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DENSITY PROPERTIES OF HAUSDORFF MOMENT SEQUENCES

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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Let $d_n = \int_0^1 t^n d\chi(t)$, $\chi \in V[0, 1]$ be a moment sequence. It is shown that $d_{n_k} = O(c^{n_k})$ $(0 < c < 1$, n_k naturals) and $\sum 1/n_k = \infty$ implies $d_n = O(c^n)$ for all n, and hence $d_n = \int_0^c t^n d\chi(t)$.

1. Let $\chi \in V[0, 1]$ ($V[a, b]$ denotes the space of functions of bounded variation on $[a, b]$) and assume that $\chi(t+0) = \chi(t)$ for $t \in [0, 1)$. In what follows we use the notation

$$d_n = \int_0^1 t^n d\chi(t) \qquad n = 0, 1, 2, \cdots . \tag{1}$$

We will call

$$\rho = \inf\{v : \chi(t) = \chi(1),\ v \leqq t \leqq 1\} \tag{2}$$

the order of $\chi(t)$ and also the order of the d_n. The function

$$D(z) = \sum_0^\infty d_n z^n = \int_0^1 d\chi(t)/(1 - zt) \tag{3}$$

is regular in the complex plane apart from a cut from $1/\rho$ to ∞, and the integral representation holds in this region. Furthermore, if δ is the distance between z and the line $[1/\rho, \infty)$, we have

$$D(z) = O(\delta^{-1}) , \qquad \delta \rightarrow 0 . \tag{4}$$

Hardy [2] p. 267 states the following theorem: If $d_n = O(c^n)$ for some $0 < c < 1$, then d_n is at most of the order c.

A sharper result follows from a theorem of Pòlya [5] p. 777, see also Bieberbach [1] p. 78 (but none of the authors gives a prove):

Let $f(z) = \sum_0^\infty a_n z^n$ be analytic in the circle $|z| < a$, $a > 1$, apart from the line $[1, a]$ and let $f(z) = O(\exp(\delta^{-\alpha}))$, $\delta \rightarrow 0$, $\alpha > 0$, where δ is the distance between z and the line $[1, a]$. Let $a_{n_k} = O(c^{n_k})$ for some c, $0 < c < 1$, and a subsequence n_k. Then either $f(z)$ is analytic at $z = 1$ or n_k has density zero (i.e. $\lim k/n_k = 0$). Applying this result to the function $D(z)$, we immediately find

THEOREM 1. *If*

$$d_{n_k} = O(c^{n_k}) , \qquad 0 < c < 1 , \tag{5}$$

holds for a subsequence n_k, then either

$$\lim_{k\to\infty} k/n_k = 0 \tag{6}$$

or d_n is at most of order c.

In this paper we will show (Theorem 2) that (6) can be replaced by the stronger statement

$$\sum_0^\infty 1/n_k < \infty , \tag{7}$$

which is best possible. This result can be also applied to Laplace transforms (Theorem 3).

2. LEMMA. *Let n_k, $k = 1, 2 \cdots$, be integers with the properties $0 < n_k < n_{k+1}$,*

$$\sum_1^\infty 1/n_k = \infty . \tag{8}$$

Then there exists a subsequence n_{k_j} with density zero, i.e.

$$\lim_{j\to\infty} j/n_{k_j} = 0 , \tag{9}$$

such that

$$\sum_{j=1}^\infty 1/n_{k_j} = \infty . \tag{10}$$

PROOF. We may assume that

$$\limsup k/n_k = a > 0 . \tag{11}$$

If $\eta(m)$ is defined for $m = 1, 2, \cdots$, and if $\lim_{m\to\infty} \eta(m) = 0$, then Toeplitz' theorem implies

$$\lim_{M\to\infty} 2^{-M} \sum_1^M 2^m \eta(m) = 0 . \tag{12}$$

Let

$$I_m = (2^m + 1, 2^{m+1}) , \qquad m = 1, 2, \cdots . \tag{13}$$

It follows from (8) that infinitely many I_{m_l} exist such that the number of terms $n_k \in I_{m_l}$ is at least $\varepsilon \cdot 2^{m_l}$ for some $\varepsilon > 0$. Otherwise, the number of terms would be $\eta(m) \cdot 2^m$ with $\eta(m) \to 0$, and (12) shows that the density of n_k is zero in violation of (11).

