ON WALKER DIFFERENTIATION IN DIFFERENTIABLE MANIFOLDS WITH r-π-STRUCTURE

CHEN-JUNG HSU

(Received May 28, 1962)

After Walker [1][1)] has introduced a new covariant differentiation in almost complex spaces under which the almost complex structure F_i^h is covariant constant and applied such differentiation to the Nijenhuis tensor N_{ji}^h to obtain a new differential invariant constructed only from the tensor F_i^h, Yano [2] showed that there exist an infinitely many Walker differentiations (the one obtained by Walker in the case of almost complex space is also contained) in almost product as well as almost complex spaces which leave their structure tensor F_i^h covariant constant.

In this note we intend to give formal extensions of some results by Walker and Yano to a more general differentiable manifold with an r-π-structure [3]. Such spaces contain as special cases almost complex spaces, almost product spaces and also almost contact manifolds studied by Boothby-Wang, Gray, Sasaki and others.

As preparation we treat in §1 the hybrid and purity of tensors of differentiable manifold with r-π-structure. In §2 we prove that the torsion tensor of an r-π-structure is hybrid. In §3 we determine all Walker differentiations which leave the fundamental tensor F_i^h of the r-π-structure covariant constant. In §4 the one corresponding to that of Walker is considered and a differential invariant constructed only from F_i^h is obtained. In §5 specialization to almost contact manifold is given.

1. If in a differentiable manifold V_n there exist $r\,(2 \leqq r \leqq n)$ differentiable distributions $T_1, \cdots, T_r$ which assign r complemented subspaces of dimension $\geqq 1$ in the complexified tangent space $T_x^c(T_x^c = T_1 + \cdots + T_r$: direct sum) at each point $x \in V_n$, then V_n is said to have an r-π-structure [3].

It is known that for a manifold V_n to have an r-π-structure, it is necessary and sufficient that the manifold has a non-degenerate tensor field F_i^h satisfying

$$\overset{r}{F}{}_j^{\,i} = \lambda^r \delta_j^i, \tag{1. 1}$$

where λ is a non-zero fixed complex constant, and we have put

$$\overset{s}{F}{}_j^{\,i} = F_{h_1}{}^i F_{h_2}{}^{h_1} \cdots F_j{}^{h_{s-1}}, \quad \overset{1}{F}{}_j^{\,i} = F_j^{\,i} \quad \text{and } \overset{0}{F}{}_j^{\,i} = \delta_j^i. \tag{1. 2}$$

The tensor $F_j^{\,i}$ satisfying (1. 1) is called the fundamental tensor associated with

1) Number in bracket refers to the reference at the end of the paper.

the r-π-structure. It is evident that

$$\overset{-s}{F}{}_j{}^i = \frac{1}{\lambda^{ar}} \overset{ar-s}{F}{}_j{}^i, \ (a, s: \text{positive integers}, \ r > ar - s \geqq 0). \tag{1. 3}$$

Now let $\underset{\alpha}{p}{}_a{}^i$ be the projection tensor to the distribution $T_\alpha(\alpha = 1,\cdots,r)$, then we have

$$\underset{\alpha}{p}{}_a{}^i = \frac{1}{r}\sum_{s=0}^{r-1}\frac{1}{(\lambda w_\alpha)^s}\overset{s}{F}{}_a{}^i, \tag{1. 4}$$

where $w_\alpha(\alpha = 1,\cdots,r)$ are the r-th power roots of unity.

Define

$$\underset{1}{\Phi}{}_{aj}^{ib} = \sum_{\alpha=1}^{r}\underset{\alpha}{p}{}_a{}^i\underset{\alpha}{p}{}_j{}^b, \tag{1. 5}$$

then by use of (1. 4) and the following identities

$$\sum_{\alpha=1}^{r}\frac{1}{w_\alpha} = \sum_{\alpha=1}^{r}\frac{1}{w_\alpha{}^2} = \cdots = \sum_{\alpha=1}^{r}\frac{1}{w_\alpha{}^{r-1}} = 0,$$

we have the following expression:

$$\underset{1}{\Phi}{}_{aj}^{ib} = \frac{1}{r}\left(\delta_a^i\delta_j^b + \frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{r-t}{F}{}_a{}^i\overset{t}{F}{}_j{}^b\right). \tag{1. 6}$$

Furthermore, we define

$$\underset{2}{\Phi}{}_{aj}^{ib} = \delta_a^i\delta_j^b - \underset{1}{\Phi}{}_{aj}^{ib} = \frac{1}{r}\left\{(r-1)\delta_a^i\delta_j^b - \frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{r-t}{F}{}_a{}^i\overset{t}{F}{}_j{}^b\right\}. \tag{1. 7}$$

Then for the two tensors in (1. 6) and (1. 7), we have

$$\left\{\begin{aligned} &\underset{1}{\Phi}{}_{aj}^{ib} + \underset{2}{\Phi}{}_{aj}^{ib} = \delta_a^i\delta_j^b, \\ &\underset{1}{\Phi}{}_{hj}^{ik}\underset{1}{\Phi}{}_{ak}^{hb} = \underset{1}{\Phi}{}_{aj}^{ib}, \qquad \underset{2}{\Phi}{}_{hj}^{ik}\underset{2}{\Phi}{}_{ak}^{hb} = \underset{2}{\Phi}{}_{aj}^{ib}, \\ &\underset{1}{\Phi}{}_{hj}^{ik}\underset{2}{\Phi}{}_{ak}^{hb} = \underset{2}{\Phi}{}_{hj}^{ik}\underset{1}{\Phi}{}_{ak}^{hb} = 0. \end{aligned}\right. \tag{1. 8}$$

For, by use of

$$\underset{\alpha}{p}{}_h{}^i\underset{\beta}{p}{}_a{}^h = \begin{cases} \underset{\alpha}{p}{}_a{}^i & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta, \end{cases}$$

we have for example

$$\underset{1}{\Phi}{}_{hj}^{ik}\underset{1}{\Phi}{}_{ak}^{hb} = \left(\sum_{\alpha=1}^{r}\underset{\alpha}{p}{}_h{}^i\underset{\alpha}{p}{}_j{}^k\right)\left(\sum_{\beta=1}^{r}\underset{\beta}{p}{}_a{}^h\underset{\beta}{p}{}_k{}^b\right) = \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}(\underset{\alpha}{p}{}_h{}^i\underset{\beta}{p}{}_a{}^h)(\underset{\alpha}{p}{}_j{}^k\underset{\beta}{p}{}_k{}^b) = \sum_{\alpha=1}^{r}\underset{\alpha}{p}{}_a{}^i\underset{\alpha}{p}{}_j{}^b = \underset{1}{\Phi}{}_{aj}^{ib}.$$

Using this relation and (1. 7) other relations in (1. 8) can be easily seen.

