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# GENERALIZED PUPPE SEQUENCE AND SPANIER-WHITEHEAD DUALITY

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**Introduction.** In [3], I. M. James has discussed the following problem: Let $\xi_k (k=1,2,\cdots)$ be the orthonormal $k$-frame bundle associated with a vector bundle $\xi$ and let $p: \xi_k \to \xi_1$ be the projection. Then under what conditions does this fibration admit a cross-section?

For this purpose James has introduced his ex-homotopy theory. In the ex-homotopy theory the usual suspension $SX$ is extended to the suspension $\Sigma X$ which is an over-space. The assertion on the usual suspension homomorphism $S: \pi(X, Y) \to \pi(SX, SY)$ can be generalized to the corresponding one in the ex-homotopy theory.

James pointed out that $\Sigma$-theory, on the line of the Spanier-Whitehead $S$-theory, would be worth investigating and asked whether the duality of $S$-theory can be extended to such a $\Sigma$-theory.

In § 1 we shall review the outline of James's ex-homotopy theory and list the related definitions. In § 2 we shall define the mapping cone in ex-homotopy theory and try to generalize the Puppe sequence [4]. In § 3 we try to extend the duality of $S$-theory to $\Sigma$-theory.

**1. Preliminaries and notations.** In this section we summarize basic definitions in [3]. An over-space is a space $A$ with a map $\varphi: A \to B$, called the projection. (The base space $B$ is mainly fixed.) The notions of over-map and over-homotopy are usually defined. Let $A_i$ $(i=1,2)$ be over-spaces with projections $\varphi_i$. The direct product $A_1 \times A_2$ is the subspace of the topological product consisting of pairs $(a_1, a_2)$ such that $\varphi_1 a_1 = \varphi_2 a_2$, where $a_i \in A_i$. The projection $\varphi: A_1 \times A_2 \to B$ is given by $\varphi(a_1, a_2) = \varphi_i a_i$. The join of $A_1$ with $A_2$ is the over-space $A_1 * A_2$ defined as follows:

$$A_1 * A_2 = A_1 \cup A_2 \cup (A_1 \times A_2) \times I/(x, i) \sim p_i x \qquad i = 1, 2$$

where $A_1 \times A_2$ is the direct product in the above sense and $p_i: A_1 \times A_2 \to A_i$ is the projection.

The projection $\psi: A_1 * A_2 \to B$ is given by

$$\psi(x, t) = \varphi(x) \qquad x \in A_1 \times A_2.$$

The join of over-maps is defined as usual. The suspension of the over-space $A$ with projection $\varphi$ is the over-space $\Sigma A$ defined as follows:

$$\Sigma A = A \times I \cup B \times I/(a, i) \sim (\varphi a, i) \qquad a \in A,\ i \in I.$$

The projection $\psi: \Sigma A \to B$ is given by

$$\psi(a, t) = \varphi(a) \qquad \psi(b, i) = b \qquad b \in B,\ i \in I.$$

Taking $A = \phi$, the empty over-space, then $\Sigma\phi = B \times I$.

The $n$-fold suspension $\Sigma^n A$ can be identified $A * \Sigma^n\phi$, where $\Sigma^n\phi = B \times S^{n-1}$. In fact, define $\psi_1 : \Sigma A \to A * \Sigma\phi$ by

$$\begin{aligned} \psi_1 \langle a, t\rangle &= [a, (\varphi(a), 0), 1-2t] \qquad 0 \leqslant t \leqslant 1/2 \\ &= [a, (\varphi(a), 1), 2t-1] \qquad 1/2 \leqslant t \leqslant 1, \\ \psi_1 \langle b, i\rangle &= [b, i] \qquad b \in B,\ i = 0, 1. \end{aligned}$$

Also define $\chi_1 : A * \Sigma\phi \to \Sigma A$ by

$$\begin{aligned} \chi_1[a, (\varphi(a), 0), t] &= \langle a, 1 - t/2\rangle, \\ \chi_1[a, (\varphi(a), 1), t] &= \langle a, 1 + t/2\rangle, \\ \chi_1[b, i] = \langle b, i\rangle \qquad i &= 0, 1 \qquad \chi_1[a] = \langle a, 1/2\rangle. \end{aligned}$$

Then $\psi_1$ and $\chi_1$ are the inverse of each other. Inductively we can prove the assertion for $n > 1$.

