ON SOME ALMOST KÄHLERIAN SPACES

Dedicated to Professor K. Yano on his 50th birthday.

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1. Introduction. Let us consider a Riemannian space V^{2n} with positive definite metric tensor G_{BA} [1] and admitting a Killing vector field u^A such that the magnitude of the absolute differential $d\xi^T\nabla_T u^A$ is equal to the magnitude of the infinitesimal vector $d\xi^A$. Then we get from $(d\xi^B\nabla_B u^T)(d\xi^A\nabla_A u^S)G_{TS}= d\xi^B d\xi^A G_{BA}$ the equation

$$(1.\ 1) \qquad (\nabla_B u^T)(\nabla_A u_T) = G_{BA}$$

where $u_A = u^S G_{SA}$.

A Killing vector u^A satisfies the equation $\nabla_B u_A + \nabla_A u_B = 0$, so that, if we put

$$(1.\ 2) \qquad \nabla_B u^A = F_B^{\cdot A},$$

the equation (1. 1) is equivalent to

$$(1.\ 3) \qquad F_B^{\cdot T} F_T^{\cdot A} = -\delta_B^A.$$

Thus we obtain an almost Hermitian structure. We shall study in the present paper some local properties of spaces V^{2n} with almost complex structure $F_B^{\cdot A}$ which is derived in such a way from a Killing vector u^A. Letters such as $F_B^{\cdot A}$ and $f_j^{\cdot i}$ will always denote almost complex or complex structures.

Since we get from (1. 2) $\nabla_C\nabla_B u^A = \nabla_C F_B^{\cdot A}$ and since a Killing vector u^A satisfies

$$(1.\ 4) \qquad \nabla_C\nabla_B u^A = -K_{SCB}^{\cdot\cdot\cdot A}\ u^S,$$

we get

$$\nabla_C F_{BA} + \nabla_B F_{AC} + \nabla_A F_{CB} = 0,$$

which shows that our space V^{2n} is an almost Kählerian space.

V^{2n} being an almost Kählerian space, we get $\nabla^B F_B^{\cdot A} = 0$ and hence

1) Indices $A, B, \cdots$ would run from 1 to $2n$, but, since we prefer a special coordinate system, we use indices as follows,

$$A, B, C, \cdots, S, T, \cdots = 0, 1, \cdots, 2n-2, \infty,$$
$$\alpha, \beta, \gamma, \cdots, \lambda, \mu, \cdots = 1, \cdots, 2n-2, \infty,$$
$$p, q, r, \cdots, x, y, z = 0, 1, \cdots, 2n-2,$$
$$h, i, j, k, l, m, n = 1, \cdots, 2n-2.$$

$\nabla^B\nabla_B\, u^A = 0$. We then obtain $K_S^{\cdot A} u^S = 0$ where $K_B^{\cdot A} = K_{BS}G^{SA}$ and K_{BA} is the Ricci tensor.

We also obtain

$$\underset{u}{\pounds}\, F_B^{\cdot A} = \underset{u}{\pounds}\nabla_B u^A = (\underset{u}{\pounds}\nabla_B - \nabla_B\underset{u}{\pounds})\, u^A + \nabla_B\underset{u}{\pounds}\, u^A$$

$$= \left(\underset{u}{\pounds}\left\{ {A \atop BC} \right\}\right) u^C = 0$$

from which we find that u^A is a contravariant almost analytic vector [2].

Thus we obtain the

THEOREM 1.1. *If a Riemannian space V^{2n} admits a Killing vector field u^A such that the tensor $\nabla_B u^A$ determines an almost Hermitian structure in V^{2n}, then the space is an almost Kählerian space and the Ricci tensor is not definite. Moreover, the vector u^A is a contravariant almost analytic vector.*

2. Killing vector and some special coordinate systems. Let us consider that a Killing vector $\mathbf{u}$ is given in a Riemannian space V^{2n} and assume that its magnitude $|\mathbf{u}|$ is not a constant. We prove in the following that we can then find a coordinate system (ξ^A) such that the components u^A of the given Killing vector in this coordinate system satisfy

$$u^A = \delta_0^A, \qquad u^T u_T = G_{00} = \xi^\infty. \tag{2. 1}$$

Let (η^A) be any coordinate system in which the given Killing vector has the components $v^0, v^1, \cdots, v^{2n-2}, v^\infty$ and the fundamental tensor has the components H_{BA}. Then, if we put $\varphi^\infty = v^T v^S H_{TS}$, we get

$$\frac{\partial\varphi^\infty}{\partial\eta^A} v^A = 0$$

for we have

$$v^A \frac{\partial(v^T v^S H_{TS})}{\partial\eta^A} = 2v^A(\nabla_A v^T)\, v^S H_{TS}$$

$$= 2(\nabla_A v_T) v^A v^T = 0.$$

Evidently we can choose $2n-2$ functions $\varphi^1, \varphi^2, \cdots, \varphi^{2n-2}$ of η^A such that

$$\frac{\partial\varphi^1}{\partial\eta^A} v^A = 0, \cdots\cdots\cdots\cdots, \frac{\partial\varphi^{2n-2}}{\partial\eta^A} v^A = 0$$

and moreover such that the rank of the matrix

$$\begin{pmatrix} \dfrac{\partial\varphi^1}{\partial\eta^0} & \dfrac{\partial\varphi^1}{\partial\eta^1} & \cdots\cdots & \dfrac{\partial\varphi^1}{\partial\eta^{2n-2}} & \dfrac{\partial\varphi^1}{\partial\eta^\infty} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\varphi^{2n-2}}{\partial\eta^0} & \dfrac{\partial\varphi^{2n-2}}{\partial\eta^1} & \cdots\cdots & \dfrac{\partial\varphi^{2n-2}}{\partial\eta^{2n-2}} & \dfrac{\partial\varphi^{2n-2}}{\partial\eta^\infty} \\ \dfrac{\partial\varphi^\infty}{\partial\eta^0} & \dfrac{\partial\varphi^\infty}{\partial\eta^1} & \cdots\cdots & \dfrac{\partial\varphi^\infty}{\partial\eta^{2n-2}} & \dfrac{\partial\varphi^\infty}{\partial\eta^\infty} \end{pmatrix}$$

is $2n-1$. Then, if φ^0 is a solution of the equation

$$\frac{\partial\varphi^0}{\partial\eta^A}v^A = 1,$$

we get

$$\det\left(\frac{\partial\varphi^A}{\partial\eta^B}\right) \neq 0,$$

so that we can take $(\xi^A) = (\varphi^A)$ as a new coordinate system. The components u^A of the given Killing vector in this coordinate system satisfy

$$u^A = \frac{\partial\varphi^A}{\partial\eta^B}v^B = \delta_0^A, \qquad u^T u_T = v^T v_T = \varphi^\infty = \xi^\infty$$

and we get $G_{00} = G_{BA}u^B u^A = u^T u_T = \xi^\infty$, hence (2. 1) is satisfied.

