THE DERIVATIONS OF THE LIE ALGEBRAS

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Some results have been given by several authors as to the derivations of Lie algebras. Namely, E. Schenkman and N. Jacobson [1] proved that every non-zero nilpotent Lie algebra has an outer derivation. The author [4] sharpened this theorem and showed that every nilpotent Lie algebra over a field of characteristic 0 possesses an outer derivation in the radical of its derivation algebra. On the other hand G. Leger [2] has given a necessary and sufficient condition for a Lie algebra to have an outer derivation, and in [3] he has shown that if a Lie algebra has no outer derivations and its center is not zero then it is not solvable and its radical is nilpotent and is not quasi-cyclic. Moreover, S. Tôgô [6] proved that such a Lie algebra coincides with its derived algebra and he proceeded the studies concerning to the Lie algebra which has outer derivations. The purpose of this paper is to add some results to them.

In §2 we shall introduce the notion of the free nilpotent Lie algebra, and interpret a nilpotent Lie algebra as a quotient algebra of the free nilpotent Lie algebra by a suitable ideal. We shall also investigate the relationship between the derivations of the former and of the latter. In §3, we shall study some applications of the results by Leger [2]. As it is well known, every semi-simple Lie algebra has no outer derivations. It is also known that there exists a solvable Lie algebra with null center which has no outer derivations. However, it seems to the author that it is unknown whether there exists a Lie algebra with non-zero center which has no outer derivations. In §4 we shall give an example of such a Lie algebra of dimension 41 and with one dimensional center.

1. Preliminaries and notations. Throughout this paper, we suppose that the Lie algebras have the coefficient field of characteristic 0. For a subset M of a vector space, we denote by $\{M\}$ the subspace generated by the elements of M. When M is a subset of a Lie algebra L and k is a natural number, we denote by M^k the subspace generated by the elements of the form

$$[m_1, [m_2, [\cdots [m_{k-1}, m_k]\cdots] \qquad (m_1, m_2, \cdots, m_k \in M).$$

Furthermore we shall employ the following notations:

$Z(L)$: the center of a Lie algebra L

$\mathfrak{D}(L)$: the Lie algebra of all derivations of L

$\mathfrak{I}(L)$: the ideal of $\mathfrak{D}(L)$ consisting of all the inner derivations of L

$\mathfrak{R}(L)$: the radical of $\mathfrak{D}(L)$

$\mathfrak{S}(L)$: a maximal semi-simple subalgebra of $\mathfrak{D}(L)$.

2. The nilpotent Lie algebras generated by m elements. Let U be an m-dimensional vector space, and let $e_1, \cdots, e_m$ be its basis. We consider the tensor products $U \otimes U$, $U \otimes U \otimes U$, $\cdots, U \otimes U \otimes \cdots \otimes U$ (n-times) of U, and make a space of direct sum of them :

$$V = U \oplus (U \otimes U) \oplus \cdots \oplus (U \otimes U \otimes \cdots \otimes U).$$

We define a distributive and non-associative product $\times$ in V as follows :

$$e_i \times (e_{j_1} \otimes \cdots \otimes e_{j_k}) = e_i \otimes e_{j_1} \otimes \cdots \otimes e_{j_k} \qquad \text{for } k \leqq n-1$$

$$e_i \times (e_{j_1} \otimes \cdots \otimes e_{j_n}) = 0.$$

Next, when the product $a \times b$ has been defined already for an element $a \in V$ and for every element b of V, we define the product $(e \otimes a) \times b$ by

$$(e \otimes a) \times b = e \otimes (a \times b) - a \times (e \otimes b),$$

where e denotes an arbitrary element of U. Then the next equality holds.

$$(1) \qquad (a \times b) \times c = a \times (b \times c) - b \times (a \times c) \qquad \text{for } a, b \text{ and } c \in V$$

To prove this it is enough to show (1) in the case where a, b and c are homogeneous. We will prove this by mathematical induction. So we assume that $a = e \otimes a_1$ for $e \in U$ and our assertion holds for a_1. Then,

$$((e \otimes a_1) \times b) \times c = (e \otimes (a_1 \times b)) \times c - (a_1 \times (e \otimes b)) \times c$$

$$= e \otimes ((a_1 \times b) \times c) - (a \times b) \times (e \otimes c)$$

$$\quad - a_1 \times ((e \otimes b) \times c) + (e \otimes b) \times (a_1 \times c)$$

$$= e \otimes (a_1 \times (b \times c)) - e \otimes (b \times (a_1 \times c)) - a_1 \times (b \times (e \otimes c))$$

$$\quad + b \times (a_1 \times (e \otimes c)) - a_1 \times (e \otimes (b \times c)) + a_1 \times (b \times (e \otimes c))$$

$$\quad + e \otimes (b \times (a_1 \times c)) - b \times (e \otimes (a_1 \times c))$$

$$= (e \otimes a_1) \times (b \times c) - b \times ((e \otimes a_1) \times c).$$

Hence our assertion holds for a.

Now we denote by W the left ideal generated by the set $\{v \times v \, ; \, v \in V\}$ and make the quotient space $\widetilde{N} = V/W$. For any elements $a, b \in V$ it holds that

$$(a + b) \times (a + b) = a \times a + b \times b + a \times b + b \times a \equiv 0 \bmod W$$

Hence we get

$$a \times b \equiv - b \times a \qquad \bmod W.$$

PROPOSITION 1. $\widetilde{N}$ *is a nilpotent Lie algebra.*

PROOF. It is obvious that we can define the product uniquely in $\widetilde{N}$. We denote by $[\ ,\ ]$ this product. Let x, y, z be any elements of $\widetilde{N}$, and a, b, c be representatives of x, y, z respectively. Then the relation $[x, x] = 0$ follows from the fact that $a \times a \in W$. Moreover,

$$a \times (b \times c) + b \times (c \times a) + c \times (a \times b)$$

$$\equiv a \times (b \times c) - b \times (a \times c) + c \times (a \times b)$$

$$= (a \times b) \times c + c \times (a \times b) \qquad \equiv 0 \bmod W.$$

Hence we get the Jacobi's identity :

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Therefore $\widetilde{N}$ is a Lie algebra, and it is nilpotent because of $\widetilde{N}^{n+1} = 0$.