Now, if a tensor $T^{\cdot\,\cdot\,\cdot\, i\,\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot\,\cdot\, j\,\cdot\,\cdot}$ satisfies the relation

$$\underset{1}{\Phi}{}^{ib}_{aj}\, T^{\cdot\,\cdot\,\cdot\, a\,\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot\,\cdot\, b\,\cdot\,\cdot} = 0 \text{ (i.e., } \underset{2}{\Phi} T = T), \tag{1. 9}$$

or

$$\underset{2}{\Phi}{}^{ib}_{aj}\, T^{\cdot\,\cdot\,\cdot\, a\,\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot\,\cdot\, b\,\cdot\,\cdot} = 0 \text{ (i.e., } \underset{1}{\Phi} T = T), \tag{1.10}$$

then we say that the tensor $T^{\cdot\,\cdot\,\cdot\, i\,\cdot\,\cdot\,\cdot}_{\cdot\,\cdot\,\cdot\,\cdot\, j\,\cdot\,\cdot}$ is *hybrid* or *pure* with respect to the indices i and j.

2. A basis (e_i) in the complexified tangent space $T_x{}^c$ at each point $x \in V_n$ is called an adapted basis if $e_{a_\alpha} \in T_\alpha$ $(\alpha = 1,\cdots,r)$, where the indices take the following ranges:

$$\begin{gathered} 1 \leqq a_1, b_1, c_1, \cdots \leqq n_1, \\ n_1 + 1 \leqq a_2, b_2, c_2, \cdots \leqq n_1 + n_2, \\ \cdots\cdots\cdots\cdots \\ n_1 + \cdots + n_{r-1} + 1 \leqq a_r, b_r, c_r, \cdots \leqq n_1 + \cdots + n_r, \\ 1 \leqq a, b, c, \cdots;\ i, j, k, \cdots \leqq n, \end{gathered}$$

in which

$$n_\alpha = \dim T_\alpha \qquad \text{and} \qquad n_1 + \cdots + n_r = n.$$

Moreover, we assume that $\bar{a}_\alpha$, $\bar{b}_\alpha$, $\bar{c}_\alpha, \cdots (1 \leqq \alpha \leqq r)$ take all integers $[(n - n_\alpha)$ in number$]$ between 1 and n except for n_α integers between $n_1 + \cdots + n_{\alpha-1} + 1$ and $n_1 + \cdots + n_{\alpha-1} + n_\alpha$.

With respect to an adapted basis, the projection tensors have the following components:

$$\underset{\alpha}{p}{}^{a_\alpha}_{b_\alpha} = \delta^{a_\alpha}_{b_\alpha} \qquad \underset{\alpha}{p}{}_{b_\alpha}{}^{\bar{a}_\alpha} = \underset{\alpha}{p}{}_{\bar{b}_\alpha}{}^{a_\alpha} = \underset{\alpha}{p}{}_{\bar{b}_\alpha}{}^{\bar{a}_\alpha} = 0. \tag{2. 1}$$

By making use of which, we have

$$\underset{1}{\Phi}{}^{c_\alpha b_\alpha}_{a_\alpha d_\alpha} = \delta^{c_\alpha}_{a_\alpha}\delta^{b_\alpha}_{d_\alpha},\ \underset{1}{\Phi}{}^{c_\alpha b}_{\bar{a}_\alpha d_\alpha} = \underset{1}{\Phi}{}^{c_\alpha \bar{b}_\alpha}_{a\, d_\alpha} = 0, \quad \underset{1}{\Phi}{}^{c_\alpha b}_{a\, \bar{d}_\alpha} = 0. \tag{2. 2}$$

Hence we have

$$\underset{1}{\Phi}{}^{c_\alpha b}_{a\, d_\alpha} T_b{}^a = \delta^{c_\alpha}_{a_\alpha}\delta^{b_\alpha}_{d_\alpha} T_{b_\alpha}{}^{a_\alpha} = T_{d_\alpha}{}^{c_\alpha},\ \underset{1}{\Phi}{}^{c_\alpha b}_{a\, \bar{d}_\alpha} T_b{}^a = 0. \tag{2. 3}$$

Thus it follows from (2. 3) that *the necessary and sufficient condition for the tensor $T_j{}^i$ to be hybrid with respect to i and j is*

$$T_{\bar{d}_\alpha}{}^{c_\alpha} = 0,\ (\alpha = 1,\cdots,r) \tag{2. 4}$$

with respect to any adapted basis.

Now consider in each neighborhood U of V_n a local cross section of class

C^∞ of the principal fibre space $E_\pi(V_n)$ which consists of all adapted bases relative to all points of V_n. Then at each point of the neighborhood U an adapted basis (e_i) is assigned. Let (θ^i) be the dual cobasis of (e_i), then we have

$$d\theta^i = \frac{1}{2} C_{jk}{}^i \theta^j \wedge \theta^k, \text{ where } C_{jk}{}^i + C_{kj}{}^i = 0. \tag{2. 5}$$

Then by definition the torsion tensor $t_{jk}{}^i$ of an r-π-structure is the one with the following components with respect to the adapted basis (e_i):

$$t_{\bar{b}_\alpha \bar{c}_\alpha}{}^{a_\alpha} = C_{\bar{b}_\alpha \bar{c}_\alpha}{}^{a_\alpha}, \quad t_{jk}{}^i = 0 \text{ for other indices.} \tag{2. 6}$$

Hence

$$t_{b_\alpha c}{}^{a_\alpha} = t_{b c_\alpha}{}^{a_\alpha} = 0. \tag{2. 7}$$

Therefore, $t_{jk}{}^i$ *is hybrid both with respect to* i, j *and to* i, k. Q.E.D.