Let $A$ be an over-space (of $B$) with projection $\varphi$. An ex-space of $A$ (over $B$) is a space $X$ together with maps $A \xrightarrow{\sigma} X \xrightarrow{\rho} B$ such that $\rho\sigma = \varphi$. Then $\sigma$ and $\rho$ are called the section and the projection of the ex-structure. Let $X_i$ $(i = 1, 2)$ be an ex-space of $A$, with section $\sigma_i$ and projection $\rho_i$. An ex-map $f: X_1 \to X_2$ is a map such that $f\sigma_1 = \sigma_2$, $\rho_2 f = \rho_1$. An ex-homotopy $f_t : X_1 \to X_2$ is an ordinary homotopy which is an ex-map for all values of $t$. The set of ex-homotpy classes of ex-maps is denoted by $\pi(X_1, X_2)$. An ex-homotopy equivalence is similary defined. The direct product and the join in the category of ex-spaces are obtained from those notions in the category of over-spaces by defining the sections appropriately.

An ex-space $K$ of $A$ is called an ex-complex if $K$ is a $CW$-complex with $A$ as a subcomplex and the inclusion as section.

**2. Generalized Puppe sequence.** Let $X, Y$ be ex-spaces of $A$ (over $B$) with sections $\sigma_0, \sigma_1$ and projections $\rho_0, \rho_1$ respectively, and let $f: X \to Y$ be an ex-map. We define the mapping cone of $f$, denoted by $C_f$, as follows:

$$C_f = B \cup (X \times I) \cup Y/(x, 0) \sim \rho_0(x),\ (x, 1) \sim f(x).$$

$$\begin{tikzcd}
& X \arrow[dd, "f"] \arrow[dr, "\rho_0"] & \\
A \arrow[ur, "\sigma_0"] \arrow[dr, "\sigma_1"'] & & B \\
& Y \arrow[ur, "\rho_1"'] &
\end{tikzcd}$$

Then $C_f$ is an ex-space of $A$ with section $\sigma$, projection $\rho$ given by

$$\sigma(a) = (\sigma_0 a, 1),\ \ \rho(b) = b,\ \ \rho(y) = \rho_1(y),\ \ \rho(x, t) = \rho_0(x).$$

Note that if $B$ is a point, then $C_f$ is an ordinary (unreduced) mapping cone. Here, different from [3], we shall regard the suspension $\Sigma X$ as an ex-space by giving the following section and projection, namely

$$\hat{\sigma}(a) = \langle \sigma_0 a, 1 \rangle,\ \ \psi\langle x, t \rangle = \rho_0(x),\ \ \psi\langle b, i \rangle = b,\quad i \in \dot{I}.$$

Thus, taking $Y = B$ and $f = \rho_0$ in $C_f$, then we have $C_f = \Sigma X$. Now define $\alpha_f: Y \to C_f$ by $\alpha_f(y) = y$ and let $i: B \to C_f$ be the inclusion.

PROPOSITION 2.1. *Let $X, Y$ be ex-complexes of $A$. Assume that $\rho: C_f \to B$ is a fibration. Then we have $\alpha_f \circ f \simeq i \circ \rho_0$ (ex-homotopic).*

PROOF. We define $m_s: X \times \dot{I} \cup A \times I \to C_f$ by

$$m_s(x, 0) = \rho_0 x,\quad m_s(x, 1) = \alpha_f(fx),\quad m_s(a, t) = (\sigma_0 a, s + t - st).$$

Also define $k_s: X \times I \to B$ by $k_s(x, t) = \rho_0 x$ and $h_0: X \times I \to C_f$ by $h_0(x, t) = (x, t)$. Then $\rho h_0 = k_0$ and $\rho m_s = k_s | X \times \dot{I} \cup A \times I$. Since $\rho$ is a fibration, there is a homotopy $h_s: X \times I \to C_f$ such that $\rho h_s = k_s$ and $h_s | X \times \dot{I} \cup A \times I = m_s$. If we now define $\eta_t: X \to C_f$ by $\eta_t(x) = h_1(x, t)$, then $\eta_t$ provides an ex-homotopy between $\alpha_f \circ f$ and $i \circ \rho_0$.

REMARK. If $B$ is a point, then the assertion in Proposition 2.1 reduces to $\alpha_f \circ f \simeq 0$.

PROPOSITION 2.2. *Let $Z$ be a ex-space of $A$ with a cross-section $s$ and let $g: Y\to Z$ be an ex-map. If $g\circ f\simeq s\circ\rho_0$ (ex-homotopic), then there is an ex-map $l: C_f\to Z$ such that $l\circ\alpha_f\simeq g$ (ex-homotopic)*

PROOF. Let $\varphi_t$ be an ex-homotopy between $g\circ f$ and $s\circ\rho_0$. We define

$$l': B\cup(X\times I)\cup Y\to Z$$

by

$$l'(b)=s(b),\quad l'(x,t)=\varphi_t(x),\quad l'(y)=g(y).$$

Then $l'$ induces an ex-map $l: C_f\to Z$ satisfing $l\circ\alpha_f\cong g$ (*ex-homotopic*).

