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ON THE SEMIGROUP OF DIFFERENTIABLE MAPPINGS ON MONTEL SPACE

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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Let E be a locally convex Hausdorff space over the real number field R. Let $\mathscr{D}$ be the set of all continuous mappings which are Fréchet-differentiable at every point of E. If we define the product fg of two mappings $f, g: E \to E$ by the composition:

$$(fg)(x) = f(g(x)) \qquad \text{for every } x \in E\,,$$

then the set $\mathscr{D}$ is a semigroup.

A one-to-one mapping ϕ of $\mathscr{D}$ onto itself such that

$$(1) \qquad \phi(fg) = \phi(f)\phi(g) \qquad \text{for all } f, g \in \mathscr{D}$$

is called an *automorphism*. An automorphism is said to be *inner* if there is an invertible element $h \in \mathscr{D}$ such that

$$(2) \qquad \phi(f) = hfh^{-1} \qquad \text{for every } f \in \mathscr{D}\,.$$

The purpose of this paper is to prove the following theorem.

THEOREM. *Let E be a Montel space. Then, every automorphism of the semigroup $\mathscr{D}$ is inner.*

Preliminaries. Let E be a locally convex Hausdorff space over R and let $\mathscr{L}$ be the set of all continuous linear mappings of E into itself with the topology of uniform convergence on each bounded set. The real numbers will be denoted by Greek letters. A mapping $f: E \to E$ is said to be *Fréchet-differentiable at* $a \in E$ if there exists $u \in \mathscr{L}$ such that

$$f(a + x) - f(a) = u(x) + r(f, a, x) \qquad \text{for every } x \in E\,,$$

where the "*remainder*" $r(f, a, x)$ of f at a satifies the following condition:

$$(3) \qquad \lim_{\varepsilon \to 0} \varepsilon^{-1} r(f, a, \varepsilon x) = 0$$

uniformly with respect to x on each bounded set. The continuous linear mapping u is determined uniquely and is called the *Fréchet-derivative of f at a.* We denote it by $f'(a)$.

The proof of the chain rule which implies the fact that $\mathscr{D}$ is a semigroup with respect to the composition can be found in [1, p. 216]. It is known that every Fréchet-differentiable mapping is continuous if and only if the space E is sequential, a proof of which can be found in [2, p. 105].

The problem of the type considered here has been started by M. Eidelheit [3] who has proved that, when E is a real Banach space, every ring automorphism of the ring $\mathscr{L}$ is inner. This was improved in [11] to the effect that every semigroup automorphism of the semigroup $\mathscr{L}$ is inner.

On the other hand, K. D. Magill, Jr. has published a series of papers in which he has shown that there are many semigroups of functions and relations on topological spaces which have the property that every automorphism is inner. Let us say that a semigroup has the *Magill property* if every automorphism is inner. One of Magill's result [5], in which we have been especially interested, is that the semigroup $\mathscr{D}$ has the Magill property if E is one-dimensional. We still do not know whether or not $\mathscr{D}$ has the Magill property when E is an arbitrary real Banach space. In [8], we have given a necessary and sufficient condition for an automorphism of the semigroup $\mathscr{D}$ to be inner, and in [9] we have shown that $\mathscr{D}$ has the Magill property if E is finite-dimensional. In fact, in the latter paper, we have shown that, if E is a real Banach space, every automorphism ϕ of the semigroup $\mathscr{D}$ can be expressed in the form (2) where h satisfies the following condition:

$$\operatorname*{weak\text{-}lim}_{||x||\to 0} ||x||^{-1}r(h, a, x) = 0 \qquad \text{at every } a \in E\,.$$

Since ϕ is inner if this weak limit can be replaced by the strong limit, for any real Banach space in which the sequential weak convergence is equivalent to the strong convergence, the semigroup $\mathscr{D}$ has the Magill property. This fact has been used in [10] to prove that the semigroup of all Hadamard-differentiable mappings on any real Banach space has the Magill property. A mapping $f\colon E \to E$ is said to be *Hadamard-differentiable at* $a \in E$ if the convergence in (3) is uniform on each sequentially compact set. Since the weak and strong convergences for a sequence in a compact set coincide, this result is a natural consequence of the above mentioned result of [9].