In a previous paper [3] of the present author it is shown that

$$t_{jk}{}^m = \frac{2(r-1)^2}{r^2} \Phi\Phi' S_{jk}{}^m, \tag{2. 8}$$

where $S_{jk}{}^m$ is the torsion tensor of a π-connection $\Gamma_{jk}{}^m$ and the operation Φ and Φ' are defined as follows:

$$\Phi S_{jk}{}^m = S_{jk}{}^m - \frac{1}{r-1} \frac{1}{\lambda^r} \sum_{s=1}^{r-1} \overset{s}{F}_k{}^{k_1} S_{jk_1}^{h} h \overset{r-s}{F}_h{}^m = \frac{r}{r-1} \underset{2}{\Phi}{}_{hk}^{mk} S_{jk_1}{}^h, \tag{2. 9}$$

$$\Phi' S_{jk}{}^m = S_{jk}{}^m - \frac{1}{r-1} \frac{1}{\lambda^r} \sum_{s=1}^{r-1} \overset{s}{F}_j{}^{j_1} S_{j_1 k}{}^h \overset{r-s}{F}_h{}^m = \frac{r}{r-1} \underset{2}{\Phi}{}_{hj}^{m j_1} S_{j_1 k}{}^h. \tag{2.10}$$

From these relations we have the following by virtue of $(1. 8)_3$:

$$\underset{1}{\Phi} t_{jk}{}^m = 0. \tag{2.11}$$

Since Φ and Φ' commute, it follows that $t_{jk}{}^m$ is hybrid both with respect to m, j and m, k.

3. Hereafter we assume that the torsion tensor of the r-π-structure $t_{jk}{}^i$ does not vanish.

Let U and V be two local coordinate neighborhoods of V_n such that $U \cap V \neq \phi$ and let (x^i) and (x^α) be respectively the local coordinates in U and V.

Suppose $\Gamma_{jkl}{}^i$ and $\Gamma_{\beta\gamma\delta}{}^\alpha$ be respectively the components of a geometric object having the following transformation law which is associative and reversible:

$$\Gamma_{jkl}{}^i = X_\alpha^i X_j^\beta X_k^\gamma X_l^\delta \Gamma_{\beta\gamma\delta}{}^\alpha - X_{\alpha\beta}{}^i X_j^\beta X_k^\gamma X_l^\delta t_{\gamma\delta}{}^\alpha, \tag{3. 1}$$

where $t_{jk}{}^i$ is the torsion tensor of the considered r-π-structure of V_n and

$$X^i_\alpha = \frac{\partial x^i}{\partial x^\alpha}, \quad X^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial x^\alpha \partial x^\beta} \text{ and } X^\alpha_i = \frac{\partial x^\alpha}{\partial x^i} \tag{3. 2}$$

are formed from the coordinate transformation $x^i = x^i(x^\alpha)$.

Then the following are tensors provided that v^i, w_j and $\phi_j{}^i$ are tensors:

$$\begin{cases} \nabla_{kl} f = \dfrac{\partial f}{\partial x^m} t_{kl}{}^m, \quad (f: \text{scalar}), \\ \nabla_{kl} v^i = \dfrac{\partial v^i}{\partial x^m} t_{kl}{}^m + \Gamma_{jkl}{}^i v^j, \\ \nabla_{kl} w_j = \dfrac{\partial w_j}{\partial x^m} t_{kl}{}^m - \Gamma_{jkl}{}^i w_i, \\ \nabla_{kl} \phi_j{}^i = \dfrac{\partial \phi_j{}^i}{\partial x^m} t_{kl}{}^m + \Gamma_{mkl}{}^i \phi_j{}^m - \Gamma_{jkl}{}^m \phi_m{}^i, \text{ etc.} \end{cases} \tag{3. 3}$$

This sort of covariant differentiation is a formal extension of the one first introduced by Walker and following Yano we shall call the left hand side terms of these formulas as *Walker derivatives*. The same calculation rules as those of ordinary covariant differentiation hold also in the present case.

We shall now determine all Walker differentiations which satisfy[2)]

$$\nabla_{kl} F_j{}^i = \frac{\partial F_j{}^i}{\partial x^m} t_{kl}{}^m + \Gamma_{mkl}{}^i F_j{}^m - \Gamma_{jkl}{}^m F_m{}^i = 0, \tag{3. 4}$$

where $F_j{}^i$ is the fundamental tensor of the considered r-π-structure.

Let ∇_m denote the covariant derivative with respect to a fixed affine connection $\Gamma_{jk}{}^i$, then (3. 4) may be written as

$$(\nabla_m F_j{}^i) t_{kl}{}^m + T_{mkl}{}^i F_j{}^m - T_{jkl}{}^m F_m{}^i = 0, \tag{3. 5}$$

where we put

$$T_{jkl}{}^i = \Gamma_{jkl}{}^i - \Gamma_{jh}{}^i t_{kl}{}^h. \tag{3. 6}$$

From (1. 1) and (3. 5) we have

$$T_{jkl}{}^n - \frac{1}{\lambda^r} F_j{}^m T_{mkl}{}^i \overset{r-1}{F_i}{}^n = \frac{1}{\lambda^r} t_{kl}{}^m (\nabla_m F_j{}^i) \overset{r-1}{F}{}_i{}^m. \tag{3. 7}$$

Since from $\nabla_{kl} F_j{}^i = 0$ it follows that

$$\nabla_{kl} \overset{s}{F_j}{}^i = 0 \qquad (s = 1, \cdots, r-1), \tag{3. 8}$$

so we have the following:

$$T_{jkl}{}^n - \frac{1}{\lambda^r} \overset{s}{F_j}{}^m T_{mkl}{}^i \overset{r-s}{F_i}{}^n = \frac{1}{\lambda^r} t_{kl}{}^m (\nabla_m \overset{s}{F_j}{}^i) \overset{r-s}{F_i}{}^n, \qquad (s = 1, \cdots, r-1). \tag{3. 9}$$

---

2) This method is quite the same as that of determination of all π-connections of an r-π-structure which was reported (unpublished) by the present author in a seminar conducted by Prof. S. Sasaki at Tôhoku University, Sendai, Japan in the spring of 1960.