REMARK. If $A=\phi$ and $B$ is a point, then $\pi(\ ,\ )$ reduces to the ordinary set of homotopy classes and $\pi(C_f, Z)\xrightarrow{\alpha_f^*}\pi(Y,Z)\xrightarrow{f^*}\pi(X,Z)$ is exact in the usual sense.

Let $X$ be an ex-space of $A$ with section $\sigma_0$ and projection $\rho_0$. Then the cone $CX$ is defined by

$$CX=B\cup(X\times I)/(x,0)\sim\rho_0(x).$$

Define $\rho_0^c: CX\to B$ by $\rho_0^c(b)=b$ and $\rho_0^c(x,t)=\rho_0(x)$ and $\sigma_0^c: A\to CX$ by $\sigma_0^c(a)=(\sigma_0 a, 1)$ (here $X$ is embedded in $CX$ as $x\to(x,1)$). Then $CX$ is an ex-space of $A$.

PROPOSITION 2.3. *Let $f: X\to Y$ be the injection where $X$ and $Y$ are over-spaces. Assume that $(Y,X)$ satisfies the over-homotopy extension property. Then we have an over-homotopy $\chi_s: C_f\to C_f$ with $\chi_0=1$ and $\chi_1|CX\cong\rho_0^c$.*

PROOF. We define $\varphi_s: CX\to C_f$ by $\varphi_s(x,t)=(x,(1-s)t)$ and $\varphi_s(b)=b$. Let $j: X\to CX$ be an embedding given by $j(x)=(x,1)$. Since $(Y,X)$ satisfies the over-homotopy extension property, there is an over-homotopy $\psi_s: Y\to C_f$ such that $\psi_s|X=\varphi_s\circ j$ and $\psi_0$ is the injection. Now define $\chi_s': CX\cup Y\to C_f$ by

$$\chi_s'(b)=b\quad b\in B,\quad \chi_s'(x,t)=\varphi_s(x,t),\quad (x,t)\in CX,\quad \chi_s'(y)=\psi_s(y),\quad y\in Y.$$

Then $\chi_s'$ induces an over-homotopy $\chi_s: C_f\to C_f$ satisfying the required properties.

PROPOSITION 2.4. *The inclusion $\alpha_f: Y\to C_f$ satisfies the ex-homotopy extension property.*

PROOF. Let $Z$ be any ex-space of $A$ (over $B$) and $h_0: C_f\to Z$ an ex-map. Let $g_s: Y\to Z$ be an ex-homotopy with $h_0\alpha_f=g_0$. According to Puppe [4], we define a homotopy $h_s: C_f \to Z$ by

$$h_s(b) = b \quad b \in B \qquad h_s(y) = g_s(y) \quad y \in Y$$

$$h_s(x,t) = \begin{cases} h_0(x, t+s/2) & t \geqslant 1/2 \quad s \leqslant 2-2t \\ g_{s+2t-2}(fx) & t \geqslant 1/2 \quad s \geqslant 2-2t \\ h_0(x, t+st) & t \leqslant 1/2. \end{cases}$$

Then it is easily checked that $h_s$ is a required ex-homotopy.

Let $X, Y$ be ex-spaces of $A$ (over $B$) and let $f: X \to Y$ be an ex-map. Now we consider the generalized Puppe sequence:

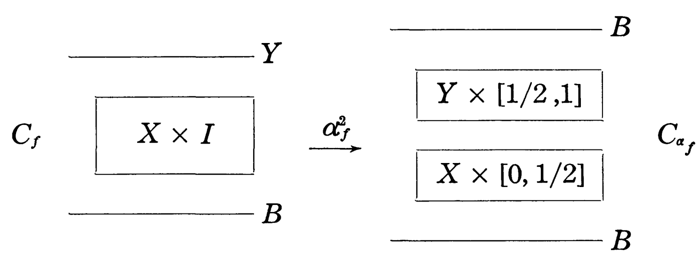
$$X \xrightarrow{f} Y \xrightarrow{\alpha_f} C_f \xrightarrow{\beta_f} \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

where $\beta_f$ is defined by the following conditions

$$\beta_f(b) = \langle b, 0\rangle \quad b \in B, \quad \beta_f(x,t) = \langle x, t\rangle \quad (x,t) \in CX,$$
$$\beta_f(y) = \langle \rho_1 y, 1\rangle \quad y \in Y.$$

We easily see that the mapping cone $C_{\alpha_f}$ may be considered as the quotient space of $CX \cup CY$ by the relations:

$$(x,0) \sim (\rho_0 x, 0), \quad (x, 1/2) \sim (fx, 1/2), \quad (y,1) \sim (\rho_1 y, 1)$$




Then $\alpha_f^2: C_f \to C_{\alpha_f}$ may be defined by $\alpha_f^2(x,t) = (x, t/2)$, $\alpha_f^2(y) = (y, 1/2)$, $\alpha_f^2(b) = (b, 0)$.