We now take from each interval I_{m_l} the first $[\varepsilon\cdot 2^{m_l}/l]$ terms and define the subsequence n_{k_j} as the union of these terms. It follows that

$$\sum_{j=1}^{\infty} 1/n_{k_j} \geqq \sum_{l=1}^{\infty} (\varepsilon\cdot 2^{m_l}/l)\cdot(1/2^{m_l+1}) = \infty$$

which is (10). Using (12) we finally get

$$\lim_{j\to\infty} j/n_{k_j} \leqq \lim_{L\to\infty} 2^{-m_L}\sum_{l=1}^{L} \varepsilon\cdot 2^{m_l}/l = 0\ ,$$

i.e. n_{k_j} has density zero.

THEOREM 2. *Let d_n be Hausdorff moments. If*

$$(5)\qquad d_{n_k} = O(c^{n_k})\ ,\qquad 0 < c < 1$$

holds for a subsequence n_k, then either d_n is at most of order c or

$$(7)\qquad \sum_{k=1}^{\infty} 1/n_k < \infty\ .$$

PROOF. Assume that

$$(14)\qquad \sum_{k=1}^{\infty} 1/n_k = \infty\ .$$

Without loss of generality we may assume that

a) n_k has density zero (by the preceding lemma)

b) $n_0 = 0$, $n_1 = 1$ (otherwise add these terms to the sequence n_k)

c) $d\chi(t) = \phi(t)dt$ with $\phi(t)$ continuous on $[0, 1]$ (otherwise take the moment sequence $d'_n = d_{n+2}/((n+2)(n+1))$)

d) the order of $\chi(t)$ is $\rho = 1$ (otherwise take $d''_n = d_n/\rho^n$).

We define the functions

$$(15)\qquad \phi_b(t) = \begin{cases} 0 & 0 \leqq t < b \\ \phi(t) & b \leqq t \leqq 1 \end{cases}.$$

Next we construct the generalized Bernstein polynomials (Lorentz [4] p. 44, Gelfand [3] p. 71) associated to the sequence n_k

$$(16)\qquad p_{k\nu}(t) = (-1)^{k-\nu}\sum_{\mu=\nu}^{k} t^{n_\mu}(n_\mu/n_\nu)\prod_{\substack{j=\nu\\ j\neq\mu}}^{k} n_j/(n_\mu - n_j)\ .$$

The condition (14) guarantees that for any function $f(t)$ continuous on $(a, b)\subset[0, 1]$ and bounded on $[0, 1]$ we have

$$(17)\qquad \sum_{\nu=0}^{k} p_{k\nu}(t)f(\tau_{k\nu}) \longrightarrow f(t)\quad\text{for}\quad k\longrightarrow\infty$$

uniformly on every compact subset of (a, b), where

$$\begin{cases} \tau_{k\nu} = \prod_{j=\nu+1}^{k} (1 - 1/n_j) & 0 \leqq \nu < k \\ \tau_{kk} = 1 \,. \end{cases} \tag{18}$$

We apply the approximation (17) to the function $\phi_b(t)$. The sequence of functions

$$\phi_b^k(t) = \sum_{\nu=0}^{k} p_{k\nu}(t)\phi_b(\tau_{k\nu}) \tag{19}$$

converges to $\phi_b(t)$ uniformly for t outside of an arbitrary small neighbourhood of b. Furthermore, since the $\phi_b^k(t)$ are uniformly bounded on $[0, 1]$ we have

$$\sigma_k = \left|\int_0^1 \phi_b^k(t)\phi(t)dt\right| \rightarrow \int_0^1 \phi_b(t)\phi(t)dt = \int_b^1 \phi^2(t)dt \,. \tag{20}$$

In what follows we will show that

$$\sigma_k = o(1) \tag{21}$$

holds, if b is sufficiently close to 1, which implies $\phi(t) = 0$ on $[b, 1]$. This is a contradiction to the defition of the order ρ of $\chi(t)$.

Let $\nu_0 = \nu_0(k)$ be the minimum of the numbers ν with $\tau_{k\nu} \geqq b$. Then we have

$$\begin{aligned} \sigma_k &= \left|\sum_{\nu=\nu_0}^{k} \int_0^1 \phi_b(\tau_{k\nu}) p_{k\nu}(t)\phi(t)dt\right| \\ &\leqq C \sum_{\nu=\nu_0}^{k} \sum_{\mu=\nu}^{k} |d_{n_\mu}|(n_\mu/n_\nu) \prod_{\substack{j=\nu \\ j\neq\mu}}^{k} n_j/|n_j - n_\mu| \\ &\leqq C \sum_{\nu=\nu_0}^{k} \sum_{\mu=\nu}^{k} c^{n_\mu}(n_\mu/n_\nu) \prod_{\substack{j=\nu \\ j\neq\mu}}^{k} n_j/|n_j - n_\mu| \,. \end{aligned}$$