Adding these relations, we have by virtue of (1. 7)

$$\underset{2}{\Phi}{}^{nb}_{aj} T_{bkl}{}^{a} = t_{kl}{}^{m} P_{mj}{}^{n}, \tag{3.10}$$

where

$$P_{mj}{}^{n} = \frac{1}{r}\frac{1}{\lambda^{r}} \sum_{t=1}^{r-1} (\nabla_m \overset{t}{F}_j{}^{i}) \overset{r-t}{F}_i{}^{n}. \tag{3.11}$$

Thus (3.10) is a necessary condition to be satisfied by $T_{jkl}{}^{i}$ so that the Walker differentiation with coefficients $\Gamma_{jkl}{}^{i}$ to satisfy (3. 4).

First of all, we shall solve the system of equations (3.10) for unknown tensor $T_{jkl}{}^{i}$.

It is evident that

$$\underset{1}{\Phi}{}^{hb}_{ai}(t_{kl}{}^{m} P_{mb}{}^{a}) = t_{kl}{}^{m} \underset{1}{\Phi}{}^{hb}_{ai} P_{mb}{}^{a}. \tag{3.12}$$

However we show that

$$\underset{1}{\Phi}{}^{hb}_{ai} P_{mb}{}^{a} = 0. \tag{3.13}$$

For, each term of $P_{mb}{}^{a}$ is written as

$$(\nabla_j \overset{t}{F}_i{}^{h_t}) \overset{r-t}{F}_{h_t}{}^{h} = \sum_{s=0}^{r-1} \overset{s}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r-s-1}{F}_{h_{s+1}}{}^{h} \tag{3.14}$$

and we have

$$\begin{aligned} &\underset{1}{\Phi}[\overset{s}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r-s-1}{F}_{h_{s+1}}{}^{h} \\ &\quad = \frac{1}{r}\left\{ \overset{r}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r-s-1}{F}_{h_{s+1}}{}^{h} + \frac{1}{\lambda^{r}} \sum_{t=1}^{r-1} \overset{r+s-1}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) F_{h_{s+1}}{}^{h_s} \right\}. \end{aligned} \tag{3.15}$$

However,

$$\begin{aligned} &\frac{1}{\lambda^{r}} \sum_{t=1}^{r-1} \overset{r+s-t}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r+t-s-1}{F}_{h_{s+1}}{}^{h} \\ &\quad = \sum_{t=1}^{s} \overset{s-t}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r+t-s-1}{F}_{h_{s+1}}{}^{h} + \sum_{t=s+1}^{r-1} \overset{r+s-t}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{t-s-1}{F}_{h_{s+1}}{}^{h} \\ &\quad = \sum_{u=0}^{s-1} \overset{u}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r-u-1}{F}_{h_{s+1}}{}^{h_s} + \sum_{v=s+1}^{r-1} \overset{v}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{t-s-1}{F}_{h_{s+1}}{}^{h}, \end{aligned} \tag{3.16}$$

so we have

$$\underset{1}{\Phi}[\overset{s}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r-s-1}{F}_{h_{s+1}}{}^{h}] = \frac{1}{r} \sum_{s=0}^{r-1} \overset{s}{F}_i{}^{h_s} (\nabla_j F_{h_s}{}^{h_{s+1}}) \overset{r-s-1}{F}_{h_{s+1}}{}^{h} = \frac{1}{r} \nabla_j \overset{r}{F}_i{}^{h} = 0. \tag{3.17}$$

Thus we have (3.13) and consequently from (3.12) and $(1.8)_1$

$$\underset{2}{\Phi}{}_{aj}^{hb}(t_{kl}{}^{m}P_{mb}{}^{a}) = t_{kl}{}^{m}P_{mj}{}^{h}, \tag{3.18}$$

that is $\overset{0}{T}{}_{jkl}{}^{h} = t_{kl}{}^{m}P_{mj}{}^{h}$ is a special solution of (3.10).

Let $T_{jkl}{}^{h}$ be any solution of (3.10) and put

$$W_{jkl}{}^{h} = T_{jkl}{}^{h} - t_{kl}{}^{m}P_{mj}{}^{h}, \tag{3.19}$$

then from (3.10) and (3.18) we have

$$\underset{2}{\Phi}W_{jkl}{}^{h} = 0 \quad \text{or} \quad \underset{1}{\Phi}W_{jkl}{}^{h} = W_{jkl}{}^{h}. \tag{3.20}$$

Conversely for any tensor $W_{jkl}{}^{h}$, $t_{kl}{}^{m}P_{mj}{}^{h} + \underset{1}{\Phi}W_{jkl}{}^{h}$ is a solution of (3.10). Thus *the most general solution of* (3.10) *are expressed as*

$$T_{jkl}{}^{i} = t_{kl}{}^{m}P_{mj}{}^{h} + \underset{1}{\Phi}{}_{aj}^{ib}\, W_{bkl}{}^{a}, \tag{3.21}$$

where $W_{jkl}{}^{h}$ is an arbitrary tensor.