We can derive from (2. 1) the following equations ([1], p.209; [3],p. 31; [4],p.49):

(2. 2) $$\partial_0 G_{BA} = 0.$$

Now consider the equations

(2. 3) $$\begin{aligned} G^{0\infty} + G^{i\infty}\frac{\partial f}{\partial\xi^i} + G^{\infty\infty}\frac{\partial f}{\partial\xi^\infty} &= 0, \\ G^{i\infty}\frac{\partial f^j}{\partial\xi^i} + G^{\infty\infty}\frac{\partial f^j}{\partial\xi^\infty} &= 0. \end{aligned}$$

As the tensor G_{BA} is positive definite, we have $G^{\infty\infty} > 0$. On the other hand, since we have (2. 2), the functions G^{BA} do not involve the variable ξ^0. Consequently we can find functions f, f^j of $\xi^1,\cdots,\xi^{2n-2}$, ξ^∞ satisfying (2. 3) and

$$\det\left(\frac{\partial f^j}{\partial\xi^i}\right) \neq 0.$$

Then

$$\begin{cases} \xi^{0'} = \xi^0 + f(\xi^1, \cdots, \xi^{2n-2}, \xi^\infty), \\ \xi^{j'} = f^j(\xi^1, \cdots, \xi^{2n-2}, \xi^\infty), \\ \xi^{\infty'} = \xi^\infty \end{cases}$$

is a coordinate transformation such that the components $u^{A'}$ of $\mathbf{u}$ and the components $G_{B'A'}$, $G^{B'A'}$ of the fundamental tensor in the new coordinate system $(\xi^{A'})$ satisfy

$$u^{A'} = \delta^{A'}_{0'}, \qquad \partial_{0'} G^{B'A'} = 0,$$
$$G_{0'0'} = \xi^{\infty'}, \quad G^{0'\infty'} = G_{0'\infty'} = 0, \quad G^{j'\infty'} = G_{j'\infty'} = 0.$$

Thus we obtain the

THEOREM 2.1. *If a Riemannian space V^{2n} admits a Killing vector $\mathbf{u}$ whose magnitude $|\mathbf{u}|$ is not a constant in the V^{2n}, then there exists a coordinate system (ξ^A) such that the components u^A of $\mathbf{u}$ and the components G_{BA} of the fundamental tensor in this coordinate system satisfy the equations*

$$u^A = \delta^A_0, \qquad \partial_0 G_{BA} = 0,$$
$$G_{00} = \xi^\infty, \qquad G_{0\infty} = G_{i\infty} = 0.$$

Such a coordinate system will be called a *favourable coordinate system* in the present paper.

There are many favourable coordinate systems. If (ξ^A) and $(\xi^{A'})$ are such ones, then we have

$$(2.\ 4) \qquad \begin{cases} \xi^{0'} = \xi^0 + f(\xi^1, \cdots, \xi^{2n-2}), \\ \xi^{i'} = f^{i'}(\xi^1, \cdots, \xi^{2n-2}), \\ \xi^{\infty'} = \xi^\infty. \end{cases}$$

REMARK 1. If (2. 1) is satisfied, we have

$$(\nabla_B(u^T u_T))(\nabla_A \varphi)\ G^{BA} = (\nabla_B \xi^\infty)(\nabla_A \varphi) G^{BA} = G^{\infty A} \nabla_A \varphi$$

for any scalar field φ. Hence we can think that (2. 3) means that $\operatorname{grad} \varphi$ and $\operatorname{grad}(u^T u_T)$ are orthogonal to each other whenever we put $\varphi = \xi^0 + f$ or $\varphi = f^j$.

REMARK 2. As $G^{0\infty} = G^{i\infty} = 0$ and $G_{0\infty} = G_{i\infty} = 0$ are equivalent, we find that the hypersurfaces $\xi^x = \text{const.}$ are intersected orthogonally by the hypersurfaces $u^T u_T = \text{const.}$ and that the parametric curves ξ^x are orthogonal to the parametric curves ξ^∞ for every number $x = 0, 1, \cdots, 2n-2$ if we take a favourable coordinate system.

3. A Riemannian space V^{2n} which admits a Killing vector field u^A satisfying the equation $(\nabla_B u^T)(\nabla_A u_T) = G_{BA}$. At first we prove the

THEOREM 3.1. *In a Riemannian space V^{2n} in which $\nabla_B u^A$ determines* *an almost Kählerian structure*[2] *the Killing vector* u^A *can not have constant length.*

PROOF. If we assume $u^T u_T = \text{const.}$, we get $F_A^{\ T} u_T = 0$ from $(\nabla_A u^T)u_T = 0$. But, as we have $\det(F_B^{\ A}) = \pm 1$, this leads to contradiction.

The following corollary is a direct consequence of Theorems 2.1 and 3.1.

COROLLARY. *In a Riemannian space* V^{2n} *in which* $\nabla_B u^A$ *determines an almost Kählerian structure we can choose a coordinate system* $(\xi^0, \xi^i, \xi^\infty)$ *such that the components* u^A *of the Killing vector* $\mathbf{u}$ *satisfy*

$$u^A = \delta_0^A \tag{3. 1}$$

and the components G_{BA} *of the metric tensor satisfy*

$$G_{00} = \xi^\infty, \quad \partial_0 G_{BA} = 0, \quad G_{0\infty} = G_{i\infty} = 0. \tag{3. 2}$$

In such a favourable coordinate system we can write (1. 1) in the form

$$[0B, T][0A, S]G^{TS} = G_{BA} \tag{3. 3}$$

or in the form

$$(\partial_B G_{0T} - \partial_T G_{0B})(\partial_A G_{0S} - \partial_S G_{0A})G^{TS} = 4\, G_{BA}. \tag{3. 4}$$

Taking (2. 2) into account we get from (3. 4)

$$\begin{aligned}&(\partial_B G_{00})(\partial_A G_{00})G^{00} + (\partial_B G_{00})(\partial_A G_{0\alpha} - \partial_\alpha G_{0A})G^{0\alpha}\\ &+ (\partial_B G_{0\beta} - \partial_\beta G_{0B})(\partial_A G_{00})G^{0\beta}\\ &+ (\partial_B G_{0\beta} - \partial_\beta G_{0B})(\partial_A G_{0\alpha} - \partial_\alpha G_{0A})G^{\beta\alpha} = 4\, G_{BA}\end{aligned}$$

and taking $G_{00} = \xi^\infty$, $G_{0\infty} = G_{i\infty} = G^{0\infty} = G^{i\infty} = 0$ into account we find that these equations split into the following three sets of equations,

$$G^{\infty\infty} = 4G_{00}, \tag{3. 5}$$

$$G^{\infty\infty}\partial_\infty G_{0\lambda} = 4G_{0\lambda}, \tag{3. 6}$$

$$\begin{aligned}&\delta_\mu^\infty \delta_\lambda^\infty G^{00} + \delta_\mu^\infty(\partial_\lambda G_{0h} - \partial_h G_{0\lambda})G^{0h} + \delta_\lambda^\infty(\partial_\mu G_{0k} - \partial_k G_{0\mu})G^{0k}\\ &+ (\partial_\mu G_{0\beta} - \partial_\beta G_{0\mu})(\partial_\lambda G_{0\alpha} - \partial_\alpha G_{0\lambda})G^{\beta\alpha} = 4G_{\mu\lambda}.\end{aligned} \tag{3. 7}$$

We obtain $G^{\infty\infty} = 4\xi^\infty$ from (3. 5) and substituting this into (3. 6) we have

$$\xi^\infty \partial_\infty G_{0\lambda} = G_{0\lambda}, \tag{3. 8}$$

hence the functions $f_A = (\xi^\infty)^{-1} G_{0A}$ depend only upon the variables $\xi^1, \cdots, \xi^{2n-2}$. Thus we get

$$G_{0A} = \xi^\infty f_A(\xi^1, \cdots, \xi^{2n-2}) \tag{3. 9}$$

2) It follows immediately that u^A is then a Killing vector.

and also

$$G_{\infty\infty} = \frac{1}{4\xi^{\infty}}, \qquad G^{\infty\infty} = 4\xi^{\infty}. \tag{3.10}$$

Equations obtained from (3. 7) by putting $\lambda = \mu = \infty$ and also by putting $\lambda = \infty$, $\mu = j$ are satisfied identically because of (3. 8) and (3.10). But, if we put $\lambda = i$, $\mu = j$ in (3. 7), we get

$$(\partial_j G_{0k} - \partial_k G_{0j})(\partial_i G_{0h} - \partial_h G_{0i})G^{kh} + (\partial_\infty G_{0j})(\partial_\infty G_{0i})G^{\infty\infty} = 4G_{ji}$$

which we can write also in the following form,

$$\frac{1}{4}(\partial_j G_{0k} - \partial_k G_{0j})(\partial_i G_{0h} - \partial_h G_{0i})G^{kh} = G_{ji} - (\xi^{\infty})^{-1} G_{0j} G_{0i}. \tag{3.11}$$

Thus we have the

THEOREM 3.2. *Let V^{2n} be a Riemannian space with an almost Kählerian structure determined by G_{BA} and $F_B{}^A = \nabla_B u^A$. Then if we take in V^{2n} a favourable coordinate system (ξ^A), that is, a coordinate system in which* (3. 1) *and* (3. 2) *hold, we get* (3. 8), (3. 10) *and* (3.11).

