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ON PROPERTY P OF VON NEUMANN ALGEBRAS

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Many results are known for the unitary invariants of von Neumann algebras, but, unfortunately, very little is known for algebraic one, especially for continuous von Neumann algebras, even in the case of factors. Recently, J.Schwartz has shown the existence of the new factor of type II_1 ([1]), by introducing a new property, called property P. The definition of property P is not given in the algebraic form, however, it is very suggestive that the isomorphic images of known factors with property P have also property P. The purpose of this paper is to show that property P is actually an algebraic invariant, i.e. invariant under isomorphism.

1. We shall start with our key lemma in rather general form. Let E be a Banach space which is the conjugate space of a Banach space and G a uniformly bounded group of continuous linear transformations on E with respect to the weak* topology. Let E_0 be the subset of E whose elements are G-invariant, i.e. $E_0 = \{x \in E : sx = x \text{ for all } s \in G\}$. By $\mathfrak{C}$, we shall denote the set of all non-negative real valued function f on G such that $f(s)=0$ for $s \in G$ with finite numbers of exceptions and $\sum_{s \in G} f(s) = 1$. $\widetilde{f}(x)$ will be defined as follows: $\widetilde{f}(x) = \sum_{s \in G} f(s)sx$. For arbitrary $x \in E$, K_x will denote the closure of the set of all $\widetilde{f}(x)$ for $f \in \mathfrak{C}$ with respect to the weak* topology. Then K_x is compact with respect to the weak* topology. Hereafter, we use the weak* topology in the argument on E without any proviso.

LEMMA 1.1. *If there exists a directed sequence $(f_\alpha)_{\alpha \in A} \subset \mathfrak{C}$ such that $\widetilde{f}_\alpha(x)$ converges to some $\tau(x) \in K_x$ for every $x \in E$, then, for arbitrary $h \in \mathfrak{C}$, there exists a directed sequence $(h_\alpha)_{\alpha \in A} \subset \mathfrak{C}$ such that $\widetilde{h}_\alpha(x)$ converges to $\widetilde{h}(\tau(x))$ for every $x \in E$.*

PROOF. We define the directed sequence $(h_\alpha)_{\alpha \in A} \subset \mathfrak{C}$ as follows:

$$h_\alpha(r) = \sum_{s \in G} h(s) f_\alpha(s^{-1}r) \quad \text{for} \quad r \in G.$$

By the continuity of every elment of G, $\sum_{r \in G} f_\alpha(s^{-1}r)rx$ converges to $s\tau(x) \in K_x$.

Therefore

$$\sum_{r\in G} h_\alpha(r)rx = \sum_{r,s\in G} h(s)f_\alpha(s^{-1}r)rx = \sum_{s\in G} h(s)\sum_{r\in G} f_\alpha(s^{-1}r)rx$$

coverges to $\sum_{s\in G} h(s)s\tau(x)$. q. e. d.

LEMMA 1.2. *If, for every $x\in E$, there exists a directed sequence $(f_\alpha)_{\alpha\in A}\subset\mathfrak{G}$ such that $\widetilde{f_\alpha}(x)$ converges to some element of E_0, then there exists a directed sequence $(g_\beta)_{\beta\in B}\subset\mathfrak{G}$ such that $\widetilde{g_\beta}(x)$ converges to some element of E_0 for every $x\in E$.*

PROOF. Let $\mathfrak{F}=\{\widetilde{f}=(\widetilde{f}(x))_{x\in E}:f\in\mathfrak{G}\}$, $\overline{\mathfrak{F}}$ the closure in $\prod_{x\in E} K_x$ and $\mathfrak{T}_x=(K_x \cap E_0)\times\left(\prod_{y\neq x} K_y\right)$.

(i) We shall show $\overline{\mathfrak{F}}\cap\mathfrak{T}_{x_0}\neq\emptyset$ for every $x_0\in E$. From the hypothesis, there exists a directed sequenc e$(f_\alpha)_{\alpha\in A}\subset\mathfrak{G}$ such that $\widetilde{f_\alpha}(x_0)$ converges to some element of E_0. Since $\prod_{x\in E} K_x$ is compact, $(\widetilde{f_\alpha})_{\alpha\in A}\subset\prod_{x\in E} K_x$, and $(\overline{\Phi}_\alpha:\Phi_\alpha=\{\widetilde{f_\beta}:\beta\geqq\alpha\})_{\alpha\in A}$ has the finite intersection property, we have $\bigcap_{\alpha\in A}\overline{\Phi}_\alpha\neq\emptyset$, where $\overline{\Phi}_\alpha$ is the closure of Φ_α in $\prod_{x\in E} K_x$. Therefore, there exists a convergent directed subsequence $(\widetilde{f}_{\alpha(\beta)})_{\beta\in B}(\alpha(\beta)\in A)$. Since $\widetilde{f}_{\alpha(\beta)}(x_0)$ converges to some element of E_0, and this implies $\overline{\mathfrak{F}}\cap\mathfrak{T}_{x_0}\neq\emptyset$.