Now, the motivation of this paper may be quite clear. A Montel space is a locally convex Hausdorff space which is barrelled and satisfies the condition that every bounded set is relatively compact. Therefore, we shall at first show that the arguments in [9] can be carried over to the case when E is a real locally convex Hausdorff space. Then, the

Fréchet-differentiability of h will follow from the compactness of bounded closed sets.

Since this is the first paper on the non-normed case, we shall make it self-contained by not depending on the previous papers. Throughout, let E be a locally convex Hausdorff space over R, $\bar{E}$ be the set of all continuous linear functionals on E and ϕ be an automorphism of the semigroup $\mathscr{D}$.

We shall start with the following lemma. A continuous mapping is said to be weakly Fréchet-differentiable at a point if it satisfies the condition of the Fréchet-differentiability given above in which the topology is replaced by the weak topology $\sigma(E, \bar{E})$.

LEMMA. *There is a bijection h such that h and h^{-1} are weakly Fréchet-differentiable and satisfy the condition* (2).

The proof is divided into nine steps.

Proof of Lemma. A mapping $f: E \rightarrow E$ is said to be constant if there is $a \in E$ such that $f(x) = a$ for every $x \in E$. We denote it by c_a. Then, it is obvious that

$$(4) \qquad c_a f = c_a \quad \text{and} \quad f c_a = c_{f(a)}$$

for any mapping $f: E \rightarrow E$. Since all constant mappings are continuous, Fréchet-differentiable and $c'_a(x) = 0$ for any $x \in E$, they belong to $\mathscr{D}$.

The following lemma is essentially due to Magill [5]. The case when E is a Banach space was treated in [7].

1. *There is a bijection $h: E \rightarrow E$ which satisfies* (2).

PROOF. For any $x \in E$ we define $h(x)$ by

$$(5) \qquad \phi(c_x) = c_{h(x)} .$$

To do this, we have to show that the images of constant mappings are again constant. Let us take an arbitray $y \in E$. Then, since ϕ is onto, there is $f \in \mathscr{D}$ such that

$$\phi(f) = c_y .$$

Then, since $c_y(z) = y$ for any $z \in E$, by (1) and (4),

$$\begin{aligned} \phi(c_x)(y) &= \phi(c_x)c_y(z) = \phi(c_x)\phi(f)(z) \\ &= \phi(c_x f)(z) = \phi(c_x)(z) , \end{aligned}$$

which means that $\phi(c_x)$ is constant and, hence, h can be defined.

To show that h is onto, let us take arbitrary $y \in E$ and $f \in \mathscr{D}$ such that $\phi(f) = c_y$. Then, since $f = \phi^{-1}(c_y)$, by the same method as above we see that f is constant, or $f = c_x$ for some $x \in E$, and, hence, $h(x) = y$.

To show that h is one-to-one, assume that $h(x) = h(y)$. Then, it follows from (5) that $\phi(c_x) = \phi(c_y)$, which implies that $c_x = c_y$, and, hence, $x = y$.

Thus, h is a bijection. Moreover, for any $f \in \mathscr{D}$, by (4) and (5),

$$\begin{aligned}\phi(f)(x) &= \phi(f)c_x(y) = \phi(fc_{h^{-1}(x)})(y)\\ &= \phi(c_{fh^{-1}(x)})(y) = c_{hfh^{-1}(x)}(y) = hfh^{-1}(x)\end{aligned}$$

for any $x \in E$ and $y \in E$, which implies (2).

Needless to say, a bijection is not always continuous. However, in our case, we can show that h is weakly continuous, which is enough for the following discussion. The following fact was proved by Magill [5] for the case when E is one-dimensional. The case when E is normed was treated in [8, p. 506].

2. *For any $\bar{a} \in \bar{E}$, the function $\langle h(x), \bar{a}\rangle$ of E into R is continuous with respect to $x \in E$*, where $\langle h(x), \bar{a}\rangle$ denotes the value of $\bar{a}$ at $h(x)$.