Define $R: C_{\alpha_f} \to \Sigma X$ by $R(b,i) = (b,i)$ $i = 0,1$, $R(x,t) = \langle x, 2t\rangle$ $0 \leqslant t \leqslant 1/2$, $R(y,t) = \langle \rho_1 y, 1\rangle$ $1/2 \leqslant t \leqslant 1$ and also define $S: \Sigma X \to C_{\alpha_f}$ by $S\langle b, i\rangle = (b,i)$ $i = 0,1$, $S\langle x,t\rangle = \begin{cases} (x,t) & 0 \leqslant t \leqslant 1/2 \\ (fx,t) & 1/2 \leqslant t \leqslant 1. \end{cases}$

If we define a homotopy $\psi_s : \Sigma X \to \Sigma X$ by

$$\psi_s<x, t> = \begin{cases} <x, 2t/2-s> & 0 \leqslant t \leqslant 1-s/2 \\ <\rho_0 x, 1> & 1-s/2 \leqslant t \leqslant 1, \end{cases}$$

$$\psi_s<b, i> = <b, i>,$$

then $\psi_s$ provides an ex-homotopy between $1_{\Sigma X}$ and $RS$. Also define a homotopy $\chi_s: C_{\alpha_f} \to C_{\alpha_f}$ by

$$\chi_s(x, t/2) = \begin{cases} (x, t/2-s) & 0 \leqslant t \leqslant 1-s/2 \\ (fx, s+2t-1/2) & 1-s/2 \leqslant t \leqslant 1 \end{cases}$$

$$\chi_s(y, 1+t/2) = (y, 1+s+t-st/2) \quad 0 \leqslant t \leqslant 1,$$

Then $\chi_s$ provides an ex-homotopy between $1_{C_{\alpha_f}}$ and $SR$. Thus $C_{\alpha_f}$ and $\Sigma X$ have the same ex-homotopy type.

A similar consideration as in $C_{\alpha_f}$ can be applied to $C_{\alpha_f^2}$ and $C_{\alpha_f^2}$ may be considered as the quotient space of $CY \cup C(C_f)$ by the relations: $(y, 0) \sim (\rho_1 y, 0)$ $(y, 1/2) \sim (\alpha_f(y), 1/2)$, $(u, 1) \sim (\rho u, 1)$ $u \in C_f$.
Define $R_1 : C_{\alpha_f^2} \to Y$ by

$$R_1(b, i) = (b, i) \quad i = 0, 1, \quad R_1(y, t/2) = (y, t) \quad 0 \leqslant t \leqslant 1,$$

$$R(u, 1+t/2) = (\rho u, 1) \quad 0 \leqslant t \leqslant 1 \quad u \in C_f,$$

and also define $S_1 : \Sigma Y \to C_{\alpha_f^2}$ by

$$S_1(b, i) = (b, i) \quad i = 0, 1,$$

$$S_1(y, t) = \begin{cases} (y, t) & 0 \leqslant t \leqslant 1/2 \\ (\alpha_f y, t) & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then we have $R_1 \circ S_1 \simeq 1_{\Sigma Y}$ (ex-homotopic) and $S_1 \circ R_1 \simeq 1_{C_{\alpha_f^2}}$ (ex-homotopic). On the other hand, it is easily checked that $R \circ \alpha_f^2 = \beta_f$ and $R_1 \circ \alpha_f^3 = \Sigma f \circ R$.