We call the Lie algebra $\widetilde{N}$ the *free nilpotent Lie algebra* of length n generated by U. Assume that a and b are any monomials in V. W is spanned by the elements of the following types :

$$a \times a, \ a \times b + b \times a, \ C(a \times a), \ C(a \times b + b \times a),$$

where C means the product of repeated multiplications by elements of U. Therefore W is homogeneous and also is $\widetilde{N}$.

From now on we regard the elements of U as imbedded in $\widetilde{N}$.

PROPOSITION 2. *Any linear mapping from U to $\widetilde{N}$ may be extended to a derivation of $\widetilde{N}$ uniquely.*

PROOF. Let D be a linear mapping from U to $\widetilde{N}$, and $e_1, \cdots, e_m$ be a basis of U. At first we extend D to linear mapping from V to $\widetilde{N}$ as follows.

$$D(e_{i_1} \times (e_{i_2} \times (\cdots \times (e_{i_{k-1}} \times e_{i_k}) \cdots))$$

$$= \sum_{j=1}^{k} [e_{i_1}, [e_{i_2}, [\cdots, [De_{i_j}, [\cdots, [e_{i_{k-1}}, e_{i_k}] \cdots].$$

Then it holds that

(2) $$D(a \times b) = [Da, b] + [a, Db] \qquad \text{for } a, b \in V.$$

This can be samely proved as in (1) by induction on the degree of a. For any $a, b_1, b_2, \cdots, b_k \in V$,

$$D(b_1 \times (b_2 \times \cdots \times (b_k \times (a \times a)) \cdots)$$

$$= \sum_{j=1}^{k} [b_1, [b_2, [\cdots, [Db_j, [\cdots, [b_k, [a, a]] \cdots]$$

$$+ [b_1, [b_2, [\cdots, [b_k, [Da, a] + [a, Da]] \cdots] = 0.$$

Hence D maps W into 0. Therefore D defines a linear endomorphism of the Lie algebra $\widetilde{N} = V/W$, which is a derivation of $\widetilde{N}$ by (2).

PROPOSITION 3. *Every linear mapping from U into $\widetilde{N}$ may be extended to an endomorphism of the Lie algebra $\widetilde{N}$. In particular if $f_1, \cdots, f_m$ are linearly independent elements of U and $g_1, \cdots, g_m \in U^2 + \cdots + U^n$, then the linear mapping which sends e_i to $f_i + g_i$ $(i = 1, \cdots, m)$ defines an automorphism of $\widetilde{N}$.*

PROOF. We may prove the first half samely as the preceeding proposition. So we shall prove the latter half. We denote by σ_1 or σ a linear mapping from U into $\widetilde{N}$ which maps e_i to f_i or $f_i + g_i$ respectively, and we take the inverse transformation σ_1^{-1} of σ_1 in U. We denote by σ_1 again the extention of σ_1 to an endomorphism of $\widetilde{N}$. Because of

$$[e_{j_1}, [e_{j_2}, [\cdots, [e_{j_{k-1}}, e_{j_k}] \cdots]$$

$$= \sigma_1[\sigma_1^{-1}e_{j_1}, [\sigma_1^{-1}e_{j_2}, [\cdots, [\sigma_1^{-1}e_{j_{k-1}}, \sigma_1^{-1}e_{j_k}] \cdots],$$

σ_1 sends $\widetilde{N}$ onto $\widetilde{N}$. Next, we assign to f_i the element $\sigma_2 f_i = f_i + g_i$, then the

extension σ_2 to an endomorphism of $\widetilde{N}$ must transform $\widetilde{N}$ onto $\widetilde{N}$. In fact, at first it is obvious that $\sigma_2 U^n = U^n$. So we suppose that

$$\sigma_2(U^k + U^{k+1} + \cdots + U^n) = U^k + U^{k+1} + \cdots + U^n.$$

Then as it follows that

$$\sigma_2[f_{i_1}, [f_{i_2}, [\cdots, [f_{i_{k-2}}, f_{i_{k-1}}] \cdots]$$
$$\equiv [f_{i_1}, [f_{i_2}, [\cdots, [f_{i_{k-2}}, f_{i_{k-1}}] \cdots] \qquad \mathrm{mod}\ U^k + U^{k+1} + \cdots + U^n,$$

σ_2 transforms $U^{k-1} + U^k + \cdots + U^n$ onto itself. Hence by induction we can see that σ_2 is a surjection, and $\sigma = \sigma_2 \circ \sigma_1$ is also, whence σ is an automorphism of N.

Now we investigate the structure of the derivation algebra $\mathfrak{D}(\widetilde{N})$ of $\widetilde{N}$. By Proposition 2 every endomorphism of U may be extended to a derivation of $\widetilde{N}$. Now we denote by E the extension of the identity transformation of U, and by $\widetilde{\mathfrak{S}}$ the collection of all extensions of linear endomorphisms of U whose traces are 0. $\widetilde{\mathfrak{S}}$ is a simple Lie algebra of type A_{m-1}. Let $\widetilde{\mathfrak{N}}$ be the ideal consisting of all the extensions of linear transformations which map U to $[\widetilde{N}, \widetilde{N}]$. Then any element of $\widetilde{\mathfrak{N}}$ is a derivation which transforms $\widetilde{N}$ into $[\widetilde{N}, \widetilde{N}]$, whence $\widetilde{\mathfrak{N}}$ is a nilpotent ideal of $\mathfrak{D}(\widetilde{N})$. E is commutative with the elements of $\widetilde{\mathfrak{S}}$ and $\mathfrak{D}(\widetilde{N}) = \widetilde{\mathfrak{S}} + \{E\} + \widetilde{\mathfrak{N}}$. Therefore $\{E\} + \widetilde{\mathfrak{N}}$ is the radical of $\mathfrak{D}(\widetilde{N})$, and $\widetilde{\mathfrak{S}}$ is a maximal semi-simple Lie algebra.

PROPOSITION 4. *Let N be a nilpotent Lie algebra of length n generated by m linearly independent elements. Then there exists an ideal $\widetilde{A}$ of $\widetilde{N}$ such that N is isomorphic to $\widetilde{N}/\widetilde{A}$.*

PROOF. Let $f_1, \cdots, f_m$ be linearly independent, and $N = \{f_1, \cdots, f_m\} \oplus [N, N]$. Then $f_1, \cdots, f_m$ generate the Lie algebra N. This may be proved similarly as in the proof of Proposition 3. Hence the set $f_1, \cdots, f_m$ is a minimal system of generators of the Lie algebra N. We extend the linear transformation τ from U to N which sends e_i to f_i, to the one from V to N as follows:

$$e_{i_1} \times (e_{i_2} \times (\cdots \times (e_{i_{k-1}} \times e_{i_k})) \cdots) \to [f_{i_1}, [f_{i_2}, [\cdots [f_{i_{k-1}}, f_{i_k}] \cdots].$$

Then we may prove the fact $\tau(a \times b) = [\tau a, \tau b]$ samely as in the proof of Proposition 1. Therefore τ maps W to zero and τ induces a homomorphism from $\widetilde{N}$ onto N.

