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LACUNARY TRIGONOMETRIC SUM AND PROBABILITY

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1. Introduction. In the present note let $\{\lambda_k\}$ denote a *lacunary* sequence of *positive* numbers, that is,

$$\lambda_{k+1} \geqq q\lambda_k,\ k = 1, 2, \cdots, \quad \text{where } q > 1\,. \tag{1.1}$$

Then the limiting distributions of lacunary trigonometric sums

$$T_n(x) = B_n^{-1}\sum_{k=1}^{n} a_k\cos(\lambda_k x + \alpha_k), \quad \text{where} B_n = \left(2^{-1}\sum_{k=1}^{n} a_k{}^2\right)^{1/2},$$

were considered by Salem and Zygmund [2] who proved that, over any fixed set of positive Lebesgue measure, $T_n(x)$ converges in law to the normal distribution with mean zero and variance 1, provided if $B_n \to +\infty$ and $a_n = o(B_n)$, as $n \to +\infty$.

In this note we discuss the problem whether the Salem-Zygmund result is valid or not, if we replace the *Lebesgue* measure by a *probability* measure on $(-\infty, \infty)$. If the probability measure is absolutely continuous with respect to the Lebesgue measure, then the Salem-Zygmund result implies that $T_n(x)$ is asymptotically normal with respect to this probability measure. But nothing like the Salem-Zygmund result is necessarily valid, even if the probability measure is continuous.

However, if we treat the series in (x, t)

$$\sum a_k\cos(\lambda_k xt + \alpha_k)\,,$$

we can obtain a similar result, under fairly general conditions. This problem was considered by Kaufman when $a_k = 1$, $k = 1, 2, \cdots$ [1].

In the following P is a probability measure on $(-\infty, \infty)$ satisfying

$$\begin{cases} P\{[x, x+h]\} \leqq Mh^\beta, & \text{for all } x \in (-\infty, \infty) \text{ and } h > 0, \\ \text{where } \beta,\ 0<\beta<1, \text{ and } M \text{ are positive constants.} \end{cases} \tag{1.2}$$

THEOREM. *Let us put*

$$S_n(x,t)=\sum_{k=1}^{n}a_k\cos(\lambda_k xt+\alpha_k)\equiv\sum_{k=1}^{n}A_k(x,t).$$

Then under the assumptions

$$(1.3)\qquad \begin{cases} B_n=\left(2^{-1}\sum_{k=1}^{n}a_k{}^2\right)^{1/2}\to+\infty, \\ a_n=o(B_n/\log\log B_n), \qquad \textit{as } n\to+\infty, \end{cases}$$

one has, for a. e. t in $(-\infty,\infty)$, *with respect to the Lebesgue measure,*

$$\lim_{n\to\infty}P\{x\,;\,S_n(x,t)\leqq yB_n\}=(2\pi)^{-1/2}\int_{-\infty}^{y}\exp\,(-u^2/2)\,du,\quad -\infty<y<+\infty.$$

2. Some Lemmas. Let $\hat{P}(\lambda)$ denote the Fourier-Stieltjes transform of the probability measure P, that is, $\hat{P}(\lambda)=\int_{-\infty}^{\infty}\exp\,(i\lambda x)P(dx)$. The next lemma is due to Kaufman.

LEMMA 1. *If* $\lambda\neq0$, *then we have*

$$\int_{v}^{v+1}|\hat{P}(\lambda t)|\,dt\leqq C|\lambda|^{-\beta/2},$$

where v *is any real number and* C *a constant independent of* v.

