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# HEREDITARY PROPERTIES OF PRODUCT SPACES

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All spaces considered in this paper are Hausdorff spaces. Let $X$ be a normal space and $Y$ a metric space. Then K. Morita [4, Theorem 2.2] proved that the countable paracompactness of $X\times Y$ implies the normality of $X\times Y$. He proved also, in another paper [3, Theorem 5.4], that if $X$ is perfectly normal, then $X\times Y$ is perfectly normal. Inspired by these results and the method of proofs this note proves that if $X$ is hereditarily normal and every subset of $X\times Y$ is countably paracompact, then $X\times Y$ is hereditarily normal. Analogous statements for the case when $X$ is hereditarily paracompact or totally normal will be proved.

The following three facts will illustrate the circumstances of the present study:

(1) (An well-known example due to E.Michael) There exist a hereditarily paracompact space $X$ and a metric space $Y$ such that $X\times Y$ is not normal.

(2) Let $X$ be an ordered space consisting of all ordinals less than or equal to the first uncountable ordinal. Then $X$ is hereditarily normal. Let $Y$ be an infinite compact metric space. Then $X\times Y$ is not hereditarily normal but countably paracompact (and hence normal).

(3) (M.Katětov [5]) Let $Y$ be a metric space and $X\times Y$ be hereditarily normal. Then either $X$ is perfectly normal or $Y$ is discrete.

LEMMA 1 (F.Ishikawa [1]). *Let $X$ be a countably paracompact space and $G_1\subset G_2\subset \cdots$ an increasing sequence of open sets of $X$ whose sum is $X$. Then there exists a sequence $H_1, H_2,\cdots$ of open sets of $X$ such that $\overline{H}_i\subset G_i$ for each $i$ aud such that $\cup H_i=X$.*

A subset $C$ of a space $X$ is a *cozero*-set of $X$ if there exists a real-valued non-negative continuous function $f$ defined on $X$ such that $C=\{x: f(x)>0\}$. A *cozero covering* is a covering all of whose elements are cozero-sets.

LEMMA 2 (K.Morita [3, Theorem 1.2]). *A $\sigma$-locally finite cozero covering of an arbitrary space is normal.*

LEMMA 3 (K.Morita [3, Theorem 1.1]). *Let $X$ be a space and $\mathfrak{G}$ and $\mathfrak{H}$*

*be open coverings of $X$. If $\mathfrak{G}$ is normal and $\mathfrak{H}|G$, the restriction of $\mathfrak{H}$ to $G$, is normal for each element $G$ of $\mathfrak{G}$, then $\mathfrak{H}$ is normal.*

A *perfect* mapping $f: Y_0 \to Y$ is a closed continuous transformation such that $f^{-1}(y)$ is compact for each point $y \in Y$.

LEMMA 4 (K.Morita [2, Added in proof]). *If $Y$ is a metric space, then there exist a metric space $Y_0$ with $\dim Y_0 \leqq 0$ and a perfect mapping of $Y_0$ onto $Y$.*

LEMMA 5. *Let $X$ be a space, $S$ a cozero-set of $X$ and $T$ a cozero-set of $S$. Then $T$ is a cozero-set of $X$.*

PROOF. Let $f$ be a non-negative continuous function defined on $X$ such that

$$S = \{x : f(x) > 0\}.$$

Let $g$ be a non-negative continuous function defined on $S$ such that

$$T = \{x : g(x) > 0\},$$

$$g(x) \leqq 1, \ x \in S.$$

Let $h$ be a function defined on $X$ as follows:

$$h(x) = f(x) \cdot g(x), \ x \in S,$$

$$h(x) = 0, \ x \in X - S.$$

Then as can easily be seen $h$ is continuous and

$$T = \{x : h(x) > 0\}.$$

Thus $T$ is cozero in $X$.