From now on, we identify the element of U the corresponding one of N.

PROPOSITION 5. *Let a nilpotent Lie algebra N be isomorphic to $\widetilde{N}/\widetilde{A}$, and $\mathfrak{D}_1$ be the subalgebra of all the derivations of $\widetilde{N}$ which leave $\widetilde{A}$ invariant. We denote by $\mathfrak{D}_0$ the collection of all the derivations which map $\widetilde{N}$ into $\widetilde{A}$. Then $\mathfrak{D}_0$ is an ideal of $\mathfrak{D}_1$ and $\mathfrak{D}_1/\mathfrak{D}_0$ is isomorphic to the derivation algebra $\mathfrak{D}(N)$ of N.*

PROOF. We may easily see that $\mathfrak{D}_0$ is an ideal of $\mathfrak{D}_1$ and we can define naturally an isomorphism from $\mathfrak{D}_1/\mathfrak{D}_0$ into $\mathfrak{D}(N)$. Conversely let D be an arbitrary derivation of N. Now we take a basis $f, \cdots, f_k$ on N, and let $\widetilde{f}_1, \cdots, \widetilde{f}_k$ be mapped to $f_1, \cdots, f_k$ respectively by the natural homomorphism ρ of $\widetilde{N}$ onto N. Let $e_1, \cdots, e_m$ be the minimal system of generators of N. If we set $De_i = \sum_{j=1}^{k} \alpha_{ji} f_j$, then by Proposition 2 there exists a derivation $\widetilde{D}$ of $\widetilde{N}$ such that $\widetilde{D}e_i = \sum_{j=1}^{k} \alpha_{ji} \widetilde{f}_j$. For any $a_1, \cdots, a_l \in U$ it holds that

$$\rho \circ \widetilde{D}[a_1, [a_2, [\cdots, [a_j, [\cdots, [a_{l-1}, a_l]\cdots]$$

$$= \sum_{j=1}^{l} [\rho a_1, [\rho a_2, [\rho \circ \widetilde{D}a_j, [\cdots [\rho a_{l-1}, \rho a_l]\cdots]$$

$$= \sum_{j=1}^{l} [\rho a_1, [\rho a_2, [\cdots, [\widetilde{D} \circ \rho a_j, [\cdots, [\rho a_{l-1}, \rho a_l]\cdots]$$

$$= D \circ \rho [a_1, [a_2, [\cdots, [a_j, [\cdots [a_{l-1}, a_l]\cdots].$$

Hence we get $\rho \circ \widetilde{D} = D \circ \rho$, especially $\widetilde{D}\widetilde{A} \subset \widetilde{A}$. Thus $\mathfrak{D}_1/\mathfrak{D}_0$ is isomorphic to $\mathfrak{D}(N)$.

Let $\mathfrak{D}(N)$ be the derivation algebra of a nilpotent Lie algebra N, and $\mathfrak{S}$ be a maximal semi-simple subalgebra of $\mathfrak{D}(N)$. N is a completely reducible $\mathfrak{S}$-module and $[N, N]$ is an $\mathfrak{S}$-invariant submodule, therefore N is decomposed into a direct sum of $[N, N]$ and an $\mathfrak{S}$-invariant subspace U. Then U generates the whole Lie algebra N. $\mathfrak{S}$ is isomorphic to a subalgebra of $\mathfrak{sl}(U)$. When we take the free nilpotent Lie algebra $\widetilde{N}$ generated by U, $\mathfrak{S}$ may be imbedded in the maximal semi-simple subalgebra $\widetilde{\mathfrak{S}}$ of $\mathfrak{D}(\widetilde{N})$.

When the nilpotent Lie algebra N has a subspace U such that

$$N = U + [N, N], \qquad U \cap [N, N] = \{0\},$$

and N is represented as direct sum of subspaces U^i, it was called *quasi-cyclic* by Leger [3], and he showed the following property:

Let N be isomorphic to $\widetilde{N}/\widetilde{A}$. Then N is quasi-cyclic if and only if $\widetilde{A}$ is a homogeneous ideal. If N is quasi-cyclic, then there exists an outer derivation of N which is commutative with the elements of the maximal semi-simple subalgebra $\mathfrak{S}$ of $\mathfrak{D}(N)$.

We can understand the latter half of this assertion as follows. The above-mentioned derivation E of $\widetilde{N}$ is an i-multiple of the unit transformation on U^i, therefore E belongs to $\mathfrak{D}_1$, as $\widetilde{A}$ is homogeneous. Hence E induces a derivation of N, and we denote this again by E. E and the elements of $\mathfrak{S}$ are commutative with each other in the space U of generators of N, and therefore on the whole space N. On the other hand, E is an outer derivation as it is not nilpotent.

Now we consider in the sequel that the maximal semi-simple subalgebra $\mathfrak{S}$ of $\mathfrak{D}(N)$ is imbedded in the maximal semi-simple subalgebra $\widetilde{\mathfrak{S}}$ of $\mathfrak{D}(\widetilde{N})$.

PROPOSITION 6. *If $\widetilde{A}$ is an $\mathfrak{S}$-irreducible ideal then it is contained in the center of $\widetilde{N}$.*

PROOF. As $[U, \widetilde{A}]$ is an $\mathfrak{S}$-invariant subspace contained in $\widetilde{A}$, it is either $\{0\}$ or $\widetilde{A}$ itself. If $[U, \widetilde{A}] = \widetilde{A}$, then $\widetilde{A} = [U, \widetilde{A}] = [U, [U, \widetilde{A}]] = \cdots = [U, [U, [\cdots, [U, \widetilde{A}]\cdots] = \{0\}$, namely $\widetilde{A}$ is $\{0\}$. On the other hand if $[U, \widetilde{A}] = \{0\}$, $\widetilde{A}$ is contained in the center of $\widetilde{N}$.