In the next place, we shall show that the solutions (3.21) of (3.10) all satisfy (3. 7) which is equivalent to (3. 4). Since

$$t_{kl}{}^{p}P_{pj}{}^{n} - \frac{1}{\lambda^r}F_j{}^{m}(t_{kl}{}^{p}P_{pm}{}^{i})\overset{r-1}{F_i}{}^{n} = t_{kl}{}^{p}(P_{pj}{}^{n} - \frac{1}{\lambda^r}F_j{}^{n}P_{pm}{}^{i}\overset{r-1}{F_i}{}^{n}), \tag{3.22}$$

it needs only to show that

$$P_{pj}{}^{n} - \frac{1}{\lambda^r}F_j{}^{m}P_{pm}{}^{i}\overset{r-1}{F_i}{}^{n} = \frac{1}{\lambda^r}(\nabla_p F_j{}^{i})\overset{r-1}{F_i}{}^{n} \tag{3.23}$$

and

$$\frac{1}{\lambda^r}F_j{}^{m}(\underset{1}{\Phi}W_{mkl}{}^{i})\overset{r-1}{F_i}{}^{n} = \underset{1}{\Phi}W_{jkl}{}^{n}. \tag{3.24}$$

For the proof of (3.23), if we take account of

$$(\nabla_j \overset{s}{F_i}{}^{h_1})\overset{r-s}{F_{h_1}}{}^{h} = (\nabla_j F_i{}^{m_1})\overset{r-1}{F_{m_1}}{}^{h} + F_i{}^{m_1}(\nabla_j \overset{s-1}{F_{m_1}}{}^{h_1})\overset{r-s}{F_{h_1}}{}^{h}, \tag{3.25}$$

then by virtue of

$$(\nabla_j F_i{}^{m_1})\overset{r-1}{F_{m_1}}{}^{h} = -F_i{}^{m_1}(\nabla_j \overset{r-1}{F_{m_1}}{}^{h}), \tag{3.26}$$

$P_{ji}{}^{h}$ can be written as follows:

$$P_{ji}{}^{h} = \frac{1}{\lambda^r}\left\{(\nabla_j F_i{}^{h_1})\overset{r-1}{F_{h_1}}{}^{h} + \frac{1}{r}\sum_{s=1}^{r-1} F_i{}^{m_1}(\nabla_j \overset{s}{F_{m_1}}{}^{h_1})\overset{r-s-1}{F_{h_1}}{}^{h}\right\}. \tag{3.27}$$

Furthermore, from (3.11) we have

$$F_i{}^{i_1}P_{ji_1}{}^{h_1}\overset{r-1}{F_{h_1}}{}^{h} = \frac{1}{r}\sum_{s=1}^{r-1} F_i{}^{i_1}(\nabla_j \overset{s}{F_{i_1}}{}^{h_1})\overset{r-s-1}{F_{h_1}}{}^{h}. \tag{3.28}$$

Hence, from (3.27) and (3.28) we have (3.23).

For the proof of (3.24), we have by the definition of $\underset{1}{\Phi}$:

$$\frac{1}{\lambda^r}\underset{1}{F_i}{}^{i_1}(\underset{1}{\Phi}W_{i_1kl}{}^{h_1})\overset{r-1}{F}_{h_1}{}^{h}$$

$$=\frac{1}{r}\frac{1}{\lambda^r}\left\{F_i{}^{i_1}W_{i_1kl}{}^{h_1}\overset{r-1}{F}_{h_1}{}^{h}+\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{r+t-1}{F}_a{}^{h}W_{bkl}{}^{a}\overset{r-t+1}{F}_i{}^{b}\right\}$$

$$=\frac{1}{r}\frac{1}{\lambda^r}\left\{\overset{r-1}{F}_{h_1}{}^{h}W_{i_1kl}{}^{h_1}F_i{}^{i_1}+\sum_{s=0}^{r-2}\overset{s}{F}_a{}^{h}W_{bkl}{}^{a}\overset{r-s}{F}_i{}^{b}\right\}; \qquad s=t-1$$

$$=\frac{1}{r}\left\{W_{ikl}{}^{h}+\frac{1}{\lambda^r}\sum_{s=1}^{r-1}\overset{s}{F}_a{}^{h}W_{bkl}{}^{a}\overset{r-s}{F}_i{}^{b}\right\}=\underset{1}{\Phi}W_{ikl}{}^{h}.$$

Thus *the most general* $T_{jkl}{}^{i}$ *which gives rise to a Walker differentiation satisfying* (3. 4) *is given by* (3.21).

4. We consider now a special case with the following

$$W_{jkl}{}^{i}=-\underset{1}{\Phi}{}^{ib}_{aj}(\nabla_b t_{kl}{}^{a})+\frac{1}{r}\frac{1}{\lambda^r}\left\{\sum_{t=1}^{r-1}\overset{t}{F}_a{}^{i}t_{hk}{}^{a}(\nabla_l\overset{r-t}{F}_j{}^{h})-\sum_{t=1}^{r-1}\overset{t}{F}_a{}^{i}t_{hl}{}^{a}(\nabla_k\overset{r-t}{F}_j{}^{h})\right\}. \tag{4. 1}$$

For such $W_{jkl}{}^{i}$, it can be shown that

$$\underset{1}{\Phi}{}^{ib}_{aj}W_{bkl}{}^{a}=W_{jkl}{}^{i}. \tag{4. 2}$$

For the proof of (4. 2) if we take account of $(1.\ 8)_2$, it needs only to show that

$$\underset{1}{\Phi}{}^{ib}_{aj}\left\{\sum_{t=1}^{j-1}\overset{t}{F}_\alpha{}^{a}t_{\beta k}{}^{\alpha}(\nabla_l\overset{r-t}{F}_b{}^{\beta})\right\}=\sum_{t=1}^{r-1}\overset{t}{F}_\alpha{}^{i}t^{\alpha}_{\beta k}(\nabla_l\overset{r-t}{F}_b{}^{\beta}). \tag{4. 3}$$