THEOREM 1. *Let $X$ be a hereditarily normal space, $Y$ a metric space and $G$ an open subset of $X \times Y$. If $G$ is countably paracompact, then $G$ is a normal space.*

PROOF. i) First consider the case when $\dim Y \leqq 0$. Then by Katětov-Morita's theorem $Y$ is embedded into a product of a countable number of discrete spaces. Hence there exists a sequence,

$$\mathfrak{W}_i = \{W(\alpha_1 \cdots \alpha_i)\colon \alpha_i, \cdots, \alpha_i \in \Omega\},\ i = 1, 2, \cdots,$$

of discrete open coverings of $Y$ such that a) for any finite sequence $\alpha_1, \cdots, \alpha_i$

$$W(\alpha_1 \cdots \alpha_i) = \cup \{W(\alpha_1 \cdots \alpha_i \alpha_{i+1}) : \alpha_{i+1} \in \Omega\},$$

b) $\bigcup_{i=1}^{\infty} \mathfrak{W}_i$ is a basis of $Y$. Let $\mathfrak{U} = \{U_1, \cdots, U_n\}$ be an arbitrary finite open covering of $G$. Let us prove that $\mathfrak{U}$ is a normal covering of $G$, which will imply the normality of $G$.

Let $U_j(\alpha_1 \cdots \alpha_i)$ be the maximal open set of $X$ with

$$U_j(\alpha_1 \cdots \alpha_i) \times W(\alpha_1 \cdots \alpha_i) \subset U_j.$$

Set

$$G(\alpha_1 \cdots \alpha_i) = \cup \{U_j(\alpha_1 \cdots \alpha_i)\colon j = 1, \cdots, n\},$$

$$\mathfrak{G}_i = \{G(\alpha_1 \cdots \alpha_i) \times W(\alpha_1 \cdots \alpha_i) : \alpha_1, \cdots \alpha_i \in \Omega\},$$

$$G_i = \cup \{E : E \in \mathfrak{G}_i\},$$

$$\mathfrak{G} = \cup \{\mathfrak{G}_i : i = 1, 2, \cdots\}.$$

Then every $\mathfrak{G}_i$ is a discrete collection of open sets of $X \times Y$ and $\mathfrak{G}$ is an open covering of $G$. Hence $\cup G_i = G$. Since

$$U_j(\alpha_1 \cdots \alpha_i) \subset U_j(\alpha_1 \cdots \alpha_i \alpha_{i+1}),$$

then

$$G(\alpha_1 \cdots \alpha_i) \subset G(\alpha_1 \cdots \alpha_i \alpha_{i+1}).$$

Hence

$$G_1 \subset G_2 \subset \cdots.$$

Since $G$ is by assumption countably paracompact, there exists, by Lemma 1, a sequence $H_1$, $H_2, \cdots$ of open sets of $X \times Y$ such that

$$\widetilde{H}_i = \overline{H}_i \cap G \subset G_i$$

for each $i$ and such that $\cup H_i = G$.

For each pair $i \leqq j$ let $H_i(\alpha_1 \cdots \alpha_j)$ be the maximal open set of $G$ such that

$$H_i(\alpha_1 \cdots \alpha_j) \times W(\alpha_1 \cdots \alpha_j) \subset H_i.$$

Then

$$H_i(\alpha_1 \cdots \alpha_j) \subset G(\alpha_1 \cdots \alpha_i).$$

To see that

$$\{H_i(\alpha_1 \cdots \alpha_j) \times W(\alpha_1 \cdots \alpha_j) : i \leqq j\}$$

covers $G$ let $z = (x, y)$ be an arbitrary point of $G$. Then $z \in H_i$ for some $i$. Let $D$ be an open neighborhood of $x$ and $W(\alpha_1 \cdots \alpha_j)$ be an open neighborhood of $y$ with $i \leqq j$ and with

$$D \times W(\alpha_1 \cdots \alpha_j) \subset H_i.$$

Since

$$D \subset H_i(\alpha_1 \cdots \alpha_j),$$

$z$ is contained in $H_i(\alpha_1 \cdots \alpha_j) \times W(\alpha_1 \cdots \alpha_j)$. Set

$$B = \overline{H_i(\alpha_1 \cdots \alpha_j)} \cap (\overline{G(\alpha_1 \cdots \alpha_i)} - G(\alpha_1 \cdots \alpha_i)).$$

To prove

$$(B \times W(\alpha_1 \cdots \alpha_j)) \cap G = \emptyset$$

assume the contrary. Pick a point $z$ from the left side. Then

$$z \in \overline{H}_i \cap G = \tilde{H}_i \subset G_i.$$

Therefore

$$z \in G(\alpha_1 \cdots \alpha_j) \times W(\alpha_1 \cdots \alpha_j),$$

which would imply

$$z \notin B \times W(\alpha_1 \cdots \alpha_j),$$

a contradiction.