Now for any element $b \in \widetilde{N}$, we represent b as follows.

$$b = b_1 + b_2 + \cdots + b_n, \qquad b_i \in U^i.$$

We denote by ρ_i the mapping from b to b_i.

LEMMA 1. *Let B be an $\mathfrak{S}$-invariant subspace of $\widetilde{N}$. Then $\rho_{i|B}$ is an $\mathfrak{S}$-homomorphism. Especially for the case that B is irreducible, $\rho_{i|B}$ is either an isomorphism or null mapping.*

PROOF. The operation of $\mathfrak{S}$ makes U invariant, and hence U^i also. Thus $\rho_{i|B}$ is an $\mathfrak{S}$-homomorphism. The rest is obvious.

For an $\mathfrak{S}$-invariant subspace B, let

$$\rho_1(B) = \cdots = \rho_{i-1}(B) = 0 \qquad \text{and} \quad \rho_i(B) \neq 0.$$

Then we denote the number i by $\alpha(B)$.

LEMMA 2. *There exists an $\mathfrak{S}$-isomorphism σ from $\rho_{\alpha(B)}(B)$ into B such that $\rho_{\alpha(B)} \circ \sigma$ is the identity mapping.*

PROOF. If we set $i = \alpha(B)$, by the assumption

$$B \subset U^i + U^{i+1} + \cdots + U^n \quad \text{and} \quad B \not\subset U^{i+1} + \cdots + U^n.$$

Then we may decompose B into the direct sum of $\mathfrak{S}$-spaces:

$$B = (B \cap (U^{i+1} + \cdots + U^n)) \oplus C.$$

Then it is easily proved that ρ_i maps C isomorphically onto $\rho_i(B)$. Hence we can find an isomorphism σ as in the assertion.

PROPOSITION 7. *Let $\widetilde{A}$ be an $\mathfrak{S}$-invariant ideal of $\widetilde{N}$, and be decomposed into the direct sum of $\mathfrak{S}$-irreducible subspaces as follows:*

$$(3) \qquad\qquad\qquad \widetilde{A} = B_1 \oplus \cdots \oplus B_p$$

Among such decompositions, there exists the one such that for an arbitrary q $(1 \leq q \leq p)$ the ideal generated by $\rho_{\alpha(B_q)}(B_q)$ does not contain any $\rho_j(B_i)$ but for the case either $j = \alpha(B_i)$ or $\rho_j(B_i) = 0$.

PROOF. We set $C_j = \widetilde{A} \cap (U^j + \cdots + U^n)$, then it is $\mathfrak{S}$-invariant and $\widetilde{A} = C_1 \supset C_2 \supset \cdots \supset C_n$. We can find $\mathfrak{S}$-invariant subspaces C_i' such that

$$A = C_1' \oplus \cdots \oplus C_{n-1}' + C_n, \qquad C_i = C_i' \oplus C_{i+1}.$$

Furthermore we decompose $C_1', \cdots, C_{n-1}'$ and C_n into the direct sum of irreducible subspaces, and we assume that the decomposition (3) is such a one. Now let us suppose that a nonzero $\rho_j(B_i)$ is contained in the ideal generated by $\rho_{\alpha(B_q)}(B_q)$ and $j > \alpha(B_i)$. We set $(adU)^{j-\alpha(B_q)} B_q = B$. Then $\rho_j(B_i)$ is an irreducible subspace of

$$(adU)^{j-\alpha(B_q)} \rho_{\alpha(B_q)}(B_q) = \rho_j(B).$$

Then by Lemma 2, there exists an $\mathfrak{S}$-isomorphism σ from $\rho_j(B_i)$ into B which satisfies $\rho_j \circ \sigma = $ the identity. We set

$$B_i' = \{b_i - \sigma \circ \rho_j(b_i) ; \ b_i \in B_i\}.$$

Then it is easily proved that B_i' is an $\mathfrak{S}$-irreducible space and $\rho_j(B_i') = 0$. We

may replace the subspace B_i in (3) by B_i' because $b_i \in C_{\alpha(B_i)}'$, $\sigma \circ \rho_j(b_i) \in B \subset C_j$ and $j > \alpha(B_i)$. Repeating this procedure we get the decomposition which we desired.

LEMMA 3. *Let B be an $\widetilde{\mathfrak{S}}$-invariant irreducible subspace of $\widetilde{N}$. If $\rho_j(B) \cong \rho_k(B) \neq 0$, then $j - k$ must be a multiple of m.*

PROOF. Let us denote by E_q the derivation of $\widetilde{N}$ such that $e_p \to \delta_{pq} e_p$ ($p = 1, \cdots, m$). The collection of the derivations $\sum_q \lambda_q E_q$ (where $\sum_q \lambda_q = 0$) makes up a Cartan subalgebra of $\widetilde{\mathfrak{S}}$. For a monomial a in U^i, we denote by m_p the number of e_p contained in expression of a. Then we have $\sum_{p=1}^{m} m_p = i$, and

$$\left(\sum \lambda_q E_q\right) \cdot a = \left(\sum \lambda_q m_q\right) a,$$

whence a is a weight vector of the representation of $\widetilde{\mathfrak{S}}$.

Now we denote the highest weights in both representation spaces $\rho_j(B)$ and $\rho_k(B)$ by $\sum \lambda_q m_q$ and $\sum \lambda_q m_q'$ respectively. Then they must coincide, whence

$$\sum \lambda_q (m_q - m_q') = \sum_{q=2}^{m} \lambda_q (m_q - m_q' - m_1 + m_1') = 0.$$

This holds identically for any λ_q ($q \geq 2$), and therefore

$$m_2 - m_2' = \cdots = m_m - m_m' = m_1 - m_1'.$$

This implies $j - k \equiv 0 \bmod m$, because $\sum m_q = j$ and $\sum m_q' = k$.

PROPOSITION 8. *Let N be a nilpotent Lie algebra generated by m elements, and the maximal semi-simple subalgebra $\mathfrak{S}$ of its derivation algebra be of dimension $m^2 - 1$. Further we assume that $N = \widetilde{N}/\widetilde{A}$ and B is an $\mathfrak{S}$-irreducible subspace of $\widetilde{A}$. If $\rho_j(B) \cong \rho_k(B) \neq 0$, then $j - k$ must be a multiple of m, and N is either quasi-cyclic or its length is greater than $m + 1$.*

PROOF. The first half is obvious in virtue of Lemma 3. So we assume that N is not quasi-cyclic. Then there exists $\mathfrak{S}$-irreducible subspace B of $\widetilde{A}$ such that $\rho_j(B) \neq 0$ and $\rho_k(B) \neq 0$ for $j > k$. Of course $j \geq 2$ and it must be $k \geq j + m \geq m + 2$ by the first half. Thus the latter half is also proved.