THEOREM 3.3. *If a Riemannian space V^{2n} with the fundamental tensor G_{BA} admits a coordinate system (ξ^A) such that*

$$\partial_0 G_{BA} = 0,\ G_{00} = \xi^{\infty},\ G_{0\infty} = G_{i\infty} = 0,\ G_{\infty\infty} = (4\xi^{\infty})^{-1},$$

and moreover such that (3. 8) *and* (3.11) *hold, then a vector* **u** *with the components $u^A = \delta_0^A$ is a Killing vector which satisfies* (1. 1) *and this V^{2n} becomes an almost Kählerian space by virtue of the tensor $\nabla_B u^A$.*

THEOREM 3.4. *A necessary and sufficient condition that, in a Riemannian space V^{2n} which admits a Killing vector field* **u**, *$\nabla_B u^A$ can determine an almost Hermitian structure is that the following two conditions* (I) *and* (II) *be fulfilled.* (I) $|\mathbf{u}|$ *is not constant in V^{2n} and consequently we can take a favourable coordinate system.* (II) *The metric tensor G_{BA} of V^{2n} satisfies* (3.8), (3.10) *and* (3.11) *in this coordinate system.*

4. A family of almost Kählerian spaces $\overset{*}{V}{}^{2n-2}$ induced by the Killing vector field u^A of V^{2n}. Let us define $g_{\mu\lambda}$ by

$$g_{\mu\lambda} = \frac{G_{\mu\lambda}}{G_{00}} - \frac{G_{0\mu}G_{0\lambda}}{(G_{00})^2}. \tag{4. 1}$$

As det $(g_{\mu\lambda}) \neq 0$, we can define $g^{\mu\lambda}$ by $g_{\nu\alpha}g^{\alpha\lambda} = \delta_\nu^\lambda$. Then we get

$$g^{\mu\lambda} = G_{00}G^{\mu\lambda} \tag{4. 2}$$

and

$$
\begin{aligned}
G^{00} &= \frac{1}{G_{00}} + \frac{g^{\beta\alpha}G_{0\beta}G_{0\alpha}}{(G_{00})^3}, \\
G^{0\lambda} &= -\frac{g^{\lambda\alpha}G_{0\alpha}}{(G_{00})^2}, \\
G_{\mu\lambda} &= G_{00}g_{\mu\lambda} + \frac{G_{0\mu}G_{0\lambda}}{G_{00}}.
\end{aligned} \tag{4. 3}
$$

Using these equations and (3. 9) we can write (3.11) in the form

$$\frac{1}{4}(\partial_j f_k - \partial_k f_j)(\partial_i f_h - \partial_h f_i)g^{kh} = g_{ji}. \tag{4. 4}$$

Since we have $g_{i\infty} = 0$ because of $G_{0\infty} = G_{i\infty} = 0$, we get $g^{kh}g_{hi} = \delta^k_i$. g_{ji} do not involve the variable ξ^0.

Now consider for each value of ξ^∞ a Riemannian space of dimension $2n-2$ in which any point is denoted by the coordinates $\xi^1,\cdots,\xi^{2n-2}$ and the fundamental tensor by g_{ji}. As we have (4. 4), this Riemannian space admits an almost Hermitian structure determined by

$$f_{ji} = \frac{1}{2}(\partial_j f_i - \partial_i f_j). \tag{4. 5}$$

We denote this almost Hermitian space by $\overset{*}{V}{}^{2n-2}(\xi^\infty)$.

In V^{2n} let T denote a trajectory of the group of motions induced by the Killing vector u^A. If a trajectory T passes a point P, it will be denoted by $T(P)$. Let the coordinates of a point P be ξ^A and the coordinates of a point P' be $\xi^A + d\xi^A$. Then we can define the infinitesimal distance between the two trajectories $T(P)$ and $T(P')$ by the length of the infinitesimal vector

$$d\xi^A - \frac{u_S d\xi^S}{u^T u_T}u^A,$$

hence by

$$\left(G_{BA}d\xi^B d\xi^A - \frac{u_B u_A}{u^T u_T}d\xi^B d\xi^A\right)^{1/2}. \tag{4. 6}$$

The distance thus defined depends only upon the trajectories themselves and does not depend upon the position of the points P,P' in the trajectories. This is one of the direct consequences of the definition of group of motions.

Consequently we can derive a Riemannian space of dimension $2n-1$ by regarding each trajectory as a point. This space is denoted by V_B. Substituting $u_A = G_{BA}u^B = G_{0A}$ into (4. 6) we find that the fundamental tensor of V_B is given by

(4. 7)
$$G_{\mu\lambda} - \frac{G_{0\mu}G_{0\lambda}}{G_{00}}$$
if each point of V_B is denoted by the coordinates $\xi^1,\cdots,\xi^{2n-2},\xi^\infty$.

Since ξ^∞ is constant along each trajectory on account of $\xi^\infty = u^T u_T$, V_B admits a family of hypersurfaces $\xi^\infty =$ const. According to (4. 1) and (4. 7) the fundamental tensor of each hypersurface $\xi^\infty = c$ is given by cg_{ji} where g_{ji} is the fundamental tensor of $\overset{*}{V}{}^{2n-2}(c)$.

As the functions $f_i = (\xi^\infty)^{-1}G_{0i}$ do not involve ξ^∞, the skew symmetric tensor (4. 5) does not depend upon ξ^∞. But, as g_{ji} depends in general upon ξ^∞,

(4. 8)
$$f^j = g^{ji}f_i$$
and

(4. 9)
$$f_j^{\cdot i} = f_{jk}g^{ki} = \frac{1}{2}(\overset{*}{\nabla}_j f^i - \overset{*}{\nabla}{}^i f_j)$$
involve ξ^∞. In (4. 9) $\overset{*}{\nabla}_j$ denotes covariant differentiation with respect to the Christoffel symbols $\left\{ {h \atop j\ i} \right\}^*$ formed from g_{ji}.

$\overset{*}{V}{}^{2n-2}$ is such an almost Hermitian space.

But, since we have
$$\overset{*}{\nabla}_k f_{ji} + \overset{*}{\nabla}_j f_{ik} + \overset{*}{\nabla}_i f_{kj} = 0$$
from (4. 5), $\overset{*}{V}{}^{2n-2}$ is an almost Kählerian space.

We have deduced in such a way a family of almost Kählerian spaces $\overset{*}{V}{}^{2n-2}(\xi^\infty)$ from a Riemannian space V^{2n} which admits a Killing vector u^A satisfying the equation (1. 1).