PROOF. We may assume that $|\lambda|>1$. We have

$$\int_{v}^{v+1}|\hat{P}(\lambda t)|^2dt=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{v}^{v+1}\exp\{i\lambda t(x_1-x_2)\}\,dt\,P(dx_1)\,P(dx_2)$$

$$\leqq\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\min\,(1,2|\lambda x_i-\lambda x_2|^{-1})\,P(dx_1)\,P(dx_2).$$

Let $r>0$ be the integer defined $2^{-r}<|\lambda|^{-1}\leqq2^{1-r}$; we sum the integrand over the sets:

$$\{|x_1-x_2|\geqq1\},\ \{1>|x_1-x_2|\geqq2^{-1}\},\ \cdots,\ \{2^{1-r}>|x_1-x_2|\geqq2^{-r}\}$$

and finally over the set $\{2^{-r}>|x_1-x_2|\}$. In each case the product measure can be estimated by (1.2) and the theorem of Fubini, summing up we obtain

$$\int_v^{v+1} |\hat{P}(\lambda t)|^2 \quad dt \leqq C^2 |\lambda|^{-\beta}$$

and this completes the proof of the lemma.

Next, let us put $p(m) = \max\{n; B_n \leqq \exp(m/\log m)\}$. Then by (1.3), we have

$$a^2_{p(m)+1} = o(B^2_{p(m)+1}/(\log m)^2), \quad \text{as} \quad m \to +\infty.$$

Further, since $B_n \sim B_{n+1}$, as $n \to +\infty$, we have

$$\exp\left\{\frac{2(m+1)}{\log(m+1)}\right\} - B^2_{p(m)} \geqq \exp\left\{\frac{2(m+1)}{\log(m+1)}\right\} - \exp\left\{\frac{2m}{\log m}\right\} \geqq \frac{1}{2\log(m+1)} \exp\left\{\frac{2m}{\log m}\right\}$$

$$\geqq \frac{1}{3\log m} B^2_{p(m)+1} > a^2_{p(m)+1}, \quad \text{for} \quad m \geqq m_0.$$

Therefore, we can find an integer m_0 such that if $m > m_0$, then

$$\exp\left\{\frac{m-1}{\log(m-1)}\right\} < B_{p(m)} \leqq \exp\left\{\frac{m}{\log m}\right\}. \tag{2.1}$$

In the following, we put

$$q(m) = \left[\frac{3}{\beta'} \log m\right], \quad \text{where} \quad \beta' = \beta \log q. \tag{2.2}$$

LEMMA 2. *For any real number v, there exists a set S_v in $[v, v+1)$ of Lebesgue measure zero such that if $t \notin S_v$, then*

$$\lim_{m\to\infty} B^{-2}_{p(m)} \sum_{k=1}^{p(m)} A_k^2(x, t) = 1, \quad \textit{in probability.}$$

PROOF. It is sufficient to show that for a.e. t in $[v, v+1)$[1],

$$\lim_{m\to\infty} B^{-2}_{p(m)} \sum_{k=1}^{p(m)} a_k^2 \cos 2(\lambda_k x t + \alpha_k) = 0, \quad \text{in probability.}$$

Since $B_n \uparrow +\infty$ with n, (1.3) implies that

$$\max_{1\leqq k\leqq n} |a_k| = o(B_n/\log\log B_n), \text{ as } n \to \infty. \tag{2.3}$$

1) The term a.e. t means that a.e. t with respect to the Lebesgue meaeure.

Since $p(m)>m-m_0$ for $m>m_0$,we have, by (2. 2) and (2. 3),

$$B_{p(m)}^{-2}\sum_{k=1}^{q(m)-1}a_k{}^2 = o(1) \quad\text{and}\quad B_{p(m)}^{-4}\sum_{k=1}^{p(m)}a_k{}^4 = o(1), \quad\text{as}\quad m\to\infty.$$