Let the left hand side of (4. 3) be denoted by A, then we have

$$A=\sum_{t=1}^{r-1}F_\alpha{}^{i}t_{\beta k}{}^{\alpha}(\nabla_l\overset{r-t}{F}_j{}^{\beta})+\frac{1}{\lambda^r}\sum_{s=1}^{r-1}\sum_{t=1}^{r-1}\overset{s+t}{F}_\alpha{}^{i}t_{\beta k}{}^{\alpha}(\nabla_l\overset{r-t}{F}_b{}^{\beta})\overset{r-t}{F}_j{}^{b}. \tag{4. 4}$$

Let the second block of the right hand side of (4. 4) be denoted as B, then we have by virtue of $\overset{2r-t-s}{F}_j{}^{\beta}=\overset{r-t}{F}_b{}^{\beta}\overset{r-s}{F}_j{}^{b}$

$$B=-\frac{1}{\lambda^r}\sum_{s=1}^{r-1}\sum_{t=1}^{r-1}\overset{s+t}{F}_\alpha{}^{i}t_{\beta k}{}^{\alpha}\overset{r-t}{F}_b{}^{\beta}(\nabla_l\overset{r-s}{F}_j{}^{b})+\frac{1}{\lambda^r}\sum_{s=1}^{r-1}\sum_{t=1}^{r-1}\overset{s+t}{F}_\alpha{}^{i}t_{\beta k}{}^{\alpha}(\nabla_l\overset{2r-t-s}{F}_j{}^{\beta}). \tag{4. 5}$$

Now denote the first and second block of the right hand side of (4. 5) respectively as C and D. Then by making use of the fact that $t_{jk}{}^{i}$ is hybrid with respect to i and j, that is

$$\underset{1}{\Phi}{}^{ih}_{ab}t_{hl}{}^{a}=t_{bl}{}^{a}+\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}_a{}^{i}t_{hl}{}^{a}\overset{r-t}{F}_b{}^{h}=0, \tag{4. 6}$$

we have

$$(4.7)\qquad C = -\frac{1}{\lambda^r}\sum_{s=1}^{r-1} F_\gamma{}^i\left(\sum_{t=1}^{r-1}\overset{t}{F}{}^\gamma_\alpha t_{\beta k}{}^\alpha \overset{r-t}{F}{}_b{}^\beta\right)(\nabla_l \overset{r-s}{F}{}_j{}^b) = \sum_{s=1}^{r-1}\overset{s}{F}{}_\gamma{}^i t_{bk}{}^\gamma(\nabla_l \overset{r-s}{F}{}_j{}^b).$$

However, by virtue of (1. 2) and $\nabla_l \overset{r}{F}{}_j{}^\beta = 0$ we have

$$(4.8)\qquad D = \frac{1}{\lambda^r}\sum_{s=1}^{r-1}\sum_{u=s+1}^{r+s-1}\overset{u}{F}{}_\alpha{}^i t_{\beta k}{}^\alpha(\nabla_l \overset{2r-u}{F}{}_j{}^\beta) \qquad (u = s+t)$$

$$= \frac{1}{\lambda^r}\sum_{s=1}^{r-1}\left(\sum_{u=s+1}^{r-1} + \sum_{u=r+1}^{r+s-1}\right)\overset{u}{F}{}_\alpha{}^i t_{\beta k}{}^\alpha(\nabla_l \overset{2r-u}{F}{}_j{}^\beta)$$

$$= \sum_{s=1}^{r-1}\left\{\sum_{u=s+1}^{r-1}\overset{u}{F}{}_\alpha{}^i t_{\beta k}{}^\alpha(\nabla_l \overset{r-u}{F}{}_j{}^\beta) + \sum_{v=1}^{s-1}\overset{v}{F}{}_\alpha{}^i t_{\beta k}{}^\alpha(\nabla_l \overset{r-v}{F}{}_j{}^\beta)\right\}, (v = u-r).$$

Substitute (4. 7) and (4. 8) into (4. 5), we have

$$(4.9)\qquad B = \sum_{s=1}^{r-1}\sum_{u=1}^{r-1}\overset{u}{F}{}_\alpha{}^i t_{\beta k}{}^\alpha(\nabla_l \overset{r-u}{F}{}_j{}^\beta) = (r-1)\sum_{u=1}^{r-1}\overset{u}{F}{}_\alpha{}^i t_{\beta k}{}^\alpha(\nabla_l \overset{r-u}{F}{}_j{}^\beta).$$

Put (4. 9) into (4. 4) we get (4. 3).

Finally, under the assumption that the affine connection $\Gamma_{jk}{}^i$ is symmetric, we shall calculate the following $\Gamma_{jkl}{}^i$ obtained by putting (3.21) and (4. 1) into (3. 6):

$$(4.10)\qquad \Gamma_{jkl}{}^i = \Gamma_{jh}{}^i t_{kl}{}^h + P_{mj}{}^i t_{kl}{}^m$$

$$- \underset{1}{\Phi}{}^{ib}_{aj}\,(\nabla_b t_{kl}{}^a) + \frac{1}{r}\frac{1}{\lambda^r}\left\{\sum_{t=1}^{r-1}\overset{t}{F}{}_a{}^i t_{hk}{}^a(\nabla_l \overset{r-t}{F}{}_j{}^h) - \sum_{u=1}^{r-1}\overset{t}{F}{}_a{}^i t_{kl}{}^a(\nabla_k \overset{r-t}{F}{}_j{}^h)\right\}.$$