Let us consider conversely a family of $(2n-2)$-dimensional spaces labelled by a parameter ξ^∞, so that each space is denoted by $\overset{*}{M}(\xi^\infty)$ and the family by $\{\overset{*}{M}(\xi^\infty);\ \xi^\infty \in D_1\}$ where D_1 is a domain of positive numbers. Every $\overset{*}{M}(\xi^\infty)$ and the family are differentiable of class, for example, C^∞ and in each $\overset{*}{M}(\xi^\infty)$ a fundamental tensor is introduced by the differentiable functions $g_{ji}(\xi^1,\cdots,\xi^{2n-2},\xi^\infty)$, the coordinates of a point being denoted by $\xi^1,\cdots,\xi^{2n-2}$. We assume that $\overset{*}{M}(\xi^\infty)$ becomes an almost Kählerian space by virtue of g_{ji} and the almost complex structure $f_j^{\cdot i}$ formed from a vector f_i by (4. 8) and (4. 9). We assume furthermore that the components f_i do not depend upon ξ^∞ for some choice of the coordinate system $(\xi^1,\cdots,\xi^{2n-2})$ and consider the space $M = \{\overset{*}{M}(\xi^\infty) \times \xi^0 \times \xi^\infty;\ \xi^0 \in D_0,\ \xi^\infty \in D_1\}$[3] where the points are indicated by the coordinates $\xi^0,\xi^1,\cdots,\xi^{2n-2},\ \xi^\infty$.

3) We understand in the right hand side of this equation a union of the point sets $\overset{*}{M}(\xi^\infty) \times \xi^0 \times \xi^\infty$ over $\xi^0 \in D_0$, $\xi^\infty \in D_1$. In this connection it must be especially emphasized that we are studying only local properties.

D_0 is a domain of real numbers. Though each $\overset{*}{M}(\xi^\infty)$ is a Riemannian space, we introduce the space M only as a differentiable manifold until we introduce a metric anew.

Now we introduce the fundamental tensor G_{BA} into M by

$$
\begin{aligned}
&G_{00} = \xi^\infty,\ G_{0\infty} = G_{i\infty} = 0,\ G_{\infty\infty} = (4\xi^\infty)^{-1}, \\
&G_{0k} = \xi^\infty f_k,\ G_{ji} = \xi^\infty(g_{ji} + f_j f_i).
\end{aligned}
\tag{4.10}
$$

Then the space becomes a Riemannian space V^{2n} which admits evidently a Killing vector $u^A = \delta_0^A$ and this Killing vector satisfies (1. 1). We find immediately that $\overset{*}{M}(\xi^\infty)$ can be identified with $\overset{*}{V}{}^{2n-2}(\xi^\infty)$.

Thus we can construct an almsot Kählerian space V^{2n} with $F_B{}^A = \nabla_B u^A$ starting from a family of some almost Kählerian spaces $\overset{*}{V}{}^{2n-2}(\xi^\infty)$. We can even construct an almost Kählerian space V^{2n} from only one almost Kählerian space $\overset{*}{V}{}^{2n-2}$. We need only to consider that g_{ji} do not involve ξ^∞. There is an intrinsic difference between the two cases, for the variables $\xi^0, \xi^1, \cdots, \xi^{2n-2}$ and the variable ξ^∞ are completely separated in (2. 4).

Consequently we have the following theorems.

THEOREM 4.1. *Let V^{2n} be a Riemannian space which becomes an almost Kählerian space by virtue of a tensor field $\nabla_B u^A$. Then we get a family of almost Kählerian spaces $\overset{*}{V}{}^{2n-2}$. If we take in V^{2n} a favourable coordinate system (ξ^A), then the members of the family are labelled by ξ^∞ and $\overset{*}{V}{}^{2n-2}(\xi^\infty)$ has the fundamental tensor*

$$
g_{ji} = \frac{G_{ji}}{G_{00}} - \frac{G_{0j}G_{0i}}{(G_{00})^2}
$$

and the almost complex structure (4. 9). *If we consider a hypersurface $\xi^\infty = \text{const.}$ in V^{2n} and regard in it each trajectory of the group of motions induced by the Killing vector u^A as a point, we get a Riemannian space of dimension $2n-2$. This spxce is homothetic to the Riemannian space $\overset{*}{V}{}^{2n-2}(\xi^\infty)$, the ratio of the metric tensor being $\xi^\infty : 1$.*

THEOREM 4.2. *Let us consider a family of Riemannian spaces $\overset{*}{M}(\xi^\infty)$ of dimension $2n-2$, where ξ^∞ is a parameter indicating the space and $\xi^1, \cdots, \xi^{2n-2}$ are coordinates of a point. The fundamental tensor of $\overset{*}{M}(\xi^\infty)$ is denoted by $g_{ji}(\xi^1, \cdots, \xi^{2n-2}; \xi^\infty)$ and we assume that $\overset{*}{M}(\xi^\infty)$ admits an almost complex structure*

$$
f_j{}^i = f_{jk} g^{ki}, \qquad f_{ji} = \frac{1}{2}(\partial_j f_i - \partial_i f_j)
$$

derived from a covariant vector field f_i, where f_i do not involve the variable ξ^∞. Then, if G_{BA} satisfies (4.10), *a Riemannian space of dimension 2n obtained by introducing the fundamental tensor G_{BA} into the space $\{\overset{*}{M}(\xi^\infty) \times \xi^0 \times \xi^\infty; \xi^0 \in D_0, \xi^\infty \in D_1\}$ admits a Killing vector $u^A = \delta_0^A$ and it becomes an almost Kählerian space by virtue of the tensor $\nabla_B u^A$, $\overset{*}{M}(\xi^\infty)$ playing the role of $\overset{*}{V}{}^{2n-2}(\xi^\infty)$. We may consider a special case such that g_{ji} do not involve ξ^∞.*

We remark that we have in favourable coordinates

$$F_j{}^i = f_j{}^i,\ F_{\dot\infty}{}^i = 0,\ F_{\dot 0}{}^i = 0. \tag{4.11}$$

This is proved as follows.

$$F_j{}^i = \nabla_j u^i = \begin{Bmatrix} i \\ j\,0 \end{Bmatrix} = G^{ih}[j\,0,h] = \frac{1}{2} G^{ih}(\partial_j G_{0h} - \partial_h G_{0j})$$

$$= \frac{1}{2} g^{ih}(\partial_j f_h - \partial_h f_j),$$

$$F_{\dot\infty}{}^i = \begin{Bmatrix} i \\ \infty 0 \end{Bmatrix} = G^{ip}[\infty\,0,p] = G^{i0}[\infty\,0,0] + G^{ih}[\infty\,0,h]$$

$$= \frac{1}{2} G^{i0} + \frac{1}{2\xi^\infty} G^{ih} G_{0h} = 0,$$

$$F_{\dot 0}{}^i = \begin{Bmatrix} i \\ 0\,0 \end{Bmatrix} = G^{ip}[0\ 0,p] = 0.$$

(1. 3) and (4.11) also bring about the formula $f_j{}^k f_k{}^i = F_j{}^k F_k{}^i = F_j{}^S F_S{}^i = -\delta_j^i$.

5. A Riemannian space V^{2n} which admits a Killing vector u^A satisfying $(\nabla_B u^T)(\nabla_A u_T) = G_{BA}$ and such that the hypersurfaces $u^T u_T =$ const. are totally umbilical. A necessary and sufficient condition that the hypersurfaces $\varphi(x^1,\cdots,x^m)$ = const. in a Riemannian space V^m be totally umbilical is that φ satisfy equations of the form

$$\nabla_j \nabla_i \varphi = \alpha g_{ji} + (\nabla_j \varphi) B_i + (\nabla_i \varphi) B_j \qquad i, j = 1,\cdots,m.$$

Hence, if the hypersurfaces $u^T u_T$ = const. in V^{2n} are totally umbilical, we have

$$\nabla_B \nabla_A (u^T u_T) = \alpha G_{BA} + (\nabla_B (u^T u_T))\Phi_A + (\nabla_A (u^T u_T))\Phi_B.$$

Since $u^T u_T = \xi^\infty$, the left hand side becomes

$$\partial_B \partial_A \xi^\infty - \begin{Bmatrix} C \\ BA \end{Bmatrix} \partial_C \xi^\infty = - \begin{Bmatrix} \infty \\ B\,A \end{Bmatrix} = - G^{\infty\infty}[BA, \infty]$$
$$= -2\xi^\infty(\partial_B G_{A\infty} + \partial_A G_{B\infty} - \partial_\infty G_{BA})$$

and the right hand side becomes

$$\alpha G_{BA} + \delta_B^\infty \Phi_A + \delta_A^\infty \Phi_B.$$

Hence a necessary and sufficient condition that $u^T u_T$ = const. be totally umbilical is that the following equations

$$\partial_\infty G_{yx} = \frac{\alpha}{2\xi^\infty} G_{yx}, \tag{5. 1}$$

which are obtained by putting $B = y$, $A = x$, be satisfied. But, as we have (3. 8) already, we get $\alpha = 2$, and (5. 1) becomes

$$\partial_\infty G_{yx} = (\xi^\infty)^{-1} G_{yx}. \tag{5. 2}$$

Thus we obtain the following theorem.