Further, by the fact that $\left|\int_{-\infty}^{\infty}\cos\,(\lambda tx+\alpha)P(dx)|\leqq|\hat{P}(\lambda t)\right|$, we have

$$\text{(2. 4)}\qquad \int_{-\infty}^{\infty}\left|\sum_{k=1}^{p(m)}a_k{}^2\cos 2(\lambda_k xt+\alpha_k)\right|^2P(dx)$$

$$= o(B_{p(m)}^4)+\sum_{k=q(m)+1}^{p(m)}a_k{}^2\sum_{n=q(m)}^{k-1}a_n{}^2[|\hat{P}\{2t(\lambda_k-\lambda_n)\}|+|\hat{P}\{2t(\lambda_k+\lambda_n)\}|]$$

uniformly in t, as $m\to+\infty$. On the other hand from (1. 1) and (2. 2), we obtain that if $k>n\geqq q(m)$, then

$$\lambda_k\pm\lambda_n>c_0q^{3\log m/\beta'}, \quad\text{for some}\quad c_0>0.$$

Therefore, we have, by Lemma 1 and (2. 2),

$$\sum_{m=m_0}^{\infty}B_{p(m)}^{-4}\int_v^{v+1}\sum_{k=q(m)+1}^{p(m)}a_k{}^2\sum_{n=q(m)}^{k-1}a_n{}^2[|\hat{P}\{2t(\lambda_k-\lambda_n)\}|+|\hat{P}\{2t(\lambda_k+\lambda_n)\}|]dt$$

$$=\sum_{m=m_0}^{\infty}O(e^{-\frac{3}{2}\log m}) = O(1).$$

By (2. 4) and the above relation we can prove the lemma.

3. Proof of the Theorem. For the proof of the theorem it is sufficient to show that for any fixed v, we have, for a. e. t in $[v, v+1)$,

$$\lim_{n\to\infty}P\{x\,;\,S_n(x, t)\leqq yB_n\} = (2\pi)^{-1/2}\int_{-\infty}^{y}\exp\,(-u^2/2)\,du.$$

To this end, we first prove that for a. e. t in $[v, v+1)$,

$$\text{(3. 1)}\qquad \lim_{m\to\infty}P\{x\,;\,S_{p(m)}(x, t)\leqq yB_{p(m)}\} = (2\pi)^{-1/2}\int_{-\infty}^{y}\exp(-u^2/2)du,$$

which is equivalent to the following relation

$$(3.1') \qquad \lim_{m\to\infty}\int_{-\infty}^{\infty}\exp\left\{\frac{i\lambda S_{p(m)}(x,t)}{B_{p(m)}}\right\}P(dx)=\exp(-\lambda^2/2), \quad \text{for all real } \lambda.$$

i. Let λ be any fixed real number and $t \notin S_v$, where S_v is the null set in Lemma 2. Since

$$e^z=(1+z)\exp\left(\frac{z^2}{2}+O(|z|^3)\right), \quad \text{as } |z|\to 0,$$

we have, by (2. 3),

$$\exp\left\{\frac{i\lambda S_{p(m)}(x,t)}{B_{p(m)}}\right\}$$

$$=\prod_{k=1}^{p(m)}\left\{1+\frac{i\lambda A_k(x,t)}{B_{p(m)}}\right\}\exp\left\{\frac{-\lambda^2}{2B^2_{p(m)}}\sum_{k=1}^{p(m)}A_k^2(x,t)+o(1)\right\},$$

uniformly in (x, t), as $m\to+\infty$. Putting

$$J_{n,m}(x,t,\lambda)=\prod_{k=1}^{n}\left\{1+\frac{i\lambda A_k(x,t)}{B_{p(m)}}\right\}, \quad n=1,2,\cdots,p(m),$$

then we have

$$(3.2) \qquad |J_{n,m}(x,t,\lambda)|\leqq\prod_{k=1}^{n}\left(1+\frac{\lambda^2a_k^2}{B^2_{p(m)}}\right)^{1/2}<\exp(\lambda^2).$$