Let $C_i(\alpha_1 \cdots \alpha_j)$ be a cozero-set of $X-B$ with

$$\overline{H_i(\alpha_1 \cdots \alpha_j)}-B \subset C_i(\alpha_1 \cdots \alpha_j) \subset G(\alpha_1 \cdots \alpha_i).$$

The existence of such a set is assured by the hereditary normality of $X$. Then

$$C_i(\alpha_1 \cdots \alpha_j) \times W(\alpha_1 \cdots \alpha_j) \ (=S)$$

is a cozero-set of $(X-B) \times W(\alpha_1 \cdots \alpha_j)$. Since $X \times W(\alpha_1 \cdots \alpha_j)$ is an open and closed set of $X \times Y$, $S$ is a cozero-set of $X \times Y - B \times W(\alpha_1 \cdots \alpha_j)$. Since $G \subset X \times Y - B \times W(\alpha_1 \cdots \alpha_j)$, $S$ is a cozero-set of $G$. Thus we have a $\sigma$-discrete cozero covering

$$\mathfrak{H} = \{C_i(\alpha_1 \cdots \alpha_j) \times W(\alpha_1 \cdots \alpha_j) \colon i \leqq j\}$$

of $G$. Therefore by Lemma 2 $\mathfrak{H}$ is normal. Set

$$\mathfrak{U}(\alpha_1 \cdots \alpha_i) = \{U_k(\alpha_1 \cdots \alpha_i) \times W(\alpha_1 \cdots \alpha_i) \colon k=1, \cdots, n\}.$$

Then it refines $\mathfrak{U}$ and covers $C_i(\alpha_1 \cdots \alpha_j) \times W(\alpha_1 \cdots \alpha_j)$ for each $j$ with $j \geqq i$. Since $X$ is hereditarily normal,

$$\{U_k(\alpha_1 \cdots \alpha_i) \colon k=1, \cdots, n\}$$

is normal. Hence $\mathfrak{U}(\alpha_1 \cdots \alpha_i)$ is normal. Thus we can conclude that the restriction of $\mathfrak{U}$ to each element of $\mathfrak{H}$ is normal. Therefore by Lemma 3 $\mathfrak{U}$ itself is normal. Hence $G$ is a normal space.

ii) When $Y$ is a general metric space, there exist by Lemma 4 a metric space $Y_0$ with $\dim Y_0 \leqq 0$ and a perfect mapping $f$ of $Y_0$ onto $Y$. Let $g$ be the identity mapping of $X$ onto $X$ and set

$$h = g \times f.$$

Then $h$ is a perfect mapping of $X \times Y_0$ onto $X \times Y$. Hence

$$h \mid h^{-1}(G) \colon h^{-1}(G) \to G$$

is also perfect. Thus $h^{-1}(G)$ is as can easily be seen countably paracompact by the countable paracompactness of $G$. By the first step we have already known that $h^{-1}(G)$ is normal. Then $G$ is normal as a closed continuous image of a normal space $h^{-1}(G)$. Now the theorem is completely proved.

The following is a direct corollary of this theorem.