THEOREM 5.1. *A necessary and sufficient condition that in an almost Kählerian space V^{2n} with $F_B^{\cdot A} = \nabla_B u^A$ the hypersurfaces $u^T u_T$ = const. be totally umbilical is that $(\xi^\infty)^{-1} G_{ji}$ do not involve the variable ξ^∞ in a favourable coordinate system.*

We have proved also the next theorem.

THEOREM 5.2. *Let us assume that a Riemannian space V^{2n} admits a Killing vector feld* **u** *such that* $|\mathbf{u}|$ *is not constant in V^{2n}. Then a necessary and sufficient condition that the hypersurfaces $u^T u_T$ = const. be totally umbilical is that the equation $\partial_\infty G_{yx} = \alpha G_{yx}$ be satisfied for some function α in a favourable coordinate system.*

Let us calculate some components of the curvature tensor by the formula

$$K_{DCBA} = \frac{1}{2}(\partial_D \partial_B G_{CA} + \partial_C \partial_A G_{DB} - \partial_D \partial_A G_{CB} - \partial_C \partial_B G_{DA}) + G^{TS}\{[DB,T][CA,S] - [DA,T][CB,S]\}.$$

Then, using (3. 2), (3. 3), (3. 5), we get

$$\begin{aligned} K_{0BA0} &= G^{TS}\{[0A,T][0B,S] - [00,T][BA,S]\} \\ &= G_{BA} - G^{\infty\infty}[00,\infty][BA,\infty] \\ &= G_{BA} + \xi^\infty(\partial_B G_{A\infty} + \partial_A G_{B\infty} - \partial_\infty G_{BA}), \end{aligned}$$

hence

$$K_{0\infty\infty 0} = 0,\ K_{0x\infty 0} = 0,\ K_{0yx0} = G_{yx} - \xi^\infty \partial_\infty G_{yx}. \tag{5. 3}$$

Consequently a necessary and sufficient condition that the hypersurfaces ξ^∞ = const. be totally umbilical can be written as $K_{0yx0} = 0$. We can also obtain the same result directly if we use (1. 4), get $\nabla_B \nabla_A (u^T u_T) = 2G_{BA} - 2K_{TBAS} u^T u^S = 2G_{BA} - 2K_{0BA0}$ and substitute the latter into $\nabla_y \nabla_x (u^T u_T) = \alpha G_{yx}$.

Hence we get the

COROLLARY OF THEOREM 5.1. *A necessary and sufficient condition* *that in an almost Kählerian space V^{2n} with $F_{\dot{B}}^{\ A} = \nabla_B u^A$ the hypersurfaces $u^T u_T =$ const. be totally umbilical is that, among the components of the curvature tensor, K_{0BA0} vanish in favourable coordinates.*

6. A Kählerian space V^{2n} with the complex structure $F_{\dot{B}}^{\ A} = \nabla_B u^A$. We calculate components K_{0CBA} when $(\xi^\infty)^{-1} G_{yx}$ do not contain the variable ξ^∞.

We have already

$$K_{0BA0} = 0 \tag{6. 1}$$

and we can prove

$$K_{0BA\infty} = 0 \tag{6. 2}$$

as follows.

If a Riemannian space of dimension $2n$ admits a Killing vector u^A satisfying (1. 1), we get

$$\frac{1}{2} \nabla_C \nabla_B \nabla_A (u^T u_T) = \nabla_C (G_{BA} - K_{TBAS} u^T u^S) = - \nabla_C (K_{TBAS} u^T u^S).$$

If (6. 1) holds in addition, we get $\nabla_C \nabla_B \nabla_A (u^T u_T) = 0$ because of $K_{TBAS} u^T u^S = 0$. Hence we obtain $K_{\dot{C}\dot{B}\dot{A}}^{\ \ \ \ S} \nabla_S (u^T u_T) = 0$. Since $\nabla_S (u^T u_T) = \delta_S^\infty$, this equation can be written as follows,

$$K_{\dot{C}\dot{B}\dot{A}}^{\ \ \ \ \infty} = K_{CBA\infty} = 0, \tag{6. 3}$$

which contains (6. 2).

Thus, in order to get expressions of K_{0CBA}, we only need to calculate K_{0jih}.

Using equations such as (3. 2), (3. 8), (3.10) and (5. 2), the right hand side of

$$K_{0jih} = \frac{1}{2} (\partial_j \partial_h G_{0i} - \partial_j \partial_i G_{0h}) + G^{TS} \{[0i,T][jh,S] - [0h,T][ji,S]\}$$

becomes

$$\begin{aligned}
&\frac{1}{2} \partial_j (\partial_h G_{0i} - \partial_i G_{0h}) + (\xi^\infty)^{-1} (G_{0i} G_{jh} - G_{0h} G_{ji}) \\
&+ \frac{1}{4} G^{k0} [(\partial_i G_{0k} - \partial_k G_{0i})(\partial_j G_{0h} + \partial_h G_{0j}) \\
&\qquad - (\partial_h G_{0k} - \partial_k G_{0h})(\partial_j G_{0i} + \partial_i G_{0j})] \\
&+ \frac{1}{4} G^{lk} [(\partial_i G_{0l} - \partial_l G_{0i})(\partial_j G_{hk} + \partial_h G_{jk} - \partial_k G_{jh}) \\
&\qquad - (\partial_h G_{0l} - \partial_l G_{0h})(\partial_j G_{ik} + \partial_i G_{jk} - \partial_k G_{ji})].
\end{aligned}$$

On the other hand, from (3. 9), (4. 2) and (4. 3) we have

$$G^{lk}=(\xi^{\infty})^{-1}g^{lk},\ G_{ji}=\xi^{\infty}(g_{ji}+f_jf_i),\ G_{0i}=\xi^{\infty}f_i \tag{6. 4}$$

where f_i, g_{ji}, g^{lk} do not contain ξ^{∞}. Hence, denoting the Christoffel symbols of the first kind formed from g_{ji} by $[ji,h]^*$ we get

$$\begin{aligned}\partial_jG_{hk}+\partial_hG_{jk}-\partial_kG_{jh}=\xi^{\infty}\{2[jh,k]^*&+f_k(\partial_jf_h+\partial_hf_j)\\&+f_h(\partial_jf_k-\partial_kf_j)+f_j(\partial_hf_k-\partial_kf_h)\}.\end{aligned} \tag{6. 5}$$

We substitute these equations into the formula obtained above, take (4. 4) into account, and get

$$\begin{aligned}K_{0jih}=&\frac{1}{2}\xi^{\infty}\partial_j(\partial_hf_i-\partial_if_h)+\xi^{\infty}(f_ig_{jh}-f_hg_{ji})\\&+\frac{1}{4}\xi^{\infty}(\xi^{\infty}G^{0k}+f_lg^{lk})[(\partial_if_k-\partial_kf_i)(\partial_jf_h+\partial_hf_j)\\&\qquad-(\partial_hf_k-\partial_kf_h)(\partial_jf_i+\partial_if_j)]\\&+\frac{1}{2}\xi^{\infty}(\partial_if_k-\partial_kf_i)\left\{{k\atop jh}\right\}^*-\frac{1}{2}\xi^{\infty}(\partial_hf_k-\partial_kf_h)\left\{{k\atop j\,i}\right\}^*\\&+\xi^{\infty}(f_hg_{ji}+f_jg_{hi}-f_ig_{jh}-f_jg_{ih}).\end{aligned}$$