(ii) Next, we shall show that the family $\{\overline{\mathfrak{F}}\cap\mathfrak{T}_x:x\in E\}$ has the finite intersection property. If it is the case, we get $\overline{\mathfrak{F}}\cap\left(\bigcap_{x\in E}\mathfrak{T}_x\right)\neq\emptyset$ by compactness of $\prod_{x\in E} K_x$, which is the conclusion of our lemma. We shall prove this by mathematical induction. Take arbitrary finite set $\{x_1, x_2, \cdots, x_n\}\subset E$ and assume that $\bigcap_{i<n}(\overline{\mathfrak{F}}\cap\mathfrak{T}_{x_i})\neq\emptyset$. For an element $\tau_1\in\bigcap_{i<n}(\overline{\mathfrak{F}}\cap\mathfrak{T}_{x_i})$, there exists a directed sequence $(f_\alpha)_{\alpha\in A}\subset\mathfrak{G}$, such that $\widetilde{f_\alpha}(\tau_1(x_n))$ converges to some element of E_0. Let $\tau_\alpha=(\widetilde{f_\alpha}(\tau_1(x)))_{x\in E}$, then $\tau_\alpha\in\overline{\mathfrak{F}}$ for all $\alpha\in A$, because of Lemma 1.1. By the same reason as in (i), we can assume that $(\tau_\alpha)_{\alpha\in A}$ is convergent. Let τ_α converges

to $\tau_2 \in \mathfrak{F}$, then $\tau_2(x_n) \in E_0$ and $\tau_2(x_i)=\tau_1(x_i)$ for $i<n$, i.e. $\tau_2 \in \bigcap_{i\leq n}(\mathfrak{F}\cap\mathfrak{T}_{x_i})$ Therefore the family $\{\mathfrak{F}\cap\mathfrak{T}_x : x \in E\}$ has the finite intersection property. q. e. d.

2. Now, we consider a von Neumann algebra $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. By $\mathfrak{L}(\mathfrak{H})$, we shall denote the algebra of all bounded linear operators on $\mathfrak{H}$.

DEFINITION 2.1. Let $\mathfrak{A}'$ be the commutant of $\mathfrak{A}$. It is said that $\mathfrak{A}$ has property P, when the weakly closed convex hull of the set $\{UTU^* : U \in \mathfrak{A}$ U unitary$\}$ has a non-void intersection with $\mathfrak{A}'$ for every $T \in \mathfrak{L}(\mathfrak{H})$.

By $K(T, \mathfrak{A})$, we shall denote the weakly closed convex hull of the set $\{UTU^* : U \in \mathfrak{A}\ \ U$ unitary$\}$.

PROPOSITION 2.2. *Property P is invariant under ampliation.*

PROOF. Let $\mathfrak{K}$ be a Hilbert space. Let $T=(T_{\iota\kappa})_{\iota,\kappa\in I}$ be a matrix representaton of $T \in \mathfrak{L}(\mathfrak{H})\otimes\mathfrak{L}(\mathfrak{K})$, where $T_{\iota,\kappa} \in \mathfrak{L}(\mathfrak{H})$. For any unitary operator $U \in \mathfrak{L}(\mathfrak{H})$, we define the transformation $\overline{U}$ on $\mathfrak{L}(\mathfrak{H})$ as follows:

$$\overline{U} : S \in \mathfrak{L}(\mathfrak{H}) \longrightarrow USU^* \in \mathfrak{L}(\mathfrak{H}).$$

Let $E=\mathfrak{L}(\mathfrak{H})$, $G\{\overline{U} : U \in \mathfrak{A}\ \ U$ unitary$\}$, then $E_0=\mathfrak{A}'$. We consider the $\mathfrak{E}$, as in section 1, about the group G. There exists a directed sequence $(f_\alpha)_{\alpha\in A} \subset \mathfrak{E}$, such that $f_\alpha(T_{\iota\kappa})$ converges weakly to some $T'_{\iota\kappa} \in \mathfrak{A}'$ for every $\iota, \kappa \in I$ from Lemma 1.2. We shall define the function F_α over the group $\{\overline{U\otimes I} : \overline{U} \in G\}$ such as

$F_\alpha(\overline{U\otimes I})=f_\alpha(\overline{U})$ for every f_α, and

$\widetilde{F}_\alpha(T)=\sum_{U\in G} F_\alpha(\overline{U\otimes I})\overline{U\otimes I}\,T$ for the F_α and $T \in \mathfrak{L}(\mathfrak{H})\otimes\mathfrak{L}(\mathfrak{K})$.