PROOF. We take arbitrary $a \in E$ and positive $\alpha \in R$ and show that there is an open set U such that $a \in U$ and

$$|\langle h(x), \bar{a}\rangle - \langle h(a), \bar{a}\rangle| < \alpha \quad \text{if} \quad x \in U .$$

Let $b \in E$ be an arbitrary non-zero element and consider the mapping $g: E \to E$ defined by

$$g(x) = \beta(\langle x - h(a), \bar{a}\rangle)b + h(a) ,$$

where $\beta: R \to R$ is a C^∞-function such that

$$\beta(\xi) = 0 \quad \text{if} \quad |\xi| \geqq \alpha; =1 \quad \text{if} \quad \xi = 0 .$$

Since g is a composition of continuous and Fréchet-differentiable mappings, g itself belongs to $\mathscr{D}$. Therefore, there is $f \in \mathscr{D}$ such that $\phi(f) = g$. Assume that $f(a) = a$. Then, by (2),

$$h(a) = hf(a) = gh(a) = b + h(a) ,$$

which contradicts the assumption that $b \neq 0$. Therefore, $f(a) \neq a$, and there is an open set V such that $f(a) \in V$ and $a \notin V$. Since f is continuous, there is an open set U such that $a \in U$, and $f(x) \in V$ whenever $x \in U$. Thus, $x \in U$ implies $f(x) \neq a$, which, since h is one-to-one, is equivalent to $gh(x) \neq h(a)$, which, by the definition of g, is equivalent to

$$|\langle h(x) - h(a), \bar{a}\rangle| < \alpha .$$

Thus, the proof is completed.

If E is one-dimensional, the above two facts imply that h is a homeomorphism of E onto E. Then, since it should be monotone, it is differentiable almost everywhere. This is the fact Magill [5] has used to prove that $h \in \mathscr{D}$. Therefore, it may be important to observe here that this phenomenon is exceptional. In fact, as we have shown in [9], if the dimension of E is not less than two, there is a homeomorphism of E onto itself which is not differentiable at any point in any direction. In other words, if h is differentiable, it is entirely because of the relation (2).

In the following, we shall assume that $h(0) = 0$. This does not restrict the generality, because, if $h(0) \neq 0$, we only have to consider $h - c_{h(0)}$ instead of h. As usual, (c_0) stands for the set of all sequences $\{\varepsilon_n\} \subset R$ such that $\lim_{n\to\infty} \varepsilon_n = 0$.

3. *For any non-zero* $a \in E$ *and* $\{\varepsilon_n\} \in (c_0)$, *the sequence* $\{\varepsilon_n^{-1} h(\varepsilon_n a)\}$ *does not converge weakly to zero.*

PROOF. Let us assume that there exist non-zero $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} \varepsilon_n^{-1} \langle h(\varepsilon_n a), \bar{x}\rangle = 0 \qquad \text{for any } \bar{x} \in \bar{E}\,.$$

Then, we shall show that, for any $\xi \in R$ and any $\eta \in R$,

$$(6) \qquad \lim_{n\to\infty} \varepsilon_n^{-1}\langle h(\xi a + \varepsilon_n \eta a) - h(\xi a), \bar{x}\rangle = 0 \qquad \text{for any } \bar{x} \in \bar{E}\,.$$

To show this, we consider the mapping $f \in \mathscr{D}$ defined by

$$f = \xi c_a + \eta\,,$$

where we denoted by η the mapping $x \to \eta x$, which obviously belongs to $\mathscr{D}$. Then,

$$f(\varepsilon_n a) = \xi a + \varepsilon_n \eta a \quad \text{and} \quad f(0) = \xi a\,,$$

and, hence, by (3),

$$\begin{aligned}
&\varepsilon_n^{-1}[h(\xi a + \varepsilon_n \eta a) - h(\xi a)] \\
&\quad = \varepsilon_n^{-1}[hf(\varepsilon_n a) - hf(0)] \\
&\quad = \varepsilon_n^{-1}[\phi(f)h(\varepsilon_n a) - \phi(f)h(0)] \\
&\quad = \varepsilon_n^{-1}[\phi(f)'(0)(h(\varepsilon_n a)) + r(\phi(f), 0, h(\varepsilon_n a))] \\
&\quad = \phi(f)'(0)(\varepsilon_n^{-1} h(\varepsilon_n a)) + \varepsilon_n^{-1} r(\phi(f), 0, h(\varepsilon_n a))\,,
\end{aligned}$$

where, since $\phi(f)'(0) \in \mathscr{L}$, we have

$$\lim_{n\to\infty} \langle \phi(f)'(0)(\varepsilon_n^{-1} h(\varepsilon_n a)), \bar{x}\rangle = 0 \qquad \text{for every } \bar{x} \in \bar{E}\,.$$