Further, since $B^{-2}_{p(m)}\sum_{k=1}^{p(m)}A_k^2(x,t)\leqq 2$ and $t\notin S_v$, we have, by Lemma 2,

$$\int_{-\infty}^{\infty}\exp\left\{\frac{i\lambda S_{p(m)}(x,t)}{B_{p(m)}}\right\}P(dx)$$

$$=\exp(-\lambda^2/2)\int_{-\infty}^{\infty}J_{p(m),m}(x,t,\lambda)P(dx)+o(1), \quad \text{as } m\to+\infty.$$

From (2. 2), (2. 3) and (3. 2) it is seen that

$$J_{p(m),m}(x,t,\lambda)=1+\sum_{k=1}^{p(m)}\frac{i\lambda A_k(x,t)}{B_{p(m)}}J_{k-1,m}(x,t,\lambda)$$

$$=1+o(1)+\sum_{k=q(m)}^{p(m)}\frac{i\lambda A_k(x,t)}{B_{p(m)}}J_{k-1,m}(x,t,\lambda)$$

$$= 1 + o(1) + T_m(x, t, \lambda), \quad \text{uniformly in } (x,t), \text{ as } m \to +\infty,$$

where $T_m(x, t, \lambda) = \sum_{k=q(m)}^{p(m)} \frac{i\lambda A_k(x, t)}{B_{p(m)}} \left\{1 + \sum_{j=k-r}^{k-1} \frac{i\lambda A_j(x, t)}{B_{p(m)}}\right\} J_{k-r-1,m}(x, t, \lambda)$

and r is an integer satisfying

$$1 - \frac{1}{q} - \frac{1}{(q-1)q^r} > \frac{1}{2}\left(1 - \frac{1}{q}\right) = c_0 > 0\,. \tag{3.3}$$

Therefore, for the proof of (3. 1') it is sufficient to show that for a. e. t in $[v, v+1)$, we have

$$\lim_{m\to\infty} \int_{-\infty}^{\infty} T_m(x, t, \lambda) P(dx) = 0\,, \quad \text{for all } \lambda. \tag{3.4}$$

Since $\left|\int_{-\infty}^{\infty} \cos(\lambda x t + \alpha) P(dx)\right| \leqq |\hat{P}(\lambda t)|$, we have

$$\left|\int_{-\infty}^{\infty} T_m(x, t, \lambda) P(dx)\right| \leqq U_m(t, \lambda),$$

where

$$U_m(t, \lambda) = \sum_{k=q(m)}^{p(m)} \frac{|\lambda a_k|}{B_{p(m)}} \left[|\hat{P}(\lambda_k t)| + \sum_{j=k-r}^{k-1} \frac{|\lambda a_j|}{2B_{p(m)}} \{ |\hat{P}(\lambda_k t + \lambda_j t)| + |\hat{P}(\lambda_k t - \lambda_j t)| \} \right]$$

$$+ \sum_{k=q(m)}^{p(m)} \frac{|\lambda a_k|}{B_{p(m)}} \sum\nolimits_1 \frac{|a_{k_1} \cdots a_{k_s}| \, |\lambda|^s}{(2B_{p(m)})^s} \sum\nolimits_2 |\hat{P}\{(\lambda_k \pm \lambda_{k_1} \pm \cdots \pm \lambda_{k_s}) t\}|$$

$$+ \sum_{k=q(m)}^{p(m)} \frac{|\lambda a_k|}{B_{p(m)}} \sum_{j=k-r}^{k-1} \frac{|\lambda a_j|}{2B_{p(m)}} \sum\nolimits_1 \frac{|a_{k_1} \cdots a_{k_s}| \, |\lambda|^s}{(2B_{p(m)})^s} \sum\nolimits_3 |\hat{P}\{(\lambda_k \pm \lambda_j \pm \lambda_{k_1} \pm \cdots \pm \lambda_{k_s}) t\}|$$

where $\sum_1$ denotes the summation over all complexes $(k_1, \cdots, k_s)$ such that $1 \leqq k_1 < \cdots < k_s \leqq k-r-1$ and $1 \leqq s \leqq k-r-1$, $\sum_2$ and $\sum_3$ over all combinations of $+$ and $-$. Since for each t, $|U_m(t, \lambda)|$ is an increasing function of $|\lambda|$, the convergence of the series.