First of all, we have

$$(4.11)\qquad P_{mj}{}^i t_{kl}{}^m = \frac{1}{r}\frac{1}{\lambda^r}t_{kl}{}^m\sum_{t=1}^{r-1}\left(\frac{\partial \overset{t}{F}{}_j{}^a}{\partial x^m}\right)\overset{r-t}{F}{}_a{}^i$$

$$- \frac{r-1}{r}\Gamma_{jm}{}^i t_{kl}{}^m + \frac{1}{r}\frac{1}{\lambda^r}t_{hl}{}^m\sum_{t=1}^{r-1}\overset{t}{F}{}_j{}^h\Gamma_{hm}{}^a\overset{r-t}{F}{}_a{}^i,$$

and

$$(4.12)\qquad -\underset{1}{\Phi}{}^{ib}_{aj}(\nabla_b t_{kl}{}^a) = -\underset{1}{\Phi}{}^{ib}_{aj}\left(\frac{\partial t_{kl}{}^a}{\partial x^b}\right)$$

$$- \frac{1}{r}\Gamma_{hj}{}^i t_{kl}{}^h - \frac{1}{r}\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}{}_a{}^i\overset{r-t}{F}{}_j{}^b\Gamma_{mb}{}^a t_{kl}{}^m$$

$$+ \frac{1}{r}\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}{}_a{}^i\overset{r-t}{F}{}_j{}^b(\Gamma_{kb}{}^h t_{hl}{}^a) + \frac{1}{r}\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}{}_a{}^i\overset{r-t}{F}{}_j{}^b(\Gamma_{lb}{}^h t_{kh}{}^a)$$

$$+ \frac{1}{r}\Gamma_{kj}{}^h t_{hl}{}^i + \frac{1}{r}\Gamma_{lj}{}^h t_{kh}{}^i.$$

The last two terms are equal to the following by the fact that $t_{jk}{}^{i}$ is hybrid with respect to i and j:

$$-\frac{1}{r}\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}_a{}^i t_{hl}{}^a \overset{r-t}{F}_b{}^h \Gamma_{kj}{}^b - \frac{1}{r}\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}_a{}^i t_{kh}{}^a \overset{r-t}{F}_b{}^h \Gamma_{lj}{}^b. \tag{4.13}$$

Moreover, we have

$$\begin{aligned} &-\frac{1}{r}\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}_a{}^i t_{hl}{}^a \nabla_k \overset{r-t}{F}_j{}^h \\ &= -\frac{1}{r}\frac{1}{\lambda^r}\sum_{t=1}^{r-1}\overset{t}{F}_a{}^i t_{hl}{}^a \Big(\frac{\partial \overset{r-t}{F}_j{}^h}{\partial x^k} + \Gamma_{bk}{}^h \overset{r-t}{F}_j{}^b - \Gamma_{jk}{}^b \overset{r-t}{F}_b{}^h\Big). \end{aligned} \tag{4.14}$$

Thus from (4.10), (4.11), (4.12), (4.13) and (4.14) we have

$$\begin{aligned} \Gamma_{jkl}{}^i = \frac{1}{r}\frac{1}{\lambda^r} t_{kl}{}^m \sum_{t=1}^{r-1}\left(\frac{\partial \overset{t}{F}_j{}^a}{\partial x^m}\right)\overset{r-t}{F}_a{}^i - \underset{1}{\Phi}{}_{aj}^{ib}\left(\frac{\partial t_{kl}{}^a}{\partial x^b}\right) \\ -\frac{1}{r}\frac{1}{\lambda^r}\left\{\sum_{t=1}^{r-1}\overset{t}{F}_a{}^i t_{hl}{}^a \frac{\partial \overset{r-t}{F}_j{}^h}{\partial x^k} - \sum_{t=1}^{r-1}\overset{t}{F}_a{}^i t_{hk}{}^a \frac{\partial \overset{r-t}{F}_j{}^h}{\partial x^l}\right\}. \end{aligned} \tag{4.15}$$

Since $t_{jk}{}^i$ is constructed only from $\overset{s}{F}_j{}^i$ and their first order partial derivatives, it is seen that $\Gamma_{jkl}{}^i$ *is expressed by* $\overset{s}{F}_j{}^i$ *and their partial derivatives up to the second order.*

Therefore, *the tensor* $\nabla_{lm} t_{jk}{}^i$ *is constructed only from* $\overset{s}{F}_j{}^i$ $(s = 1,\cdots,r-1)$ *and their first and second order partial derivatives.*

5. As an example we consider now an almost contact manifold [4] (or a manifold with a (ϕ, ξ, η)-structure), that is one over which there exists a tensor field $\phi_j{}^i$, a contravariant vector field ξ^i and a covariant vector field η_j such that the following conditions are satisfied:

$$\text{rank } |\phi_j{}^i| = 2n, \tag{5. 1}$$

$$\phi_j{}^i \xi^j = 0, \qquad \phi_j{}^i \eta_i = 0, \tag{5. 2}$$

$$\xi^i \eta_i = 1, \tag{5. 3}$$

$$\phi_j{}^i \phi_k{}^j = -\delta_k^i + \xi^i \eta_k. \tag{5. 4}$$

It is known [5] that such manifold has a 3-π-structure whose fundamental tensor is given as follows:

$$\left\{\begin{aligned} \overset{1}{F}_j{}^i &= \frac{1}{2}(-\delta^i{}_j + 3\xi^i\eta_j - \sqrt{3}\,\phi_j{}^i) \\ \overset{2}{F}_j{}^i &= \frac{1}{2}(-\delta_j{}^i + 3\xi^i\eta_j + \sqrt{3}\,\phi_j{}^i) \\ \overset{3}{F}_j{}^i &= \delta_j{}^i. \end{aligned}\right. \tag{5. 5}$$

Therefore $\nabla_{kl}F_j^{\ i} = 0$ is equivalent to $\nabla_{kl}\phi_j^{\ i} = 0$.
In this case we have

$$(5.6)\qquad \begin{cases} \underset{1}{\Phi}{}_{aj}^{ib} = \dfrac{1}{2}(\delta_a^{\ i}\delta_j^{\ b} - \delta_a^{\ i}\xi^b\eta_j - \xi^i\eta_a\delta_j^{\ b} + 3\xi^i\eta_a\xi^b\eta_j - \phi_a^{\ i}\phi_j^{\ b}) \\ \underset{2}{\Phi}{}_{aj}^{ib} = \dfrac{1}{2}(\delta_a^{\ i}\delta_j^{\ b} + \delta_a^{\ i}\xi^b\eta_j + \xi^i\eta_a\delta_j^{\ b} - 3\xi^i\eta_a\xi^b\eta_j + \phi_a^{\ i}\phi_j^{\ b}) \end{cases}$$

and

$$(5.7)\qquad P_{ji}^{l} = \frac{1}{2}\{-(\nabla_j\phi_i^{\ h})\phi_h^{\ l} - (\nabla_j\xi^l)\eta_i + 2(\nabla_j\eta_i)\xi^l + 3\xi^l\eta_a(\nabla_j\xi^a)\eta_i\}.$$