Since we have $\xi^{\infty}G^{0k}+f_lg^{lk}=0$ from (4. 3), we easily get

$$K_{0jih}=-\xi^{\infty}\overset{*}{\nabla}_jf_{ih}. \tag{6. 6}$$

Hence we obtain the

THEOREM 6.1. *If in an almost Kählerian space V^{2n} with $F_B^{\cdot A}=\nabla_Bu^A$ the hypersurfaces $u^Tu_T=$ const. are totally umbilical, the curvature tensor satisfy* (6. 1), (6. 2) *and* (6. 6) *in favourable coordinates.*

A necessary condition that the hypersurfaces $u^Tu_T=$ const. be totally umbilical in such V^{2n} is that (5. 2) hold. But we can replace this by $K_{0BA0}=0$ by virtue of (5 .3). Hence we obtain the

COROLLARY. *A necessary and sufficient condition that in an almost Kählerian space V^{2n} with $F_B^{\cdot A}=\nabla_Bu^A$ the hypersurfaces $u^Tu_T=$ const. be totally umbilical is that among the components K_{DCBA} of the curvature tensor, K_{0jih} satisfy* (6. 6) *and other components of the form K_{0CBA} all vanish in favourable coordinates.*

If $\overset{*}{\nabla}_jf_{ih}=0$, then $\overset{*}{V}{}^{2n-2}$ is a Kählerian space. Hence we get the

THEOREM 6.2. *A necessary and sufficient condition that an almost Kählerian space V^{2n} with $F_B^{\cdot A}=\nabla_Bu^A$ be a Kählerian space is that the spaces*

$\overset{*}{V}{}^{2n-2}$ *be Kählerian spaces beside that the hypersurfaces* $u^T u_T = const.$ *be totally umbilical.*

PROOF. That the space V^{2n} be Kählerian is equivalent to that the tensor $\nabla_C \nabla_B u^A$ vanish, that is, $K_{\dot{S}\dot{C}\dot{B}}{}^A u^S = 0$, or $K_{0CBA} = 0$. Hence, if V^{2n} is Kählerian, we get (5. 2) from (5. 3) and the hypersurfaces $u^T u_T = const.$ are totally umbilical. $\overset{*}{V}{}^{2n-2}$ are Kählerian by virtue of (6. 6). The converse is easily proved by using (6. 1), (6. 2) and (6. 6).

7. Holomorphic sectional curvature in the direction orthogonal to $\nabla^A(u^T u_T)$ **and** u^A. At first we prove the

LEMMA 7.1. *Let* V^{2n} *be an almost Kählerian space such that* $F_{\dot{B}}{}^A = \nabla_B u^A$. *Then, if a direction* v^A *is orthogonal to* u^A *and* $\nabla^A(u^T u_T)$, $v^B F_{\dot{B}}{}^A$ *is also orthogonal to* u^A, $\nabla^A(u^T u_T)$.

Proof is easily obtained from $v^B F_{\dot{B}}{}^A u_A = v^B(\nabla_B u^A)u_A = \frac{1}{2} v^B \nabla_B (u^T u_T) = 0$, $v^B F_{\dot{B}}{}^A \nabla_A (u^T u_T) = 2v^B F_{\dot{B}}{}^A F_{\dot{A}}{}^T u_T = -2v^B u_B = 0$.

In a Kählerian space V^{2n} with $F_{\dot{B}}{}^A = \nabla_B u^A$ we have obtained $K_{SCBA} u^S = K_{0CBA} = 0$ in favourable coordinates. Consequently we also obtain (6. 3) from $\nabla_C \nabla_B \nabla_A (u^T u_T) = 0$ and $K_{TBAS} u^T u^S = 0$. Thus we get the

THEOREM 7.1. *Let* V^{2n} *be a Kählerian space with* $F_{\dot{B}}{}^A = \nabla_B u^A$. *Then, if we take a favourable coordinate system, the components* K_{DCBA} *of the curvature tensor vanish except those of the form* K_{kjih}.

Let us study the holomorphic sectional curvature $K(v)$ with respect to a direction v^A which is orthogonal to both u^A and $\nabla^A(u^T u_T)$. According to Theorem 7.1 we get at first

$$K(v) = -\frac{K_{kjih} F_{\dot{T}}{}^k F_{\dot{S}}{}^i v^T v^S v^j v^h}{G_{DC} v^D v^C G_{BA} v^B v^A}.$$

On the other hand we have (4.11), hence we get

$$K(v) = -\frac{K_{kjih} f_{\dot{m}}{}^k f_{\dot{l}}{}^i v^m v^l v^j v^h}{G_{DC} v^D v^C G_{BA} v^B v^A}.$$

Since the vector v^A is orthogonal to vectors u^A and $\nabla^A(u^T u_T)$, we have $v_0 = 0$, $v^\infty = 0$. From $v_0 = G_{00} v^0 + G_{0l} v^l = 0$ we get

$$v^0 = -(\xi^\infty)^{-1} G_{0l} v^l = -f_l v^l \tag{7. 1}$$

and

$$\begin{aligned} G_{BA} v^B v^A &= G_{00} v^0 v^0 + 2G_{0l} v^0 v^l + G_{ml} v^m v^l \\ &= \{G_{ml} - (\xi^\infty)^{-1} G_{0m} G_{0l}\} v^m v^l \end{aligned}$$

$$= \xi^{\infty} g_{ml} v^m v^l.$$

Consequently we have

$$K(v) = -\frac{K_{kjih} f_{\dot m}{}^k f_{\dot i}{}^i v^m v^l v^j v^h}{(\xi^{\infty})^2 g_{kj} v^k v^j g_{ih} v^i v^h} \tag{7. 2}$$

and the

THEOREM 7.2. *Let V^{2n} be a Kählerian space with $F_{\dot B}{}^A = \nabla_B u^A$. Then, for any vector v^A orthogonal to vectors u^A and $\nabla^A(u^T u_T)$, the holomorphic sectional curvature $K(v)$ satisfies* (7. 2) *in a favourable coordinate system.*

Let us study a necessary and sufficient condition that $K(v)$ do not depend upon the direction v^A as long as v^A is orthogonal to u^A and $\nabla^A(u^T u_T)$. The way of deduction is similar to the one of Yano [4],p.239.

At first we have

$$K_{kjih} F_{\dot m}{}^k F_{\dot l}{}^i v^m v^l v^j v^h = -k' g_{mj} v^m v^j g_{lh} v^l v^h,$$

where k' is independent of v^i, for we have $v^{\infty} = 0$ and v^0 is determined by (7. 1), while v^i are arbitrary.

Since V^{2n} is a Kählerian space, we have

$$K_{\dot D \dot C \dot B}{}^S F_{\dot S}{}^A = K_{\dot D \dot C \dot S}{}^A F_{\dot B}{}^S,\ K_{DCBS} F_{\dot A}{}^S = K_{DCAS} F_{\dot B}{}^S,$$
$$K_{DCBA} = K_{DCTS} F_{\dot B}{}^T F_{\dot A}{}^S,$$

from which we get, using (4.11),

$$K_{\dot k \dot j \dot i}{}^l F_{\dot l}{}^h = K_{\dot k \dot j \dot i}{}^h F_{\dot i}{}^l,\ K_{kjil} F_{\dot h}{}^l = K_{kjhl} F_{\dot i}{}^l \tag{7. 3}$$

and

$$K_{kjih} = K_{kjml} F_{\dot i}{}^m F_{\dot h}{}^l. \tag{7. 4}$$

We also have the identity $K_{kjih} = K_{ihkj}$.