Moreover, since the set $\{\varepsilon_n^{-1} h(\varepsilon_n a)\}$ is bounded, if we write the remainder in the following form

$$r(\phi(f),\, 0,\, \varepsilon_n(\varepsilon_n^{-1}h(\varepsilon_n a)))\ ,$$

the condition (3) of the Fréchet-differentiability of $\phi(f)$ at 0 implies that

$$\lim_{n\to\infty} \varepsilon_n^{-1} r(\phi(f),\, 0,\, h(\varepsilon_n a)) = 0\ .$$

Thus, the proof of (6) is completed.

Now, let us consider the function $\lambda_{\bar{x}}\colon R \to R$ defined by

$$\lambda_{\bar{x}}(\xi) = \langle h(\xi a), \bar{x}\rangle \quad \text{for} \quad \xi \in R \quad \text{and} \quad \bar{x} \in \bar{E}\ .$$

Then, this function satisfies the following three conditions:

(i) it is continuous;

(ii) $\lambda_{\bar{x}}(0) = 0$;

(iii) there is $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} \varepsilon_n^{-1}[\lambda_{\bar{x}}(\xi + \varepsilon_n \eta) - \lambda_{\bar{x}}(\xi)] = 0$$

for any $\xi \in R$ and any $\eta \in R$.

From these three conditions, by a simple calculation similar to the proof of the Rolle's theorem, it follows that the function $\lambda_{\bar{x}}$ is identically zero. Since $\bar{x} \in \bar{E}$ is arbitrary, this implies $h(\xi a) = 0$ for any $\xi \in R$, which is impossible because h is one-to-one, as was shown in the step 1.

4. *For any $a \in E$ and $\{\varepsilon_n\} \in (c_0)$, the set $\{\varepsilon_n^{-1}h^{-1}(\varepsilon_n a)\}$ is bounded.*

PROOF. Let us assume that there is $\{\varepsilon_n\} \in (c_0)$ such that the set $\{\varepsilon_n^{-1}h^{-1}(\varepsilon_n a)\}$ is not bounded. Then,

$$\lim_{n\to\infty} \langle \varepsilon_n^{-1}h^{-1}(\varepsilon_n a), \bar{a}\rangle = +\infty \tag{7}$$

for some $\bar{a} \in \bar{E}$. We put

$$\delta_n = \langle h^{-1}(\varepsilon_n a), \bar{a}\rangle \qquad (n = 1, 2, \cdots)\ .$$

By the weak continuity proved in the step 2, $\lim_{n\to\infty} \delta_n = 0$. (If we start with ϕ^{-1} instead of ϕ, then, by exactly the same arguments, we have the corresponding properties of h^{-1}. We shall make free use of this fact.) We can assume that all δ_n are non-zeros.

Now, we consider the mapping $a \otimes \bar{a} \in \mathscr{L} \subset \mathscr{D}$ which is defined by

$$(a \otimes \bar{a})(x) = \langle x, \bar{a}\rangle a \qquad \text{for every } x \in E\ .$$

Since $\phi(a \otimes \bar{a}) \in \mathscr{D}$, the following limit exists:

$$\begin{aligned}\phi(a \otimes \bar{a})'(0)(a) &= \lim_{n\to\infty} \varepsilon_n^{-1}\phi(a \otimes \bar{a})(\varepsilon_n a) \\ &= \lim_{n\to\infty} (\varepsilon_n^{-1}\langle h^{-1}(\varepsilon_n a), \bar{a}\rangle)(\delta_n^{-1}h(\delta_n a))\ .\end{aligned}$$

Then, it follows from (7) that

$$\lim_{n\to\infty} \delta_n^{-1} h(\delta_n a) = 0 ,$$

which contradicts the conclusion of the step 3.