Summarizing the preceding statements, we get a following main theorem:

THEOREM 2.5. *Let $X, Y$ be ex-spaces of $A$ (over $B$) and let $f: X \to Y$ be an ex-map. Then we have the following commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "1"] & Y \arrow[r, "\alpha_f"] \arrow[d, "1"] & C_f \arrow[r, "\alpha_f^2"] \arrow[d, "1"] & C_{\alpha_f} \arrow[r, "\alpha_f^3"] \arrow[d, dashed, "R"] & C_{\alpha_f^2} \arrow[d, "R_1"] \\ X \arrow[r, "1"'] & Y \arrow[r, "\alpha_f"'] & C_f \arrow[r, "\beta_f"'] & \Sigma X \arrow[r, "\Sigma f"'] & \Sigma Y \end{tikzcd}$$

*in which $R$ and $R_1$ are ex-homotopy equivalences.*

CROLLARY 2.6. *Let $Z$ be an ex-space of $A$ (over $B$) with a cross-section $s$. Suppose that $\rho: \Sigma X \to B$ and $\rho: \Sigma Y \to B$ are fibrations. Then the similar statements as in Prop. 2.2 hold in the following sequences:*

$$\pi(\Sigma X, Z) \xrightarrow{\beta_f^*} \pi(C_f, Z) \xrightarrow{\alpha_f^*} \pi(Y, Z)$$

*and*

$$\pi(\Sigma Y, Z) \xrightarrow{(\Sigma f)^*} \pi(\Sigma X, Z) \xrightarrow{\beta_f^*} \pi(C_f, Z).$$

REMARK. If $A = \phi$ and $B$ is a point, then we have the usual Puppe exact sequence:

$$\pi(\Sigma Y, Z) \xrightarrow{(\Sigma f)^*} \pi(\Sigma X, Z) \xrightarrow{\beta_f^*} \pi(C_f, Z) \xrightarrow{\alpha_f^*} \pi(Y, Z) \xrightarrow{f^*} \pi(X, Z).$$

PROPOSITION 2.7. *Let $Z$ be an ex-space of $A$ (over $B$) with section $\sigma'$ and projection $\rho'$. Let $X, Y$ be ex-spaces of $A$ and $f: X \to Y$ an ex-map. If $g: Z \to Y$ is an ex-map such that $\alpha_f \cdot g \simeq i\rho'$ (ex-homotopic), then there exists an ex-map $h: \Sigma Z \to \Sigma X$ such that $\Sigma g \simeq (\Sigma f)h$ (ex-homotopic).*

PROOF. Let $\varphi_t: Z \to C_f$ be an ex-homotopy between $i\rho'$ and $\alpha_f g$. Now define $k: C_g \to C_f$ by $k(b) = b$, $k(z, t) = \varphi_t(z)$, $k(y) = \alpha_f(y)$. Then the desired map $h: \Sigma Z \to \Sigma X$ is given by $h<b,i> = <b,i>$, $i = 0, 1$, $h<z,t> = \beta_f k(z,t)$. Obviously $h$ is well defined and an ex-map. If we define $\chi_s: \Sigma Z \to \Sigma Y$ by

$$\chi_s<z, t> = \Sigma f \cdot h<z, t(1+t-st)/1-s+t>,$$

$$\chi_s<b, i> = <b, i> \quad i = 0, 1,$$

then $\chi_s$ is well defined and gives an ex-homotopy between $\Sigma f \cdot h$ and $\chi_1$ (where $\chi_1<z,t> = <g(z),1>$). Next define $\xi_s: \Sigma Z \to \Sigma Y$ by $\xi_s<z,t> = <g(z), t(1+st)/s+t>$ and $\xi_s<b,i> = <b,1>$ $i = 0, 1$. Then $\xi_s$ is well defined and provides an ex-homotopy between $\chi_1$ and $\Sigma g$. Thus we have $\Sigma f \cdot h \simeq \Sigma g$ (ex-homotopic).

**3. Spanier-Whitehead duality in $\Sigma$-theory.** In [3], I. M. James has proved the following theorem:

THEOREM A (Thm 2. 1 in the loc. cit.) *Let $E$ be a fibre bundle over $B$ with projection $\varphi: E \to B$. Suppose that fibre $F$ is compact and that base $B$ is regular and locally compact. Then the over-space $\Sigma E$ is a fibre bundle with projection $\psi: \Sigma E \to B$.*

THEOREM B (Thm 6. 4 in the loc. cit.) *Let $B$ be regular and locally compact. Let $A$ be a CW-complex represented as an over-space of $B$ and let $E$ be an ex-space of $A$. Let $K$ be an ex-complex of $A$. Suppose that the projection $\rho: E \to B$ is a fibre bundle with compact fibre. If the fibre is $r$-connected, then the suspension $\Sigma_*: \pi(K, E) \to \pi(\Sigma K, \Sigma E)$ is injective if* dim $K \leqslant 2r$, *surjective if* $\dim K \leqslant 2r+1$.

In the following the above theorem play essential roles. Henceforth we denote by $\pi(\ ,\ )$ the set of over-homotopy classes of over-maps. We shall use Thm B in the cace $A = \phi$.