$$\sum_{m=m_0}^{\infty} \int_{v}^{v+1} |U_m(t, \lambda)| dt$$

for any fixed λ, implies the validity of (3. 4) for all λ. On the other hand from (1. 1) and (3. 3), we have, for $\sum_2$

$$\lambda_k \pm \lambda_{k_1} \pm \cdots \pm \lambda_{k_s} \geqq \lambda_k - \sum_{j=1}^{k-r-1} \lambda_j > \lambda_k \left(1 - \sum_{j=r+1}^{\infty} q^{-j}\right)$$

$$= \lambda_k \left(1 - \frac{1}{(q-1)q^r}\right) > c_0 \lambda_k > c' q^k, \quad \text{for some} \quad c' > 0,$$

and for $\sum_3$

$$\lambda_k \pm \lambda_j \pm \lambda_{k_1} \pm \cdots \pm \lambda_{k_s} \geqq \lambda_k - \lambda_{k-1} - \sum_{j=1}^{k-r-1} \lambda_j$$

$$\geqq \lambda_k \left(1 - \frac{1}{q} - \frac{1}{(q-1)q^r}\right) > c_0 \lambda_k > c' q^k, \quad \text{for some} \quad c' > 0$$

Therefore, we have, by Lemma 1 and (2. 2),

$$\int_v^{v+1} |U_m(t, \lambda)| dt$$

$$= O\left(\sum_{k=q(m)}^{p(m)} \frac{|\lambda a_k|}{B_{p(m)} q^{k\beta/2}} \left\{ 1 + \sum_{j=k-r}^{k-1} \frac{|\lambda a_j|}{B_{p(m)}} \right\} \prod_{j=1}^{k-r-1} \left(1 + \frac{|\lambda a_j|}{B_{p(m)}}\right) \right)$$

$$= O\left(\sum_{k=q(m)}^{p(m)} \frac{|\lambda a_k|}{B_{p(m)} q^{k\beta/2}} \exp\left\{ \frac{|\lambda|}{B_{p(m)}} \left(\sum_{j=1}^{k-1} a_j^2 \right)^{1/2} \sqrt{k} \right\} \right)$$

$$= O\left(\sum_{k=q(m)}^{p(m)} \frac{|\lambda a_k|}{B_{p(m)} q^{k\beta/2}} \exp(|\lambda| \sqrt{2k}) \right) = O(q^{-2q(m)\beta/5}), \quad \text{as} \quad m \to +\infty.$$

Thus, by (2. 2), we have

$$\sum_{m=m_0}^{\infty} \int_v^{v+1} |U_m(t, \lambda)| dt = \sum_{m=m_0}^{\infty} O(e^{-\frac{6}{5} \log m}) = O(1),$$

and this proves (3. 1).

ii. If $p(m) \leqq n < p(m+1)$, then we have

$$\left| \frac{S_{p(m)}(x, t)}{B_{p(m)}} - \frac{S_n(x, t)}{B_n} \right| \tag{3. 5}$$

$$\leqq \left| \frac{1}{B_{p(m)}} - \frac{1}{B_{p(m+1)}} \right| |S_{p(m)}(x, t)| + \frac{1}{B_{p(m)}} |S_n(x, t) - S_{p(m)}(x, t)|.$$

On the other hand from (2. 1), we obtain

$$\left|\frac{1}{B_{p(m)}}-\frac{1}{B_{p(m+1)}}\right|=O\left(\frac{B^2_{p(m+1)}-B^2_{p(m)}}{B^3_{p(m)}}\right)=o\left(\frac{1}{B_{p(m)}}\right),\quad \text{as}\quad m\to+\infty .$$