Proposition 7 and a modification of Proposition 8 will be available to construct

a nilpotent Lie algebra with a certain property in § 4.

3. The Lie algebra with outer derivations. In this section we shall be concerned with investigation of the Lie algebra which has outer derivations. First of all we cite the following lemma from Tôgô [5]:

LEMMA 4. *Let the Lie algebra L be decomposed into the direct sum of ideals in the form $L = L_1 \oplus L_2$. Then the derivation algebra $\mathfrak{D}(L)$ of L may be expressed in the following form:*

$$\mathfrak{D}(L) = \mathfrak{D}(L_1) + \mathfrak{D}(L_2) + \mathfrak{D}(L_1, L_2) + \mathfrak{D}(L_2, L_1).$$

Here, $\mathfrak{D}(L_1)$ is the collection of trivial extensions of all the elements in $\mathfrak{D}(L_i)$, and $\mathfrak{D}(L_i, L_j)$ is the collection of all the linear transformations which send L_i into the center $Z(L_j)$ of L_j, and send $[L_i, L_i]$ and L_j to $\{0\}$. Moreover it holds

$$[\mathfrak{D}(L_1) + \mathfrak{D}(L_2), \mathfrak{D}(L_i, L_j)] \subset \mathfrak{D}(L_i, L_j) \qquad (i, j = 1 \text{ or } 2).$$

In particular provided that $Z(L_1) \subset [L_1, L]$, when we denote by $\mathfrak{C}(L_1)$ the collection of all the linear transformations of L which send L_1 into $Z(L_1)$ and L_2 into $\{0\}$, $\mathfrak{C}(L_1)$ is an abelian ideal of $\mathfrak{D}(L)$ and $[\mathfrak{D}(L_1, L_2), \mathfrak{D}(L_2, L_1)]$ is contained in $\mathfrak{C}(L_1)$.

PROPOSITION 9. *Let the Lie algebra L be decomposed into the direct sum of two ideals as follows.*

$$L = L_1 \oplus L_2 \quad and \quad Z(L_1) \subset [L_1, L_1].$$

We designate by $\mathfrak{S}(L_i)$ a maximal semi-simple subalgebra of derivation algebra $\mathfrak{D}(L_i)$ of L_i, and by $\mathfrak{R}(L_i)$ the radical of $\mathfrak{D}(L_i)$. Then $\mathfrak{S}(L_1) + \mathfrak{S}(L_2)$ is a maximal semi-simple subalgebra of $\mathfrak{D}(L)$ and

$$\mathfrak{R} = \mathfrak{R}(L_1) + \mathfrak{R}(L_2) + \mathfrak{D}(L_1, L_2) + \mathfrak{D}(L_2, L_1)$$

is the radical of $\mathfrak{D}(L)$.

PROOF. By the above-mentioned lemma,

$$[\mathfrak{D}(L_1, L_2), \mathfrak{D}(L_2, L_1)] \subset \mathfrak{C}(L_1) \subset \mathfrak{R}(L_1)$$

$$[\mathfrak{D}(L_1) + \mathfrak{D}(L_2), \mathfrak{D}(L_i, L_j)] \subset \mathfrak{D}(L_i, L_j)$$

$$[\mathfrak{D}(L_i, L_j), \mathfrak{D}(L_i, L_j)] = \{0\} \quad \text{for} \quad i \neq j.$$

Hence $\mathfrak{R}$ is an ideal of $\mathfrak{D}(L)$. If we set $\mathfrak{R}^{(2)} = [\mathfrak{R}, \mathfrak{R}]$, $\mathfrak{R}^{(3)} = [\mathfrak{R}^{(2)}, \mathfrak{R}^{(2)}], \cdots$, we have the following:

$$\mathfrak{R}^{(2)} \subset \mathfrak{R}(L_1)^{(2)} + \mathfrak{R}(L_2)^{(2)} + \mathfrak{C}(L_1) + \mathfrak{D}(L_1, L_2) + \mathfrak{D}(L_2, L_1)$$

$$\mathfrak{R}^{(3)} \subset \mathfrak{R}(L_1)^{(3)} + \mathfrak{R}(L_2)^{(3)} + \mathfrak{C}(L_1) + \mathfrak{D}(L_1, L_2) + \mathfrak{D}(L_2, L_1)$$

$$\cdots \cdots \cdots$$

$$\mathfrak{R}^{(k)} \subset \mathfrak{R}(L_1)^{(k)} + \mathfrak{R}(L_2)^{(k)} + \mathfrak{C}(L_1) + \mathfrak{D}(L_1, L_2) + \mathfrak{D}(L_2, L_1).$$

As $\mathfrak{R}(L_1)$ and $\mathfrak{R}(L_2)$ are solvable, for sufficiently large k, $\mathfrak{R}(L_1)^{(k)} = \mathfrak{R}(L_2)^{(k)} = 0$. Hence it follows that

$$\mathfrak{R}^{(k)} \subset \mathfrak{C}(L_1) + \mathfrak{D}(L_1, L_2) + \mathfrak{D}(L_2, L_1).$$

As $[\mathfrak{C}(L_1), \mathfrak{D}(L_i, L_j)] \subset \mathfrak{C}(L_1) \cap \mathfrak{D}(L_i, L_j) = \{0\}$, $\mathfrak{R}^{(k+1)} \subset \mathfrak{C}(L_1)$ and therefore $\mathfrak{R}^{(k+2)} = 0$. This implies that $\mathfrak{R}$ is a solvable ideal. On the other hand $\mathfrak{S}(L_1) + \mathfrak{S}(L_2)$ is a semi-simple subalgebra and $\mathfrak{D}(L)$ is expressed as $\mathfrak{S}(L_1) + \mathfrak{S}(L_2) + \mathfrak{R}$. This concludes the proof.

Let $L = S + R$ be a Levi decomposition of the Lie algebra L. For an element s in S, we denote by $ad_R s$ the restriction of ad s to R, and by $ad_R S$ all of them. Then Leger [2] has proved the following.