Throughout §3 we assume that $B$ is a finite $CW$-complex and $L, M$ are finite $CW$-complexes represented as over-spaces of $B$.

Assume that projection $M \to B$ is a fibre bundle with compact fibre $F$. Then we shall define $\{L, M\}$ to be $\varinjlim \pi(\Sigma^k L, \Sigma^k M)$. According to Theorems A and B, this is well defined. Here note that the fibre of $\Sigma^k M$ is the ordinary suspension $S^k F$ and hence $(k-1)$-connected. As in [3; p. 374], we define the functions

$$L^{\#}: \pi(Z, M) \to \pi(L * Z, L * M),$$
$$L_{\#}: \pi(Z, M) \to \pi(Z * L, M * L)$$

by taking the join with the identity on $L$, where $Z$ is an over-space. We consider maps $u: L * M \to \Sigma^{n+1}\phi$. Such a map defines two functions

$$u_q: \{\Sigma^{q+1}\phi, M\} \to \{L * \Sigma^{q+1}\phi, \Sigma^{n+1}\phi\}$$

and

$$u^q: \{\Sigma^{q+1}\phi, L\} \to \{M * \Sigma^{q+1}\phi, \Sigma^{n+1}\phi\}$$

by taking the direct limit of the compositions

$$\pi(\Sigma^{q+k+1}\phi, \Sigma^k M) \xrightarrow{L^*} \pi(L * \Sigma^{q+k+1}\phi, L * \Sigma^k M) \cong \pi(\Sigma^k(L * \Sigma^{q+1}\phi), \Sigma^k(L * M))$$
$$\xrightarrow{(\Sigma^k u)_*} \pi(\Sigma^k(L * \Sigma^{q+1}\phi), \Sigma^k(\Sigma^{n+1}\phi))$$

and

$$\pi(\Sigma^{q+k+1}\phi, \Sigma^k L) \xrightarrow{M_*} \pi(\Sigma^{q+k+1}\phi * M, \Sigma^k L * M) \cong \pi(\Sigma^k(M * \Sigma^{q+1}\phi), \Sigma^k(L * M))$$

$$\xrightarrow{(\Sigma^k u)_*} \pi(\Sigma^k(M * \Sigma^{q+1}\phi), \Sigma^k(\Sigma^{n+1}\phi))$$

respectively. An over-map $u: L * M \to \Sigma^{n+1}\phi$ is called an $(n-1)$-duality if the above $u_q$, $u^q$ are both bijections.

REMARK. If $B$ is a point, then $u$ becomes an $n$-duality in the sense of Spanier-Whitehead (see [2], [5]).

THEOREM 3.1. *Let $L$, $M$ be sphere bundles over $B$ associated with the euclidean bundles $\xi$, $\eta$ respectively. If the Whitney sum $\xi \oplus \eta$ is a trivial $(n+1)$-plane bundle, then there is an $(n-1)$-duality.*

PROOF. Since $L * M = (\xi \oplus \eta)_1$ (see [3]), it follows that $L * M$ is homeomorphic to $\Sigma^{n+1}\phi$, as an over-space. Denote this homeomorphism by $u: L*M \to \Sigma^{n+1}\phi$. We shall prove that $u$ is actually an $(n-1)$-duality. We only prove that $u^q$ is a bijection. Consider $M^*: \pi(\Sigma^{q+k+1}\phi, \Sigma^k L) \to \pi(M * \Sigma^{q+k+1}\phi, M * \Sigma^k L)$ and $L^*: \pi(M * \Sigma^{q+k+1}\phi, M * \Sigma^k L) \to \pi(L * M * \Sigma^{q+k+1}\phi, (L * M) * \Sigma^k L)$. It is well known that $(L*M)^* = L^* M^*$ and $(\Sigma^k\phi)^*$ is the $q$-fold suspension. By Thm B, the suspension $\Sigma_*: \pi(\Sigma^{q+k+1}\phi, \Sigma^k L) \to \pi(\Sigma^{q+k+2}\phi, \Sigma^{k+1}L)$ is bijective if $k \geqslant \dim B + q + 2$. Hence the composite $\{\Sigma^{q+1}\phi, L\} \xrightarrow{M^*} \{\Sigma^{q+1}M, M * L\} \xrightarrow{L^*} \{\Sigma^{q+n+2}\phi, \Sigma^{n+1}L\}$ is a bijection. This shows that $M^*$: $\{\Sigma^{q+1}\phi, L\} \to \{\Sigma^{q+1}M, M * L\}$ admits a left inverse and $L^*$: $\{\Sigma^{q+1}M, M * L\} \to \{\Sigma^{q+n+2}\phi, \Sigma^{n+1}\phi\}$ admits a right inverse.