5. *For any $a \in E$ and $\{\varepsilon_n\} \in (c_0)$, there is a subsequence $\{\varepsilon_{n_k}\}$ such that $\{\varepsilon_{n_k}^{-1} h(\varepsilon_{n_k} a)\}$ is convergent.*

PROOF. We can take $\bar{a} \in \bar{E}$ such that

$$\phi(a \otimes \bar{a})'(0)(a) \neq 0 . \tag{8}$$

To see this, let us assume that

$$\phi(a \otimes \bar{x})'(0)(a) = 0 \qquad \text{for any } \bar{x} \in \bar{E} .$$

We take a sequence $\{\delta_n\} \in (c_0)$ such that $\delta_n \neq 0$ for every n. Let M be the set of all $\bar{x} \in \bar{E}$ such that the sequence $\{\langle h^{-1}(\delta_n a), \bar{x}\rangle\}$ contains infinite non-zero members.

If $\bar{x} \notin M$, then, obviously, the sequence $\{\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{x}\rangle\}$ converges to zero.

If $\bar{x} \in M$, we have, by the assumption,

$$\begin{aligned} 0 &= \lim_{n\to\infty} \delta_n^{-1} \phi(a \otimes \bar{x})(\delta_n a) \\ &= \lim_{n\to\infty} (\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{x}\rangle)(\tau_n^{-1} h(\tau_n a)) , \end{aligned} \tag{9}$$

where $\tau_n = \langle h^{-1}(\delta_n a), \bar{x}\rangle$. If the sequence $\{\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{x}\rangle\}$ does not converge to zero, then there is a subsequence $\{n_i\}$ such that

$$|\delta_{n_i}^{-1}\langle h^{-1}(\delta_{n_i} a), \bar{x}\rangle| \geqq \gamma > 0$$

for some $\gamma \in R$. Then, from (9), the sequence $\{\tau_{n_i}^{-1} h(\tau_{n_i} a)\}$ has to converge to zero. This contradicts the conclusion of the step 3.

Therefore, for any $\bar{x} \in \bar{E}$, the sequence $\{\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{x}\rangle\}$ converges to zero. This, again, contradicts the conclusion of the step 3.

Thus, the existence of $\bar{a} \in \bar{E}$ which satisfies (8) has been proved. We note here that, if $\bar{a}$ satisfies (8), then $-\bar{a}$ also satisfies the same condition. This can be proved as follows. Since $a \otimes (-\bar{a}) = -(a \otimes \bar{a})$,

$$\phi(a \otimes (-\bar{a}))'(0) = \phi((-1)(a \otimes \bar{a}))'(0) = \phi(-1)'(0)\phi(a \otimes \bar{a})'(0) .$$

However, $\phi(-1)'(0)$ is a bijection; because

$$\phi(-1)'(0)\phi(-1)'(0) = \phi(1)'(0) = 1 .$$

Therefore, $\phi(a \otimes \bar{a})'(0)(a) \neq 0$ if and only if $\phi(a \otimes (-\bar{a}))'(0)(a) \neq 0$.

Now, (8) means that

$$0 \neq \lim_{\delta \to 0}(\delta^{-1}\langle h^{-1}(\delta a), \bar{a}\rangle)(\langle h^{-1}(\delta a), \bar{a}\rangle^{-1}h(\langle h^{-1}(\delta a), \bar{a}\rangle a)\ .$$

It is clear from this relation that the continuous function $\langle h^{-1}(\delta a), \bar{a}\rangle$ of $\delta \in R$ takes non-zero values in any small neighbourhood of zero in R, because, if this is not the case, $\phi(a \otimes \bar{a})'(0)(a)$ has to be zero. Therefore, there is a sequence $\{\delta_n\} \in (c_0)$ and n_0 such that

$$\langle h^{-1}(\delta_n a), \bar{a}\rangle = \varepsilon_n \quad \text{or} \quad -\varepsilon_n \quad \text{if} \quad n \geqq n_0\ .$$