PROPOSITION 10. *The Lie algebla has no outer derivations if and only if any derivation of R which is commutable with all the elements of $ad_R S$ is an inner derivation.*

We shall now state a number of corollaries obtained from this proposition:

COROLLARY 1. *Let S be a semi-simple Lie algebra and R be a solvable Lie algebra with outer derivations. Then the direct sum of S and R has an outer derivations.*

PROOF. It is obvious as $ad_R S = 0$.

COROLLARY 2. *Let the radical N of L be nilpotent, and the derivation algebra of N be solvable. Then L has an outer derivation.*

PROOF. Let $L = S + N$ be a Levi decomposition of L. It holds that $ad_N S = 0$ and N has an outer derivation by Jacobson [1], and so our assertion holds.

COROLLARY 3. *The notation being as in Proposition* 10, *let ad_RS be contained in a proper ideal $\mathfrak{S}_1$ of a maximal semi-simple subalgebra $\mathfrak{S}(R)$ of $\mathfrak{D}(R)$. Then L has an outer derivation.*

PROOF. Because, when we decompose $\mathfrak{S}(R)$ into a direct sum of ideals $\mathfrak{S}(R) = \mathfrak{S}_1 + \mathfrak{S}_2$, any element of $\mathfrak{S}_2$ is commutable with the elements of ad_RS and is an outer derivation.

Let $\mathfrak{D}(R)$ be the derivation algebra of a solvable Lie algebra R, and $\mathfrak{S}(R)$ be its maximal semi-simple subalgebra. If there exists an outer derivation which is commutable with all the elements of $\mathfrak{S}(R)$, we call that R *belongs to the class* $\mathfrak{D}$. It is easily seen that this definition is independent of the choice of $\mathfrak{S}(R)$.

PROPOSITION 11. *If the radical R of a Lie algebra L belongs to the class $\mathfrak{D}$, then L has an outer derivation.*

PROOF. We may verify this immediately by cosidering a maximal semi-simple subalgebra of $\mathfrak{S}(R)$ which contains ad_RS.

PROPOSITION 12. *If a solvable Lie algebra R is expressed as a direct sum of two ideals R_1 and R_2, and R_1 belongs to the class $\mathfrak{D}$, then R also belongs to the class $\mathfrak{D}$.*

PROOF. Let D_1 be an outer derivation which is commutable with the elements of $\mathfrak{S}(R_1)$. We denote by D_1 the trivial extension of D_1 on R again. At first we suppose that $Z(R_2) \subset [R_2, R_2]$. Then D_1 is commutable with the elements of $\mathfrak{S}(R_1) + \mathfrak{S}(R_2)$, which coincides with $\mathfrak{S}(R)$ by Proposition 9. As D_1 is an outer derivation, so in this case R belongs to the class $\mathfrak{D}$. When $Z(R_2)$ is not contained in $[R_2, R_2]$, we may decompose R into the direct sum of ideals such as $R = R_3 \oplus R_4$, where $Z(R_3) \subset [R_3, R_3]$ and R_4 is abelian. As the abelian Lie algebra R_4 belongs to class $\mathfrak{D}$, R also belongs to the class $\mathfrak{D}$ by the above-stated. Hence the proposition is proved.

It is known that the following Lie algebras belong to the class $\mathfrak{D}$.
        abelian Lie algebra
        nilpotent Lie algebra of dimension less than 6
        quasi-cyclic Lie algebra.
On the other hand, Leger [3] and Tôgô [6] proved the following:

PROPOSITION 13. *If a Lie algebra L possesses the non-zero center and has no outer derivations, then L is not solvable and its radical is nilpotent, and moreover $L = [L, L]$.*

By the above-stated, we can find many examples of Lie algebras with outer derivations. But there has not been found a Lie algebra with non-zero center which has no outer derivations. In the next section we shall give an example of such a Lie algebra. For the purpose of this we need to construct a nilpotent Lie algebra which does not belong to the class $\mathfrak{O}$.

4. Example of a nilpotent Lie algebra which does not belong to the class $\mathfrak{O}$.

THEOREM. *There exists a Lie algebra which has no outer derivation and has non-zero center.*

To prove the theorem, we shall construct the nilpotent Lie algebra N of dimension 38, and further the Lie algebra of dimension 41 whose radical is N. Let $x_1, x_2, \cdots, x_{38}$ be the basis of N, and let N be generated by the elements x_1, x_2, x_3 and x_4. The multiplication in N is given in the following table, but as for the multiplication which does not appear in the table, let it be commutative.

$$[x_1, x_2] = x_5 \qquad [x_1, x_4] = x_7 \qquad [x_2, x_4] = x_8$$
$$[x_1, x_3] = x_6 \qquad [x_2, x_3] = x_7 - x_5 \qquad [x_3, x_4] = x_5$$

$$[x_1, x_6] = x_9 \qquad [x_2, x_7] = x_{11} \qquad [x_3, x_8] = x_{15}$$
$$[x_1, x_7] = x_{10} \qquad [x_2, x_8] = x_{12} \qquad [x_4, x_6] = x_{14}$$
$$[x_1, x_8] = x_{11} \qquad [x_3, x_6] = x_{13} \qquad [x_4, x_7] = x_{15}$$
$$[x_2, x_6] = x_{10} \qquad [x_3, x_7] = x_{14} \qquad [x_4, x_8] = x_{16}$$

$$[x_1, x_9] = x_{17} \qquad [x_2, x_{11}] = x_{20} \qquad [x_4, x_{13}] = 30x_{35}$$
$$[x_1, x_{10}] = x_{18} \qquad [x_2, x_{12}] = x_{21} \qquad [x_4, x_{14}] = 20x_{36}$$
$$[x_1, x_{11}] = x_{19} \qquad [x_3, x_{13}] = 60x_{34} \qquad [x_4, x_{15}] = 15x_{37}$$
$$[x_1, x_{12}] = x_{20} \qquad [x_3, x_{14}] = 30x_{35} \qquad [x_4, x_{16}] = 12x_{38}$$
$$[x_2, x_9] = x_{18} \qquad [x_3, x_{15}] = 20x_{36}$$
$$[x_2, x_{10}] = x_{19} \qquad [x_3, x_{16}] = 15x_{37}$$