Then $\widetilde{F}_\alpha(T)=(\widetilde{f}_\alpha(T_{\iota\kappa}))_{\iota,\kappa\in I}$, $(\widetilde{F}_\alpha(T))_{\alpha\in A} \subset K(T, \mathfrak{A}\otimes I)$. By compactness of $K(T, \mathfrak{A}\otimes I)$ with respect to the weak topology, we can assume $(\widetilde{F}_\alpha(T))_{\alpha\in A}$ convergent by the same reason as in the proof of (i) in Lemma 1.2. Therefore $\widetilde{F}_\alpha(T)=(\widetilde{f}_\alpha(T_{\iota\kappa}))_{\iota,\kappa\in I}$ converges weakly to $(T'_{\iota\kappa})_{\iota,\kappa\in I} \in K(T, \mathfrak{A}\otimes I)\cap\mathfrak{A}'\otimes\mathfrak{L}(\mathfrak{K})$. q.e.d.

Using Proposition 2.2, we obtain the following

PROPOSITION 2.3. If $\mathfrak{A}$ *and* $\mathfrak{B}$ *have property* P, *then* $\mathfrak{A}\otimes\mathfrak{B}$ *has the same property.*

PROOF. Let $\mathfrak{A}$ (resp. $\mathfrak{B}$) act on $\mathfrak{H}$ (resp. $\mathfrak{K}$), $\mathfrak{A}\otimes I$ and $I\otimes\mathfrak{B}$ have property P

from Proposition 2.2. Hence, for an operator $T \in \mathfrak{L}(\mathfrak{H}) \otimes \mathfrak{L}(\mathfrak{K})$, $K(T, \mathfrak{A} \otimes I) \cap (\mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})) \neq \phi$, and for $T' \in K(T, \mathfrak{A} \otimes I) \cap (\mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K}))$,

$$K(T', I \otimes \mathfrak{B}) \cap (\mathfrak{L}(\mathfrak{H}') \otimes \mathfrak{B}) \neq \phi.$$

On the other hand, $I \otimes \mathfrak{B} \subset \mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})$ implies $K(T', I \otimes \mathfrak{B}) \cap \mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})$ because of $UT'U^* \in \mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})$ for any unitary operator $U \in I \otimes \mathfrak{B}$. Therefore, there exists $T_0 \in K(T, I \otimes \mathfrak{B}) \cap (\mathfrak{L}(\mathfrak{H}) \otimes \mathfrak{B}')$ such that $T_0 \in \mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})$. Since $K(T, I \otimes \mathfrak{B}) \subset K(T, \mathfrak{A} \otimes \mathfrak{B})$ and $(\mathfrak{A} \otimes \mathfrak{B})' = (\mathfrak{A}' \otimes \mathfrak{L}(\mathfrak{K})) \cap (\mathfrak{L}(\mathfrak{H}) \otimes \mathfrak{B}')$, $T_0 \in K(T, \mathfrak{A} \otimes \mathfrak{B}) \cap (\mathfrak{A} \otimes \mathfrak{B})'$. q.e.d.

DEFINITION 2.4. The algebra $\mathfrak{A}_E$ is called induction of the von Neumann algebra $\mathfrak{A}$, where E is the projection operator of $\mathfrak{A}'$.

For the case of the induction, the invariability of property P is obtained as follows.

PROPOSITION 2.5. *Property P is invariant under induction.*

PROOF. Let $\mathfrak{A}$ be a von Neumann algebra on $\mathfrak{H}$ with property P, and $\mathfrak{A}_E (E \in \mathfrak{A}')$ its induction. Then, for any $T \in \mathfrak{L}(\mathfrak{H})$, there exists a directed sequence $(f_\alpha)_{\alpha \in A} \subset \mathfrak{C}$ such that $\tilde{f}_\alpha(T)$ converges weakly to $T' \in \mathfrak{A}'$. We define the function F_α such that $F_\alpha(\overline{V}) = \sum f_\alpha(\overline{U})$, where V is a unitary operator of $\mathfrak{A}_E$ and $\overline{U}$ runs over the set $\{\overline{U} : U \in \mathfrak{A}\ U \text{ unitary, } V = (EUE)_E\}$. Then $\widetilde{F_\alpha}(T_E)$ converges weakly to some element of $(\mathfrak{A}')_E = (\mathfrak{A}_E)'$. q.e.d.

THEOREM 2.6. *Property P is invariant under isomorphism.*

PROOF. Combining Proposition 2.2, Proposition 2.5 and the structure theorem of ismorphism of von Neumann algebras [2: Chap. I, §4, 4 Thm.3], we get the theorem. q.e.d.

Still some questions remain open about property P. What is the algebraic version of property P ? Is there any difference between property P and hyperfiniteness of factors of type II_1? etc.

On the first question, we have not yet complete version. But we have made some steps with slite extended sense, which will appear later. Finally, I thank Professor M.Fukamiya and Dr. J.Tomiyama for several helpful conversations on the subject of this paper.

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