From (7. 3) we find that $F_{mjlh} = K_{kjih} F_{\dot m}{}^k F_{\dot l}{}^i$ is symmetric in m and j and also in l and h. Since we have moreover $F_{mjlh} = F_{lhmj}$, we get

$$K_{kjih} F_{\dot m}{}^k F_{\dot l}{}^i + K_{klij} F_{\dot m}{}^k F_{\dot h}{}^i + K_{khil} F_{\dot m}{}^k F_{\dot j}{}^i$$
$$= -k'(g_{mj} g_{lh} + g_{ml} g_{hj} + g_{mh} g_{jl}).$$

On the other hand we have $F_{\dot j}{}^l g_{li} = f_{\dot j}{}^l g_{li} = f_{ji}$. Hence we get

$$K_{kjih} - K_{kmlj} F_{\dot i}{}^m F_{\dot h}{}^l - K_{khml} F_{\dot j}{}^m F_{\dot i}{}^l$$
$$= -k'(f_{kj} f_{ih} + g_{ki} g_{jh} + f_{kh} f_{ij}).$$

Subtracting from this equation an equation obtained by interchanging i and h, we get

$$2K_{kjih} + K_{kjml} F_{\dot i}{}^m F_{\dot h}{}^l - K_{khml} F_{\dot j}{}^m F_{\dot i}{}^l + K_{kiml} F_{\dot j}{}^m F_{\dot h}{}^l$$

$$= -k'(2f_{kj}f_{ih} + f_{ki}f_{jh} - f_{kh}f_{ji} + g_{ki}g_{jh} - g_{kh}g_{ji}).$$

The left hand side being equal to $4K_{kjih}$ on account of (7. 4), we obtain

$$K_{kjih} = \frac{k'}{4}(g_{kh}g_{ji} - g_{jh}g_{ki}) + (f_{kh}f_{ji} - f_{jh}f_{ki}) - 2f_{kj}f_{ih}] \tag{7. 5}$$

as a necessary condition that $K(v)$ do not depend upon v^i. This is evidently a sufficient condition as long as we consider the condition at each point of V^{2n}.

Now, we shall consider k' in (7. 5) as a function of the point and prove the

THEOREM 7.3. *Let us consider a Kählerian space V^{2n} with $F_B{}^A = \nabla_B u^A$ and take a favourable coordinate system. A necessary and sufficient condition that the holomorphic sectional curvature with respect to a variable direction which is orthogonal to both u^A and $\nabla^A(u^T u_T)$ be constant at each point is that the curvature tensor K_{DCBA} satisfy the following equations in which k is a constant.*

$$K_{kjih} = \frac{1}{4}k\xi^\infty(g_{kh}g_{ji} - g_{jh}g_{ki}) + (f_{kh}f_{ji} - f_{jh}f_{ki}) - 2f_{kj}f_{ih}]. \tag{7. 6}$$

For this purpose we only need to show that we can deduce (7. 6) from (7. 5).

We write Bianchi's identity in the form

$$\begin{aligned} &\partial_E K_{DCBA} + \partial_D K_{CEBA} + \partial_C K_{EDBA} \\ &- \begin{Bmatrix} S \\ EB \end{Bmatrix} K_{DCSA} - \begin{Bmatrix} S \\ DB \end{Bmatrix} K_{CESA} - \begin{Bmatrix} S \\ CB \end{Bmatrix} K_{EDSA} \\ &- \begin{Bmatrix} S \\ EA \end{Bmatrix} K_{DCBS} - \begin{Bmatrix} S \\ DA \end{Bmatrix} K_{CEBS} - \begin{Bmatrix} S \\ CA \end{Bmatrix} K_{EDBS} = 0. \end{aligned} \tag{7. 7}$$

Noticing that the components K_{DCBA} vanish except K_{kjih} and putting $E = \infty$, $D = k$, $C = j$, $B = i$, $A = h$, we get

$$\partial_\infty K_{kjih} - \begin{Bmatrix} l \\ \infty i \end{Bmatrix} K_{kjlh} - \begin{Bmatrix} l \\ \infty h \end{Bmatrix} K_{kjil} = 0.$$

Since we have

$$\begin{aligned} \begin{Bmatrix} l \\ \infty i \end{Bmatrix} &= G^{l0}[\infty\, i,0] + G^{lk}[\infty\, i,k] \\ &= \frac{1}{2\xi^\infty}(G^{l0}G_{i0} + G^{lk}G_{ik}) = \frac{1}{2\xi^\infty}\delta^l_i, \end{aligned}$$

we get $\partial_\infty K_{kjih} = (\xi^\infty)^{-1}K_{kjih}$, which proves that $k = (\xi^\infty)^{-1}k'$ does not involve ξ^∞.

We now prove that k does not involve ξ^i.

By straightforward calculation using (6. 4) and (6. 5) we obtain

$$\partial_l K_{kjih} - \begin{Bmatrix} m \\ l\,i \end{Bmatrix} K_{kjmh} - \begin{Bmatrix} m \\ l\,h \end{Bmatrix} K_{kjim}$$
$$= \partial_l K_{kjih} - (G^{0m}[li,0] + G^{nm}[li,n])K_{kjmh} - (G^{0m}[lh,0] + G^{nm}[lh,n])K_{kjim}$$
$$= \partial_l K_{kjih} - \begin{Bmatrix} m \\ l\,i \end{Bmatrix}^* K_{kjmh} - \begin{Bmatrix} m \\ lh \end{Bmatrix}^* K_{kjim}$$
$$- \Big[-f^m \frac{1}{2}(\partial_i f_l + \partial_l f_i) + g^{nm}\frac{1}{2}\{f_n(\partial_l f_i + \partial_i f_l) + f_i(\partial_l f_n - \partial_n f_l) + f_l(\partial_i f_n - \partial_n f_i)\}\Big] K_{kjmh}$$
$$- \Big[-f^m \frac{1}{2}(\partial_h f_l + \partial_l f_h) + g^{nm}\frac{1}{2}\{f_n(\partial_l f_h + \partial_h f_l) + f_h(\partial_l f_n - \partial_n f_l) + f_l(\partial_h f_n - \partial_n f_h)\}\Big] K_{kjim}$$
$$= \partial_l K_{kjih} - \begin{Bmatrix} m \\ l\ i \end{Bmatrix}^* K_{kjmh} - \begin{Bmatrix} m \\ l\,h \end{Bmatrix}^* K_{kjim}$$
$$- \frac{g^{nm}}{2}\{f_i(\partial_l f_n - \partial_n f_l) + f_l(\partial_i f_n - \partial_n f_i)\}K_{kjmh}$$
$$- \frac{g^{nm}}{2}\{f_h(\partial_l f_n - \partial_n f_l) + f_l(\partial_h f_n - \partial_n f_h)\}K_{kjim}$$
$$= \partial_l K_{kjih} - \begin{Bmatrix} m \\ l\,i \end{Bmatrix}^* K_{kjmh} - \begin{Bmatrix} m \\ l\,h \end{Bmatrix}^* K_{kjim}$$
$$- (f_i f_{\dot{l}}{}^m + f_l f_{\dot{i}}{}^m)K_{kjmh} - (f_h f_{\dot{l}}{}^m + f_l f_{\dot{h}}{}^m)K_{kjim}.$$

Hence we obtain the following equation when we put $E = l$, $D = k$, $C = j$, $B = i$, $A = h$ in (7. 7),

$$\overset{*}{\nabla}_l K_{kjih} + \overset{*}{\nabla}_k K_{jlih} + \overset{*}{\nabla}_j K_{lkih}$$
$$- K_{kjmh}(f_{\dot{l}}{}^m f_i + f_{\dot{i}}{}^m f_l) + K_{kjmi}(f_{\dot{l}}{}^m f_h + f_{\dot{h}}{}^m f_l)$$
$$- K_{jlmh}(f_{\dot{k}}{}^m f_i + f_{\dot{i}}{}^m f_k) + K_{jlmi}(f_{\dot{k}}{}^m f_h + f_{\dot{h}}{}^m f_k)$$
$$- K_{lkmh}(f_{\dot{j}}{}^m f_i + f_{\dot{i}}{}^m f_j) + K_{lkmi}(f_{\dot{j}}{}^m f_h + f_{\dot{h}}{}^m f_j) = 0.$$

In this equation $\overset{*}{\nabla}_l K_{kjih}$ denotes a covariant derivative obtained formally by treating K_{kjih} as a tensor of $\overset{*}{V}{}^{2n-2}$ (See (8. 1)).

