$ be a lower bound for the partial sums of the latter series; since $a_0 = 0$, we have

$$\frac{1}{2}\pi \sum_{1}^{M} n^{-\alpha-1} a_n = \int_0^\pi f(x) \left\{ \sum_{1}^{M} n^{-\alpha-1}\cos nx + K \right\} dx.$$

As in Theorem 4, it now follows by Fatou's lemma that

$$\int_0^\pi f(x) \left\{ \sum_{1}^{\infty} n^{-\alpha-1}\cos nx + K \right\} dx$$

converges, and (again since $a_0 = 0$) therefore so does

$$\int_0^\pi f(x) \sum_{1}^{\infty} n^{-\alpha-1}(1 - \cos nx)\, dx.$$

But

$$\sum_{1}^{\infty} n^{-\alpha-1}(1 - \cos nx) = \int_0^x \sum_{1}^{\infty} n^{-\alpha} \sin nx\, dx \sim Ax^\alpha \ (x \to 0)$$

([5], p. 186), and so the conclusion follows.

In the case $\alpha = 0$, conditions for the convergence of (6), i.e. of $\Sigma n^{-1} a_n$, are known (cf. [5]. p.228, no.8; [3], p.96).

References

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