By taking a subsequence of $\{\varepsilon_n\}$ and replacing $\bar{a}$ by $-\bar{a}$ if necessary, we can assume that

$$\langle h^{-1}(\delta_n a), \bar{a}\rangle = \varepsilon_n \qquad \text{for all } n\ .$$

Then, we have

$$0 \neq \lim_{n \to \infty}(\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{a}\rangle)(\varepsilon_n^{-1}h(\varepsilon_n a))\ .$$

Since the sequence $\{\delta_n^{-1}\langle h^{-1}(\delta_n a), \bar{a}\rangle\}$ is bounded by the step 4, there is a subsequence $\{\delta_{n_k}\}$ of $\{\delta_n\}$ such that the limit

$$\lim_{k \to \infty} \delta_{n_k}^{-1}\langle h^{-1}(\delta_{n_k} a), \bar{a}\rangle = \alpha$$

exists. Since $\{\varepsilon_{n_k}^{-1}h(\varepsilon_{n_k} a)\}$ is bounded, α is not zero. Therefore, the following limit exists:

$$\lim_{k \to \infty} \varepsilon_{n_k}^{-1}h(\varepsilon_{n_k} a) = \alpha^{-1}\phi(a \otimes \bar{a})'(0)(a)\ .$$

6. *The limit* $\lim_{\varepsilon \to 0} \varepsilon^{-1}h(\varepsilon a)$ *exists for any* $a \in E$.

PROOF. We can assume that $a \neq 0$. At first, we shall show that, for any $\bar{a} \in \bar{E}$, the function $\lambda: R \to R$ defined by

$$\lambda(\xi) = \langle h(\xi a), \bar{a}\rangle$$

is differentiable almost everywhere. We need to show that none of the Dini derivatives of λ can not be infinite ([6, p. 270]). Now, assume that, for instance, the upper right-hand derivative is not finite at $\alpha \in R$, which implies that

$$\overline{\lim_{\varepsilon \to +0}}\, \varepsilon^{-1}|\lambda(\alpha + \varepsilon) - \lambda(\alpha)| = +\infty\ .$$

That this is impossible can been shown as follows. For any $\{\varepsilon_n\} \in (c_0)$,

$$\begin{aligned}
&\varepsilon_n^{-1}[h(\alpha a + \varepsilon_n a) - h(\alpha a)] \\
&\quad = \varepsilon_n^{-1}[h(\alpha c_a + 1)(\varepsilon_n a) - h(\alpha c_a + 1)(0)] \\
&\quad = \varepsilon_n^{-1}[\phi(\alpha c_a + 1)h(\varepsilon_n a) - \phi(\alpha c_a + 1)h(0)] \\
&\quad = \phi(\alpha c_a + 1)'(0)(\varepsilon_n^{-1}h(\varepsilon_n a)) + \varepsilon_n^{-1}r(\phi(\alpha c_a + 1), 0, h(\varepsilon_n a))\ .
\end{aligned}$$

By the step 4, the set $\{\varepsilon_n^{-1}h(\varepsilon_n a)\}$ is bounded, and, hence, by the condition (3), the second member of the last formula converges to zero as $n \to \infty$. Since $\phi(\alpha c_a + 1)'(0)$ is a continuous linear mapping, the first member is contained in a bounded set. Therefore, the set

$$\{\varepsilon_n^{-1}[h(\alpha a + \varepsilon_n a) - h(\alpha a)] \quad | n = 1, 2, \cdots\}$$

is bounded, and, hence, the set

$$\{\varepsilon_n^{-1} | \lambda(\alpha + \varepsilon_n) - \lambda(\alpha) | \quad | n = 1, 2, \cdots\}$$

is also bounded. Thus, this Dini derivative can not be infinite.

Now, we return to the proof of the statement of this step. In view of the conclusion of the step 5, what we have to show is the following: if there are $\{\varepsilon_n\} \in (c_0)$ and $\{\delta_n\} \in (c_0)$ such that

$$\lim_{n\to\infty} \varepsilon_n^{-1}h(\varepsilon_n a) = a_1 \quad \text{and} \quad \lim_{n\to\infty} \delta_n^{-1}h(\delta_n a) = a_2 ,$$

then $a_1 = a_2$.