THEOREM 2. *Let $X$ be a hereditarily normal space and $Y$ a metric space. If every subset of $X \times Y$ is countably paracompact, then $X \times Y$ is hereditarily normal.*

THEOREM 3. *Let $X$ be a hereditarily paracompact space, $Y$ a metric space and $G$ an open set of $X \times Y$. If $G$ is countably paracompact, then $G$ is paracompact.*

PROOF. Let $\mathfrak{U} = \{U_\lambda : \lambda \in \Lambda\}$ be an arbitrary open covering of $G$. Let $\mathfrak{W} = \{W_\alpha : \alpha \in A\}$ be a $\sigma$-discrete basis of $Y$. Let $G_{\alpha\lambda}$ be the maximal open set of $X$ such that

$$G_{\alpha\lambda} \times W_\alpha \subset U_\lambda .$$

Set

$$G_\alpha = \cup \{G_{\alpha\lambda} : \lambda \in \Lambda\}.$$

Then

$$\mathfrak{G} = \{G_\alpha \times W_\alpha : \alpha \in \Lambda\}$$

is a $\sigma$-discrete open covering of G. Since $G$ is a countably paracompact normal space by Theorem 1, $\mathfrak{G}$ is normal. Since $\mathfrak{U} | G_\alpha \times W_\alpha$ is refined by $\{G_{\alpha\lambda} \times W_\alpha : \lambda \in \Lambda\}$ and the latter is normal by the hereditary paracompactness of $X$, $\mathfrak{U} | G_\alpha \times W_\alpha$ is normal. Hence by Lemma 3 $\mathfrak{U}$ is normal and the proof is completed.

The following is a direct consequence of this theorem.

THEOREM 4. *Let $X$ be a hereditarily paracompact space and $Y$ a metric space. If every subset of $X \times Y$ is countably paracompact, then $X \times Y$ is hereditarily paracompact.*

THEOREM 5. *Let $X$ be a totally normal space and $Y$ a metric space. If every subset of $X \times Y$ is countably paracompact, then $X \times Y$ is totally normal.*

PROOF. Let $G$ be an arbitrary open set of $X\times Y$. Let

$$\mathfrak{W}=\{W_\alpha:\ \alpha\in\bigcup_{i=1}^{\infty}A_i\}$$

be a $\sigma$-discrete basis of $Y$ such that each $\{W_\alpha:\ \alpha\in A_i\}$ is discrete. Let $G_\alpha$ be the maximal open set of X such that

$$G_\alpha\times W_\alpha\subset G.$$

Then

$$\{G_\alpha\times W_\alpha:\ \alpha\in\cup A_i\}$$

is a $\sigma$-discrete open covering of $G$. By the total normality of $X$ every $G_\alpha$ admits an open covering

$$\mathfrak{G}_\alpha=\{G_{\alpha\lambda}:\ \lambda\in\Lambda_\alpha\}$$

such that $\mathfrak{G}_\alpha$ is locally finite in $G_\alpha$ and every $G_{\alpha\lambda}$ is a cozero-set in $X$.

Since $G$ is a countably paracompact normal space by Theorem 1, there exists, for every $\alpha\in\cup A_i$, a set $C_\alpha$ such that

a) $\overline{C}_\alpha\cap G\subset G_\alpha\times W_\alpha$,

b) $C_\alpha$ is cozero in $G$,

c) $\{C_\alpha:\ \alpha\in\cup A_i\}$ is locally finite in $G$ and covers $G$.

Then it is easy to see that

$$\mathfrak{H}=\{C_\alpha\cap(G_{\alpha\lambda}\times W_\alpha):\ \lambda\in\Lambda_\alpha,\ \alpha\in\cup A_i\}$$

is a locally finite open covering of $G$. Since $G_{\alpha\lambda}\times W_\alpha$ is cozero in $X\times Y$, $C_\alpha\cap(G_{\alpha\lambda}\times W_\alpha)$ is cozero in $X\times Y$ by Lemma 5. Since the normality of $X\times Y$ is assured by Theorem 1, $X\times Y$ is totally normal and the proof is finished.

## REFERENCES

[1] F. ISHIKAWA, On countably paracompact spaces, Proc. Japan Acad., 31(1955), 686-689.

[2] K. MORITA, A condition for the metrizability of topological spaces and for $n$-dimensionality, Sci. Rep. Tokyo Kyoiku Daigaku sect. A, 5(1955), 33-36.

[3] K. MORITA, Products of normal spaces with metric spaces, Math. Annalen, 154 (1964), 365-382.

[4] K. MORITA, Note on the product of a normal space with a metric space, to appear.
[5] M. KATĚTOV, Complete normality of Cartesian products, Fund. Math., 35(1948), 271-274.

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