The torsion tensor of the 3-π-structure is given by

$$(5.8)\qquad t_{jk}^{\ \ i} = \frac{1}{4}\{-N_{jk}^{\ \ i} - 3\xi^i(\partial_k\eta_j - \partial_j\eta_k) + 5(N_j\eta_k - N_k\eta_j) + \xi^i\phi_j^{\ p}\phi_k^{\ q}(\partial_p\eta_q - \partial_q\eta_p) - N_p^{\ i}(\phi_j^{\ p}\eta_k - \phi_k^{\ p}\eta_j)\},$$

where $\partial_j\eta_k = \dfrac{\partial\eta_k}{\partial x^j}$ and

$$(5.9)\qquad \begin{cases} N_{jk}^{\ \ i} = \phi_k^{\ q}(\partial_q\phi_j^{\ i} - \partial_j\phi_q^{\ i}) - \phi_j^{\ p}(\partial_p\phi_k^{\ i} - \partial_k\phi_p^{\ i}) - \eta_j\partial_k\xi^i + \eta_k\partial_j\xi^i, \\ N_j^{\ i} = \xi^q(\partial_q\phi_j^{\ i} - \partial_j\phi_q^{\ i}) - \phi_j^{\ q}\partial_q\xi^i, \\ N_j = \xi^p(\partial_p\eta_j - \partial_j\eta_p). \end{cases}$$

It is easily shown that

$$(5.10)\qquad \eta_a t_{hk}^{\ \ a}\xi^h = 0.$$

Then $W_{jkl}^{\ \ \ i}$ in (4.1) turns out to be the following by virtue of (5.10):

$$(5.11)\qquad W_{jkl}^{\ \ \ i} = -\underset{1}{\Phi}{}_{aj}^{ib}(\nabla_b t_{kl}^{\ \ a}) + \frac{1}{2}\phi_a^{\ i}\{(\nabla_k\phi_j^{\ \beta})t_{\beta l}^{\ \ \alpha} - (\nabla_l\phi_j^{\ \beta})t_{\beta k}^{\ \ \alpha}\} + \frac{1}{2}\eta_j\{t_{\beta l}^{\ \ i}(\nabla_k\xi^\beta) - t_{\beta k}^{\ \ i}(\nabla_l\xi^\beta)\} - \frac{3}{2}\xi^i\eta_j\eta_\alpha\{t_{\beta l}^{\ \ \alpha}(\nabla_k\xi^\beta) - t_{\beta k}^{\ \ \alpha}(\nabla_l\xi^\beta)\} + \frac{1}{2}\xi^\beta\{t_{\beta l}^{\ \ i}(\nabla_k\eta_j) - t_{\beta k}^{\ \ i}(\nabla_l\eta_j)\}.$$

Finally $\Gamma_{jkl}^{\ \ \ i}$ in (4.15) is written as follows in this case:

$$(5.12)\qquad \Gamma_{jkl}^{\ \ \ i} = -\underset{1}{\Phi}{}_{aj}^{ib}\frac{\partial t_{kl}^{\ \ a}}{\partial x^b} - \frac{1}{2}\phi_a^{\ i}\frac{\partial\phi_j^{\ a}}{\partial x^m}t_{kl}^{\ \ m} - \frac{1}{2}\eta_j\frac{\partial\xi^i}{\partial x^m}t_{kl}^{\ \ m} + \xi^i\frac{\partial\eta_j}{\partial x^m}t_{kl}^{\ \ m} + \frac{3}{2}\xi^i\eta_a\eta_j\frac{\partial\xi^a}{\partial x^m}t_{kl}^{\ \ m} + \frac{1}{2}\phi_a^{\ i}\frac{\partial\phi_j^{\ b}}{\partial x^k}t_{bl}^{\ \ a} - \frac{1}{2}\phi_a^{\ i}\frac{\partial\phi_j^{\ b}}{\partial x^l}t_{bk}^{\ \ a} + \frac{1}{2}\eta_j t_{hl}^{\ \ i}\frac{\partial\xi^h}{\partial x^k} - \frac{1}{2}\eta_j t_{hk}^{\ \ i}\frac{\partial\xi^h}{\partial x^l} - \frac{3}{2}\xi^i\eta_j\eta_a t_{hl}^{\ \ a}\frac{\partial\xi^h}{\partial x^k}$$

$$+\frac{3}{2}\xi^i\eta_j\eta_a t_{hk}{}^a\frac{\partial\xi^h}{\partial x^l}+\frac{1}{2}\xi^b t_{bl}{}^i\frac{\partial\eta_j}{\partial x^k}-\frac{1}{2}\xi^b t_{bk}{}^i\frac{\partial\eta_j}{\partial x^l}.$$

In concluding, I express my hearty thanks to Prof. S.Sasaki for his encouragement.

REFERENCES

[1] A. G. WALKER, Derivation torsionnelle et séconde torsion pour une structure complexe, C. R. Acad. Sci. Paris., 245(1957), 1213–1215.
[2] K. YANO, On Walker differentiation in almost product or almost complex spaces, Indagations Mathematicae, 20(1958), 573–580.
[3] C. J. HSU, On some properties of π-structures on differentiable manifold, Tôhoku Math. Journ., 12(1960), 429–454.
[4] S. SASAKI, On differentiable manifolds with certain structures which are closely related to almost contact structure I, Tôhoku Math. Journ., 12(1960), 456–476.
[5] C. J. HSU, Note on (ϕ, ξ, η)-structures, Tôhoku Math. Journ., 13(1961), 434–442.

NATIONAL TAIWAN UNIVERSITY.