$$[x_1, x_{17}] = x_{22} \qquad [x_3, x_{18}] = x_{29} \qquad [x_6, x_{11}] = -x_{30}$$
$$[x_1, x_{18}] = x_{23} \qquad [x_3, x_{19}] = x_{30} \qquad [x_6, x_{12}] = -x_{31}$$
$$[x_1, x_{19}] = x_{24} \qquad [x_3, x_{20}] = x_{31} \qquad [x_7, x_9] = -x_{29}$$
$$[x_1, x_{20}] = x_{25} \qquad [x_3, x_{21}] = x_{32} \qquad [x_7, x_{10}] = -x_{30}$$
$$[x_1, x_{21}] = x_{26} \qquad [x_4, x_{17}] = x_{29} \qquad [x_7, x_{11}] = -x_{31}$$
$$[x_2, x_{17}] = x_{23} \qquad [x_4, x_{18}] = x_{30} \qquad [x_7, x_{12}] = -x_{32}$$
$$[x_2, x_{18}] = \text{x}_{24} \qquad [x_4, x_{19}] = x_{31} \qquad [x_8, x_9] = -x_{30}$$
$$[x_2, x_{19}] = x_{25} \qquad [x_4, x_{20}] = x_{32} \qquad [x_8, x_{10}] = -x_{31}$$
$$[x_2, x_{20}] = x_{26} \qquad [x_4, x_{21}] = x_{33} \qquad [x_8, x_{11}] = -x_{32}$$

$$[x_2, x_{21}] = x_{27} \qquad [x_6, x_9] = -x_{28} \qquad [x_8, x_{12}] = -x_{33}$$
$$[x_3, x_{17}] = x_{28} \qquad [x_6, x_{10}] = -x_{29}$$

$$[x_1, x_{29}] = x_{34} \qquad [x_3, x_{26}] = -x_{37} \qquad [x_7, x_{20}] = (1/2)x_{37}$$
$$[x_1, x_{30}] = x_{35} \qquad [x_3, x_{27}] = -x_{38} \qquad [x_7, x_{21}] = (4/5)x_{38}$$
$$[x_1, x_{31}] = x_{36} \qquad [x_4, x_{22}] = 5x_{34} \qquad [x_8, x_{17}] = -4x_{35}$$
$$[x_1, x_{32}] = x_{37} \qquad [x_4, x_{23}] = 2x_{35} \qquad [x_8, x_{18}] = -2x_{36}$$
$$[x_1, x_{33}] = x_{38} \qquad [x_4, x_{24}] = x_{36} \qquad [x_8, x_{19}] = -x_{37}$$
$$[x_2 . x_{28}] = -5x_{34} \qquad [x_4, x_{25}] = (1/2)x_{37} \qquad [x_8, x_{20}] = (-2/5)x_{38}$$
$$[x_2, x_{29}] = -2x_{35} \qquad [x_4, x_{26}] = (1/5)x_{38} \qquad [x_9, x_{10}] = -3x_{34}$$
$$[x_2, x_{30}] = -x_{36} \qquad [x_6, x_{18}] = 2x_{34} \qquad [x_9, x_{11}] = -3x_{35}$$
$$[x_2, x_{31}] = (-1/2)x_{37} \quad [x_6, x_{19}] = 2x_{35} \qquad [x_9, x_{12}] = -3x_{36}$$
$$[x_2, x_{32}] = (-1/5)x_{38} \quad [x_6, x_{20}] = 2x_{36} \qquad [x_{10}, x_{11}] = -x_{36}$$
$$[x_3, x_{23}] = -x_{34} \qquad [x_6, x_{21}] = 2x_{37} \qquad [x_{10}, x_{12}] = (-3/2)x_{37}$$
$$[x_3, x_{24}] = -x_{35} \qquad [x_7, x_{17}] = -4x_{34} \qquad [x_{11}, x_{12}] = (-3/5)x_{38}$$
$$[x_3, x_{25}] = -x_{36} \qquad [x_7, x_{18}] = -x_{35}$$

We set $U = \{x_1, x_2, x_3, x_4\}$. After the long and tedious calculation, we may verify that the multiplication stated above satisfies Jacobi's identities. Using the facts $U^7 = 0$, $[U^2, U^2] = 0$ and that x_5 belongs to the center of N, we can abridge the calculation a little.
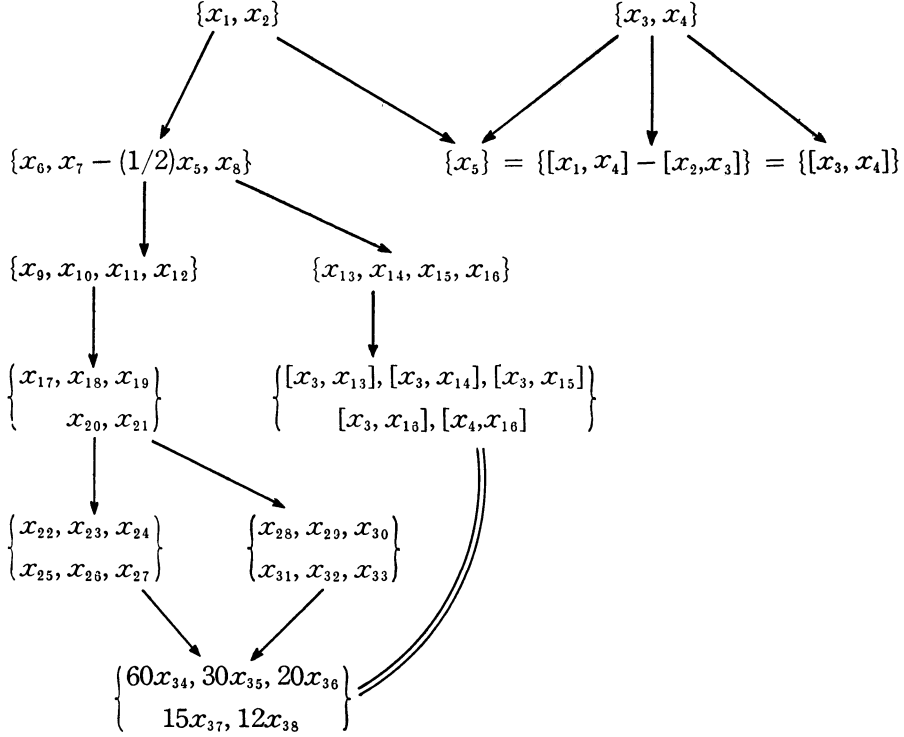
Now we take the following linear endomorphisms of U :