Now let $\zeta$ be an euclidean bundle such that $\zeta \oplus \xi$ is the trivial plane bundle and let $N$ be a sphere bundle associated with $\zeta$. Then repeating the preceding argument, it is shown that $L^*\{\Sigma^{q+1}M, M * L\} \to \{\Sigma^{q+n+2}\phi, \Sigma^{n+1}L\}$ admits a left inverse. Hence $L^*$ is a bijection. Thus $M^*$: $\{\Sigma^{q+1}\phi, L\} \to \{M * \Sigma^{q+1}\phi, M * L\}$ is a bijection.

If an $n$-duality $u: L * L' \to \Sigma^{n+2}$ exists, then $L'$ is called an $n$-dual of $L$. The following is immediate from Theorem 3.1.

PROPOSITION 3.2. *If $L$ is a sphere bundle over $B$, then there exists an $n$-dual for some $n$.*

THEOREM 3.3. *Let $L$, $L'$ and $K$ be fibre bundles having finite CW-complexes as fibres. Let $u: L * L' \to \Sigma^{n+1}\phi$ be an $(n-1)$-duality. Then*

$$u_K: \{K, L'\} \to \{L * K, \Sigma^{n+1}\phi\}$$

*and*

$$u^K: \{K, L\} \to \{K * L', \Sigma^{n+1}\phi\}$$

*are bijections.*

PROOF. First consider the situation when $B$ is an $m$-cube $I^m (m \geqslant 0)$. If $B$ is a point and $F_K, F, F'$ are fibres of $K, L, L'$ respectively, then it follows from [2; p. 207] that $u_K : \{F_K, F'\} \rightarrow \{F * F_K, S^n\}$ and $u_K : \{F_K, F\} \rightarrow \{F_K * F', S^n\}$ are bijections. If $B$ is an $m$-cube $I^m$, then $L, L'$ and $K$ are trivial bundles and hence $u_K$ and $u^K$ are both bijections.

Now consider the general case. We proceed by induction on the sections of $B$. Let $B_m$ denote $m$-section of $B$ and $K|B_m$ the restriction of the bundle $K$ on $B_m$. From the inductive assumption, $u_K : \pi(\Sigma^k K | B_m, \Sigma^k L'(B_m) \rightarrow \pi(\Sigma^k(L*K) | B_m, \Sigma^{n+k+1}\phi | B_m)$ is bijective for sufficiently large $k$. Here we may assume that $B$ is obtained from $B_m$ by adjoining one cell $\bar{e}^{m+1}$. Let $\psi : I^{m+1} \rightarrow \bar{e}^{m+1}$ be the characteristic map. Then $\psi^*(\Sigma^k K)$, $\psi^*(\Sigma^k L)$ and $\psi^*(\Sigma^k(L*K))$ are trivial and hence we may assume that $\psi^*(\Sigma^k K) = I^{m+1} \times S^k F_K$, $\psi^*(\Sigma^k L') = I^{m+1} \times S^k F'$ and $\psi^*(\Sigma^k L * K)) = I^{m+1} \times S^k(F * F_K)$. Let $\psi : I^{m+1} \times S^k(F * F_K) \rightarrow \Sigma^k(L*K) | \bar{e}^{m+1}$ etc. be the covering map of $\psi$. Take an over-map $g : \Sigma^k(L*K) \rightarrow \Sigma^{k+n+1}\phi$ and denote by $g_m$ the restriction of $g$ to $\Sigma^k(L*K) | B_m$. Then by inductive assumption, there exists an over-map $f_m : \Sigma^k K | B_m \rightarrow \Sigma^k L' | B_m$ such that $u_K[f_m] = [g_m]$, where [ ] denotes the over-homotopy classes. Let $g'$ be the restriction of $g$ to $\Sigma^k(L*K) | \bar{e}^{m+1}$. Define $\widetilde{g} : I^{m+1} \times S^k(F * F_K) \rightarrow I^{m+1} \times S^{n+k}$ by $\widetilde{\psi}\widetilde{g}(x, u) = g(\widetilde{\psi}(x, u))$, where $g(u) = \psi(x)$. Then by the former case we have an over-map $\widetilde{f} : I^{m+1} \times S^k F_K \rightarrow I^{m+1} \times S^k F'$ such that $u_k[\widetilde{f}] = [\widetilde{g}]$. Next we define an over-map $f' : \Sigma^k K | \bar{e}^{m+1} \rightarrow \Sigma^k L' | \bar{e}^{m+1}$ by $f'(\widetilde{\psi}(x, u)) = \widetilde{\psi}\widetilde{f}(x, u)$. Then we easily see that $u_K[f'] = [g']$. Since $\Sigma^k L \rightarrow B$ is a fibre bundle by Thm A, we can take $f'$ such that $f' | \dot{e}^{m+1} = f_m | \dot{e}^{m+1}$. Now we define an over-map $f : \Sigma^k K \rightarrow \Sigma^k L'$ by

$$
\begin{aligned}
f(u) &= f'(u) && \text{for} \quad u \in \Sigma^k K | \bar{e}^{m+1} \\
&= f_m(u) && \text{for} \quad u \in \Sigma^k K | B_m.
\end{aligned}
$$