At first, we see from the step 3 that $h(\xi a)$ is continuous with respect to ξ at the point $\xi = 0$. By the same technique as that we have used above, this point of continuity can be moved to any other point. Hence, this is a continuous mapping of a separable space R into E. Therefore, the smallest closed linear subspace containing the set $\{h(\xi a) | \xi \in R\}$ is also separable. Since the following argument is carried out entirely in this subspace, we may assume that E itself is separable.

Let M be any equicontinuous subset of $\bar{E}$. Then, by [4, p. 259], M is weakly sequentially separable, which means that every element of M is the limit of a subsequence of a fixed sequence $\{\bar{x}_i\}$ of elements of M. We consider the functions:

$$\lambda_i(\xi) = \langle h(\xi a), \bar{x}_i \rangle \qquad (i = 1, 2, \cdots) .$$

Since each λ_i are differentiable almost everywhere, there exists $\alpha \in R$ such that all λ_i are differentiable at α, i.e., the limits

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[\lambda_i(\alpha + \varepsilon) - \lambda_i(\alpha)]$$

exist for all i. On the other hand, we have

$$\lim_{n\to\infty} \varepsilon_n^{-1}[h(\alpha a + \varepsilon_n a) - h(\alpha a)] = \phi(\alpha c_a + 1)'(0)(a_1)$$

and

$$\lim_{n\to\infty} \delta_n^{-1}[h(\alpha a + \delta_n a) - h(\alpha a)] = \phi(\alpha c_a + 1)'(0)(a_2) .$$

Therefore,

$$\langle\phi(\alpha c_a + 1)'(0)(a_1), \bar{x}_i\rangle = \langle\phi(\alpha c_a + 1)'(0)(a_2), \bar{x}_i\rangle$$

for all i, from which it follows that

$$\langle\phi(\alpha c_a + 1)'(0)(a_1), \bar{x}\rangle = \langle\phi(\alpha c_a + 1)'(0)(a_2), \bar{x}\rangle$$

for all $\bar{x} \in M$. Since the class of all equicontinuous subsets is total ([4, p. 258]), we have

$$\phi(\alpha c_a + 1)'(0)(a_1) = \phi(\alpha c_a + 1)'(0)(a_2) .$$

On the other hand, since $(1 - \alpha c_a)(\alpha c_a + 1) = 1$, we have

$$\phi(1 - \alpha c_a)'(h(\alpha a))\phi(\alpha c_a + 1)'(0) = 1 ,$$

which means that $\phi(\alpha c_a + 1)'(0)$ is one-to-one. Therefore, $a_1 = a_2$.

We denote this limit by $h^*(0)(a)$.

7. *For any* $a \otimes \bar{a}$, $h(a \otimes \bar{a}) \in \mathscr{D}$.

PROOF. Since

$$\varepsilon^{-1}[h(a\otimes\bar{a})(\varepsilon x) - h(a\otimes\bar{a})(0)] = \varepsilon^{-1}h(\varepsilon\langle x, \bar{a}\rangle a) ,$$

it follows from the step 6 that the limit, as $\varepsilon \to 0$, exists and it is

$$\langle x, \bar{a}\rangle h^*(0)(a) ,$$

which is obviously linear and continuous with respect to x. In order to prove the Fréchet-differentiability at 0, we have to show that the condition (3) is satisfied. Let B be a bounded set. For $x \in B$ such that $\langle x, \bar{a}\rangle = 0$, the remainder is zero. Therefore, we have only to consider $x \in B$ such that $\langle x, \bar{a}\rangle \neq 0$. If the remainder divided by ε:

$$\begin{aligned}\varepsilon^{-1}r(h(a\otimes\bar{a}), 0, \varepsilon x) &= \varepsilon^{-1}h(a\otimes\bar{a})(\varepsilon x) - \langle x, \bar{a}\rangle h^*(0)(a)\\ &= \langle x, \bar{a}\rangle[(\varepsilon\langle x, \bar{a}\rangle)^{-1}h(\varepsilon\langle x, \bar{a}\rangle a) - h^*(0)(a)]\end{aligned}$$

is not uniformly convergent to zero on B, there are an open set U containing 0, a sequence $\{x_n\} \subset B$ and a sequence $\{\varepsilon_n\} \in (c_0)$ such that

$$\langle x_n, \bar{a}\rangle[(\varepsilon_n\langle x_n, \bar{a}\rangle)^{-1}h(\varepsilon_n\langle x_n, \bar{a}\rangle a) - h^*(0)(a)] \notin U .$$

However, since the sequence $\{\langle x_n, \bar{a}\rangle\}$ is bounded, it follows from the step 6 that the left-hand side is convergent to zero, which is a contradiction.