$$s_0 \; : \quad x_1 \to x_1, \; x_2 \to -x_2, \; x_3 \to x_3, \; x_4 \to -x_4$$

$$s_1 \; : \quad x_1 \to x_2, \; x_2 \to 0, \; x_3 \to x_4, \; x_4 \to 0$$

$$s_2 \; : \quad x_1 \to 0, \; x_2 \to x_1, \; x_3 \to 0, \; x_4 \to x_3$$

Then $\mathfrak{S} = \{s_0, s_1, s_2\}$ forms a simple Lie algebra. We are going to enlarge s_0, s_1, s_2 to derivations of N. This is possible because N is decomposed into the direct sum of the $\mathfrak{S}$-invariant irreducible modules as follows :

$$N = \{x_1, x_2\} \oplus \{x_3, x_4\} \oplus \{x_5\} \oplus \{x_6, x_7 - (1/2)x_5, x_8\} \oplus$$

$$\{x_9, x_{10}, x_{11}, x_{12}\} \oplus \{x_{13}, x_{14}, x_{15}, x_{16}\} \oplus \{x_{17}, x_{18}, x_{19}, x_{20}, x_{21}\}$$

$$\oplus \{x_{22}, x_{23}, x_{24}, x_{25}, x_{26}, x_{27}\} \oplus \{x_{28}, x_{29}, x_{30}, x_{31}, x_{32}, x_{33}\}$$

$$\oplus \{60x_{34}, 30x_{35}, 20x_{36}, 15x_{37}, 12x_{38}\}$$

In regard to operation of $\mathfrak{S}$, N has the structure indicated in the following diagram. Here let $\{ \; \}$ be an $\mathfrak{S}$-irreducible subspace, and we denote by $\longrightarrow$ process of generating ideals, and by $=\!=\!=$ identification of subspaces.

$$\{x_1, x_2\} \qquad \{x_3, x_4\}$$

$$\{x_6, x_7 - (1/2)x_5, x_8\} \qquad \{x_5\} = \{[x_1, x_4] - [x_2, x_3]\} = \{[x_3, x_4]\}$$

$$\{x_9, x_{10}, x_{11}, x_{12}\} \qquad \{x_{13}, x_{14}, x_{15}, x_{16}\}$$

$$\left\{ \begin{matrix} x_{17}, x_{18}, x_{19} \\ x_{20}, x_{21} \end{matrix} \right\} \qquad \left\{ \begin{matrix} [x_3, x_{13}], [x_3, x_{14}], [x_3, x_{15}] \\ [x_3, x_{16}], [x_4, x_{16}] \end{matrix} \right\}$$

$$\left\{ \begin{matrix} x_{22}, x_{23}, x_{24} \\ x_{25}, x_{26}, x_{27} \end{matrix} \right\} \qquad \left\{ \begin{matrix} x_{28}, x_{29}, x_{30} \\ x_{31}, x_{32}, x_{33} \end{matrix} \right\}$$

$$\left\{ \begin{matrix} 60x_{34}, 30x_{35}, 20x_{36} \\ 15x_{37}, 12x_{38} \end{matrix} \right\}$$

Now we will prove that the nilpotent Lie algebra N does not belong to the class $\mathfrak{O}$. Let D be a derivation of N commutable with the elements of $\mathfrak{S}$. There is no direct summand of N which is $\mathfrak{S}$-isomorphic to $\{x_1, x_2\}$ or $\{x_3, x_4\}$ but for themselves. Therefore by Schur's lemma, D must have the following form :

$$Dx_1 = \alpha x_1 + \gamma x_3 \qquad Dx_3 = \beta x_3 + \delta x_1$$

$$Dx_2 = \alpha x_2 + \gamma x_4 \qquad Dx_4 = \beta x_4 + \delta x_2$$

Then D acts on N as follows :

$$x_5 = [x_1, x_2] \rightarrow [\alpha x_1 + \gamma x_3, x_2] + [x_1, \alpha x_2 + \gamma x_4] = (2\alpha + \gamma)x_5$$

$$x_5 = [x_3, x_4] \rightarrow [\beta x_3 + \delta x_1, x_4] + [x_3, \beta x_4 + \delta x_2] = (2\beta + \delta)x_5$$

$$x_5 = [x_1, x_4] - [x_2, x_3] \rightarrow [\alpha x_1 + \gamma x_3, x_4] + [x_1, \beta x_4 + \delta x_2]$$

$$- [\alpha x_2 + \gamma x_4, x_3] - [x_2, \beta x_3 + \delta x_1] = (\alpha + \beta + 2\gamma + 2\delta)x_5$$

$$x_6 = [x_1, x_3] \rightarrow [\alpha x_1 + \gamma x_3, x_3] + [x_1, \beta x_3 + \delta x_1] = (\alpha + \beta)x_6$$

$$x_9 = [x_1, x_6] \rightarrow [\alpha x_1 + \gamma x_3, x_6] + [x_1, (\alpha + \beta)x_6]$$

$$= (2\alpha + \beta)x_9 + \gamma x_{13}$$

$$0 = [x_3, x_9] \rightarrow [\beta x_3 + \delta x_1, x_9] + [x_3, (2\alpha + \beta)x_9 + \gamma x_{13}]$$

$$= \delta x_{17} + 60\gamma x_{34}$$

Hence we have

$$2\alpha + \gamma = 2\beta + \delta = \alpha + \beta + 2\gamma + 2\delta$$

$$\gamma = \delta = 0,$$

and this implies that $\alpha = \beta$. Then,

$$x_{13} = [x_3, x_6] \rightarrow [\alpha x_3, x_6] + [x_3, 2\alpha x_6] = 3\alpha x_{13}$$

$$[x_3, x_{13}] \rightarrow [\alpha x_3, x_{13}] + [x_3, 3\alpha x_{13}] = 4\alpha[x_3, x_{13}]$$

$$x_{34} = [x_1, x_{29}] = [x_1, [x_3, [x_1, [x_1, x_4]]]] \rightarrow 6\alpha x_{34}.$$

As $[x_3, x_{13}] = 60x_{34}$, we get

$$\alpha = \beta = \gamma = \delta = 0, \qquad \text{namely} \quad D = 0.$$

Therefore a derivation of N which is commutable with the elements of $\mathfrak{S}$ is only 0, and this implies that N does not belong to $\mathfrak{O}$.

When we take a semi-direct sum $\mathfrak{S} + N$, it has a non-zero center $\{x_5\}$, and it has no outer derivations by Proposition 10. Hence the theorem is proved. We remark that $[\mathfrak{S} + N, \mathfrak{S} + N] = \mathfrak{S} + N$, which is a demand of Proposition 13.

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