Hence, we have, for a. e. t in $[v,v+1)$,

$$\lim_{m\to\infty}\left|\frac{1}{B_{p(m)}}-\frac{1}{B_{p(m+1)}}\right|\,|S_{p(m)}(x,t)|=0,\quad \text{in probability.} \tag{3. 6}$$

Since $\left|\int_{-\infty}^{\infty}\cos(\lambda xt+\alpha)P(dx)\right|\leqq|\hat{P}(\lambda t)|$, we have

$$\max_{p(m)\leqq n<p(m+1)}\int_{-\infty}^{\infty}B^{-2}_{p(m)}\,|S_n(x,t)-S_{p(m)}(x,t)|^2P(dx)$$

$$\leqq\frac{2B^2_{p(m+1)}-2B^2_{p(m)}}{B^2_{p(m)}}+\sum_{k=p(m)+1}^{p(m+1)}\frac{|a_k|}{B_{p(m)}}\sum_{j=p(m)}^{k-1}\frac{|a_j|}{B_{p(m)}}\{|\hat{P}(\lambda_kt-\lambda_jt)|+|\hat{P}(\lambda_kt+\lambda_jt)|\}$$

$$=o(1)+\left[\sum_{k=p(m)+1}^{p(m+1)}\sum_{j=p(m)}^{k-1}\{|\hat{P}(\lambda_kt-\lambda_jt)|+|\hat{P}(\lambda_kt+\lambda_jt)|\}^2\right]^{1/2},$$

uniformly in t, as $m\to+\infty$. On the other hand from (1. 1), it is seen that if $k>j$, then

$$(\lambda_k\pm\lambda_j)\geqq(\sqrt{\lambda_k}-\sqrt{\lambda_j})(\sqrt{\lambda_k}+\sqrt{\lambda_j})>\sqrt{\lambda_k}\left(1-\frac{1}{\sqrt{q}}\right)(\lambda_k\lambda_j)^{1/4}>c_0q^{(k+j)/4},$$

for some constant $c_0>0$.

Further, since (2. 1) implies that if $m\geqq m_0$, then $p(m)>m-m_0$, we have, by the same argument as in the proof of Lemma 2,

$$\sum_{m=m_0}^{\infty}\int_v^{v+1}\sum_{k=p(m)+1}^{p(m+1)}\sum_{j=p(m)}^{k-1}\{|\hat{P}(\lambda_kt-\lambda_jt)|+|\hat{P}(\lambda_kt+\lambda_jt)|\}^2\,dt$$

$$=O\left(\sum_{m=m_0}^{\infty}\sum_{k=p(m)+1}^{p(m+1)}\sum_{j=p(m)}^{k-1}q^{-(k+j)\beta/8}\right)=O(1),$$

and this proves that for a. e. t in $[v,v+1)$,

$$\lim_{m\to\infty}\max_{p(m)\leqq n<p(m+1)}\int_{-\infty}^{\infty}\frac{1}{B^2_{p(m)}}\,|S_{p(m)}(x,t)-S_n(x,t)|^2P(dx)=0\,. \tag{3. 7}$$

From (3. 5), (3. 6) and (3. 7), we obtain that for a. e. t in $[v, v+1)$,

$$\lim_{n\to\infty}\left|\frac{S_{p(m,n)}(x,t)}{B_{p(m,n)}}-\frac{S_n(x,t)}{B_n}\right|=0\,,\quad\text{in probability,}\tag{3.8}$$

where $p(m, n)$ in (3. 8) is a $p(m)$ satisfying $p(m)\leqq n<p(m+1)$.

By (3. 1) and (3. 8) we can complete the proof of the theorem.

REFERENCES

[1] R. KAUFMAN, A problem of lacunary series, Acta Sci. Math. Szeged, XXIX(1968), 313-316.
[2] R. SALEM AND A. ZYGMUND, On lacunary trigonometric series, Proc. Nat. Acad. Sci. U. S. A., 34(1948), 54-62.

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