Now that $h(a\otimes\bar{a})$ is Fréchet-differentiable at 0, the mapping $\phi(c_x + 1)h(a\otimes\bar{a})$ is also Fréchet-differentiable at 0 for any $x \in E$, and

$$(\phi(c_x + 1)h(a\otimes\bar{a}))'(0)(y) = \phi(c_x + 1)'(0)[\langle y, \bar{a}\rangle h^*(0)(a)] .$$

This means that the convergence

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}[h(x + \varepsilon\langle y, \bar{a}\rangle a) - h(x) - \varepsilon\langle y, \bar{a}\rangle\phi(c_x + 1)'(0)h^*(0)(a)] = 0$$

is uniform with respect to y on any bounded set. If we replace x in this formula by $\langle x, \bar{a}\rangle a$, this means that $h(a\otimes\bar{a})$ is Fréchet-differentiable at x, which is an arbitrary element of E. Therefore, $h(a\otimes\bar{a})\in\mathscr{D}$.

8. $h^*(0)$ *is linear and continuous in the weak topology.*

PROOF. Since

$$(a\otimes\bar{a})h = \phi^{-1}(h(a\otimes\bar{a}))\in\mathscr{D}\ ,$$

the convergence:

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}r((a\otimes\bar{a})h,\ 0,\ \varepsilon x) = 0$$

is uniform with respect to x in any bounded set. Therefore,

$$((a\otimes\bar{a})h)'(0)(x) = \langle h^*(0)(x),\ \bar{a}\rangle a \quad \text{for any} \quad x\in E\ . \tag{10}$$

Since a and $\bar{a}$ are arbitrary, the linearity of $h^*(0)$ is obvious, and, since any net convergent weakly to zero is mapped by $((a\otimes\bar{a})'(0)$ to a net which is also weakly convergent to zero, the relation (10) implies that $h^*(0)$ is continuous in the weak topology.

We put

$$r(h,\ 0,\ x) = h(x) - h^*(0)(x)\ .$$

9. *For any* $\bar{x}\in\bar{E}$ *and any bounded set* B,

$$\lim_{\varepsilon\to 0}\sup_{x\in B}|\langle\varepsilon^{-1}r(h,\ 0,\ \varepsilon x),\ \bar{x}\rangle| = 0\ .$$

PROOF. Since

$$r((a\otimes\bar{a})h,\ 0,\ x) = (a\otimes\bar{a})r(h,\ 0,\ x)\ ,$$

it follows from (10) that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}(a\otimes\bar{a})r(h,\ 0,\ \varepsilon x) = 0$$

is uniform with respect to x in any bounded set, and this is exactly what we needed.

Thus, we have proved that h is Fréchet-differentiable at 0 with respect to the weak topology. Then, by the same method as that we have used before, we can show that this point of differentiablility can be moved to any other point, and the proof of this lemma is completed.

PROOF OF THEOREM Let E be a Montel space over R. We need to show that, if $\varepsilon\to 0$, $\{\varepsilon^{-1}r(h,\ 0,\ \varepsilon x)\}$ converges to zero uniformly with respect to x on each bounded set. Let us assume that this is not the case. Then, we have a neighbourhood U of 0, a bounded set B, a sequence $\{\varepsilon_n\}\in(c_0)$

and a sequence $\{x_n\} \subset B$ such that

$$\varepsilon_n^{-1} r(h, 0, \varepsilon_n x_n) \notin U .$$

However, since every weakly convergent sequence in a Montel space is strongly convergent to the same limit ([4, p. 370]), this contradicts the conclusion of the above lemma.

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