Then $f$ is well defined and $u_k[f] = [g]$. From the above construction, we can see that the over-homotopy class of $f$ is uniquely determined by that of $g$.

THEOREM 3.4. (1) *Let* $u : K * K' \rightarrow \Sigma^{n+1}\phi$ *and* $v : L * L' \rightarrow \Sigma^{n+1}\phi$ *be* $(n-1)$-*dualities. Let* $f : K \rightarrow L$ *and* $g : L' \rightarrow K$ *be over-maps satisfying the condition*

$$u \circ (1 * g) = v \circ (f * 1).$$

*Then there exists a map* $h : C_f * C_g \rightarrow \Sigma^{n+2}\phi$ *such that the following squares are homotopy commutative.*

(1) The corresponding statement in usual homotopy theory is seen in [5, p. 463]. We do not know whether a map $h : C_f * C_g \rightarrow \Sigma^{n+2}\phi$ is an $n$-duality in the our sense.

(A)
$$\begin{array}{ccccc} L*C_g & \xrightarrow{1*\beta_g} & L*\Sigma L' & \xleftrightarrow{\eta} & \Sigma(L*L') \\ \downarrow{\scriptstyle \alpha_f*1} & & & & \downarrow{\scriptstyle \Sigma v} \\ C_f*C_g & & \xrightarrow{h} & & \Sigma^{n+2}\phi \end{array}$$

(B)
$$\begin{array}{ccccc} C_f*K' & & \xrightarrow{1*\alpha_g} & & C_f*C_g \\ \downarrow{\scriptstyle \beta_f*1} & & & & \downarrow{\scriptstyle h} \\ \Sigma K*K' & \xleftrightarrow{\omega} & \Sigma(K*K') & \xrightarrow{\Sigma u} & \Sigma^{n+2}\phi \end{array}$$

PROOF. We define $h: C_f*C_g \to \Sigma^{n+2}\phi$ by

$$h[(x,s),(y,t),l]=\begin{cases} <v[f(x),y,l], \dfrac{st(1+tl)(1+s-sl)}{(t+l)(1+s-l)}> & s\geqslant t \\ <u[x,g(y),l], \dfrac{st(1+tl)(1+s-sl)}{(t+l)(1+s-l)}> & s\leqslant t, \end{cases}$$

where $(x,s)\in C_f$ and $(y,t)\in C_g$. Now consider the diagram (A). We easily see that $h(\alpha_f*1)[x,(y,t),l]=<v[x,y,l],t(1+tl)/t+l>$. On the other hand the homeomorphism $\eta: L*\Sigma L'\to\Sigma(L*L')$ may be given by the formula

$$\eta[x,<y,t>,l]=\begin{cases} <[x,y,2tl/1-l+2tl],1-l+2tl/2> & 0\leqslant t\leqslant 1/2 \\ <[x,y,2l(1-t)/1+l-2tl],1-l+2tl/2> & 1/2\leqslant t\leqslant 1. \end{cases}$$

This formula can be deduced from the remark in §1 and [1; p. 225]. Hence in order to prove that $h\circ(\alpha_f*1)\simeq\Sigma v\circ\eta\circ(1*\beta_g)$, we have only to prove that $\eta$ is homotopic to the map $[x,<y,t>,l]\to<[x,y,l],t(1+tl)/t+l>$. But this is given by the following homotoy:

$$\varphi_s[x,<y,t>,l]=\begin{cases} <[x,y,2tl/s(1-l)+2tl],L> & 0\leqslant t\leqslant s/2 \\ <[x,y,l],L> & s/2\leqslant t\leqslant 1-s/2 \\ <[x,y,2l(1-t)l/s(1-l)+2(1-t)l],L> & 1-s/2\leqslant t\leqslant 1, \end{cases}$$

where $L=t(2-s-sl+2tl)/2st+2(1-s)(t+l)$. By the analogous argument, we can prove that the square (B) is homotopy commutative.

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