But we have from (7. 3)

$$K_{kjmh}f_i^{\cdot m} - K_{kjmi}f_h^{\cdot m} = 0$$

and

$$\begin{aligned} K_{kjmh}f_i^{\cdot m} + K_{jlmh}f_k^{\cdot m} + K_{lkmh}f_j^{\cdot m} &= (K_{kjml} + K_{jlmk} + K_{lkmj})f_h^{\cdot m} \\ &= -3K_{[kjl]m}f_h^{\cdot m} = 0, \end{aligned}$$

hence we get

$$\overset{*}{\nabla}_l K_{kjih} + \overset{*}{\nabla}_k K_{jlih} + \overset{*}{\nabla}_j K_{lkih} = 0. \tag{7. 8}$$

This equation is formally the same with Bianchi's identity in $\overset{*}{V}{}^{2n-2}$. Hence, substituting (7. 6) into this equation, we find that $k\xi^\infty$ does not contain the variables $\xi^1, \cdots, \xi^{2n-2}$. Thus we obtain $k =$ const. and Theorem 7.3 is proved.

We have also proved the following theorem.

THEOREM 7.4. *We consider a Kählerian space V^{2n} with $F_{\dot{B}}{}^A = \nabla_B u^A$ and take a favourable coordinate system. Then the functions $(\xi^\infty)^{-1}K_{kjih}$ do not involve the variable ξ^∞ and K_{kjih} satisfy* (7. 8) *if $\overset{*}{\nabla}_l$ denotes formal covariant differentiation with respect to the Christoffel symbols formed from g_{ji}.*

8. **A relation between the curvature of $\overset{*}{V}{}^{2n-2}$ and the curvature of V^{2n}.** We can prove by straightforward calculation the following theorem.

THEOREM 8.1. *Let V^{2n} be an almost Kählerian space such that $F_{\dot{B}}{}^A = \nabla_B u^A$. If $(\xi^\infty)^{-1}G_{ji}$ do not depend on ξ^∞ in favourable coordinates, then we have the following relation between the curvature tensor $\overset{*}{K}_{kjih}$ of $\overset{*}{V}{}^{2n-2}$ and the curvature tensor K_{DCBA} of V^{2n},*

$$\begin{aligned} (\xi^\infty)^{-1}K_{kjih} - \overset{*}{K}_{kjih} &= -g_{kh}g_{ji} + g_{jh}g_{ki} - (f_{kh}f_{ji} - f_{jh}f_{ki}) + 2f_{kj}f_{ih} \\ &\quad + f_j\overset{*}{\nabla}_k f_{ih} - f_k\overset{*}{\nabla}_j f_{ih} + f_h\overset{*}{\nabla}_i f_{kj} - f_i\overset{*}{\nabla}_h f_{kj}. \end{aligned} \tag{8. 1}$$

From this theorem and Theorem 7.3 we get the

THEOREM 8.2. *Let V^{2n} be a Kählerian space with $F_{\dot{B}}{}^A = \nabla_B u^A$. A necessary and sufficient condition that the holomorphic sectional curvature $K(v)$ with respect to a direction v^A orthogonal to u^A and $\nabla^A(u^T u_T)$, when considered at each point, be independent of v^A is that each Kählerian space $\overset{*}{V}{}^{2n-2}$ be a space of constant holomorphic sectional curvature.*

PROOF OF THEOREM 8.1. We start from

$$K_{kjih} = \frac{1}{2}[\partial_k\partial_i G_{jh} + \partial_j\partial_h G_{ki} - \partial_k\partial_h G_{ji} - \partial_j\partial_i G_{kh}]$$

$$
\begin{aligned}
&+ G^{\infty\infty}\{[ki,\infty][jh,\infty] - [kh,\infty][ji,\infty]\} \\
&+ G^{00}\{[ki,0][jh,0] - [kh,0][ji,0]\} \\
&+ G^{0l}\{[ki,0][jh,l] - [kh,0][ji,l]\} \\
&+ G^{0l}\{[ki,l][jh,0] - [kh,l][ji,0]\} \\
&+ G^{ml}\{[ki,m][jh,l] - [kh,m][ji,l]\}
\end{aligned}
$$

and as in §6 take into account that g_{jh} and f_j do not contain ξ^∞. Using relations such as (6. 4), (6. 5) again, we get after calculation

$$
\begin{aligned}
&(\xi^\infty)^{-1}K_{kjih} - \overset{*}{K}_{kjih} \\
&= 2f_{kj}f_{ih} + f_j\partial_k f_{ih} - f_k\partial_j f_{ih} + f_h\partial_i f_{kj} - f_i\partial_h f_{kj} + f_{kh}f_{ij} - f_{ki}f_{hj} \\
&\quad + g_{ki}g_{jh} - g_{kh}g_{ji} + f_k f_i g_{jh} + f_j f_h g_{ki} - f_k f_h g_{ji} - f_j f_i g_{kh} \\
&\quad + g^{ml}[(f_{km}f_i + f_{im}f_k)(f_{jl}f_h + f_{hl}f_j) \\
&\qquad - (f_{km}f_h + f_{hm}f_k)(f_{jl}f_i + f_{il}f_j)] \\
&\quad + \left[-\begin{Bmatrix} l \\ ki \end{Bmatrix}^* f_{lj} - \begin{Bmatrix} l \\ ji \end{Bmatrix}^* f_{kl}\right] f_h + \left[-\begin{Bmatrix} l \\ ki \end{Bmatrix}^* f_{lh} - \begin{Bmatrix} l \\ kh \end{Bmatrix}^* f_{il}\right] f_j \\
&\quad - \left[-\begin{Bmatrix} l \\ kh \end{Bmatrix}^* f_{lj} - \begin{Bmatrix} l \\ jh \end{Bmatrix}^* f_{kl}\right] f_i - \left[-\begin{Bmatrix} l \\ ji \end{Bmatrix}^* f_{lh} - \begin{Bmatrix} l \\ jh \end{Bmatrix}^* f_{il}\right] f_k
\end{aligned}
$$

where

$$
\begin{aligned}
\overset{*}{K}_{kjih} = &\frac{1}{2}[\partial_k\partial_i g_{jh} + \partial_j\partial_h g_{ki} - \partial_k\partial_h g_{ji} - \partial_j\partial_i g_{kh}] \\
&+ g^{ml}\{[ki,m]^*[jh,l]^* - [kh,m]^*[ji,l]^*\}
\end{aligned}
$$

are the components of the curvature tensor of $\overset{*}{V}^{2n-2}$. Since we have moreover $g^{ml}f_{km}f_{jl} = g_{kj}$ and $\overset{*}{\nabla}_k f_{ih} = \partial_k f_{ih} - \begin{Bmatrix} l \\ ki \end{Bmatrix}^* f_{lh} - \begin{Bmatrix} l \\ kh \end{Bmatrix}^* f_{il}$, we obtain (8. 1).

(8. 1) also proves Theorem 7.4 directly.

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