Let $j_1 = j_1(\mu)$ be the maximum of numbers j with $n_j \leqq 2n_\mu$ and $j_0 = j_0(\mu)$ be the minimum of numbers $j \geqq \nu$ with $n_j \geqq n_\mu/2$, then

$$\begin{aligned} \prod_{\substack{j=\nu \\ j\neq\mu}}^{k} n_j/|n_j - n_\mu| &\leqq \prod_{\substack{j=j_0 \\ j\neq\mu}}^{j_1} n_j/|n_j - n_\mu| \cdot \prod_{j=j_1}^{k} n_j/|n_j - n_\mu| = \Pi_1 \cdot \Pi_2 \,, \\ \Pi_1 &\leqq (2n_\mu)^{j_1-j_0}/(\Gamma(j_1 - j_0)/2 + 1)^2 \\ &\leqq [2n_\mu \cdot 4e/(j_1 - j_0)]^{(j_1-j_0)} \,. \end{aligned}$$

Since the sequence n_k has density zero, we have for large k

$$\Pi_1^{1/n_\mu} \leqq [8e \cdot n_\mu/(j_1 - j_0)]^{(j_1-j_0)/n_\mu} \leqq 1 + \varepsilon \,. \tag{22}$$

Furthermore we have

$$\Pi_2 = \prod_{j=j_1}^{k} |1 - n_\mu/n_j|^{-1} ,$$

$$|\log \Pi_2| = -\sum_{j=j_1}^{k} \log |1 - n_\mu/n_j| \leqq 2n_\mu \sum_{j_1}^{k} n_j^{-1} .$$

From the definition of ν_0 follows that b is roughly

$$\prod_{j=\nu_0}^{k} (1 - 1/n_j) \leqq \exp\left(-\sum_{j=\nu_0}^{k} 1/n_j\right)$$

and a lower bound for b is obtained from

$$\sum_{j=\nu_0}^{k} 1/n_j \leqq 2 \cdot |\log b| \text{ for large } k ,$$

and therefore

$$1/n_\mu \log \Pi_2 \leqq 4 \cdot |\log b| .$$

If b is chosen sufficiently close to 1 we have

$$\Pi_2^{1/n_\mu} \leqq 1 + \varepsilon . \tag{23}$$

Combining (22) and (23) we find

$$\begin{aligned} \sigma_k &\leqq C \sum_{\nu=\nu_0}^{k} \sum_{\mu=\nu}^{k} c^{n_\mu} n_\mu/n_\nu \Pi_1 \Pi_2 \\ &\leqq C \sum_{\nu=\nu_0}^{k} \sum_{\mu=\nu}^{k} [c(n_\mu/n_\nu)^{1/n_\mu}(1 + \varepsilon)^2]^{n_\mu} . \end{aligned}$$

The expression in the brackets is for large k less than 1, which yields

$$\begin{aligned} \sigma_k &\leqq C \sum_{\nu=\nu_0}^{k} \sum_{\mu=\nu}^{k} A^{n_\mu} \qquad (0 < A < 1) \\ &\leqq \sum_{\nu=\nu_0}^{k} O(A^{n_\nu}) = O(A^{n_{\nu_0}}) = o(1) . \end{aligned}$$

The result of theorem 2 is best possible. For let n_k be a sequence of positive integers satisfying $\sum_1^\infty 1/n_k < \infty$, $n_0 = 0$, and let C' be the space of polynomials f' spanned by the t^{n_k}. Define a bounded linear functional d' on C' by $d'(f') = f'(0)$, which can be continuated to a functional d on the space C of continuous functions f on $[0, 1]$ by $d(f) = f(0)$. By Müntz' approximation theorem C' is not dense in C and hence there exists another continuation $d_1 \neq d$ onto C,

$$d_1(f) = \int_0^1 f(t) d\chi(t) ,$$

where $\chi(t) \in V[0, 1]$ has positive order ρ, and $d_1(t^{n_k}) = 0$, $k \geqq 1$.

In terms of Laplace transforms the preceding theorem reads as follows:

THEOREM 3. *Let* $d(z) = \int_0^\infty \exp(-tz)d\phi(t)$, $\phi(t) \in V[0, \infty)$ ϕ *real and left continuous on* $(0, \infty)$.

If for a sequence n_k *of integers,* $0 < n_k < n_{k+1}$, *and a* c, $0 < c < 1$, $d(n_k) = O(c^{n_k})$ *holds, then either* $\sum_1^\infty 1/n_k < \infty$, *or* $\phi(t)$ *is constant on* $[0, |\log c|)$ *and so* $d(z) = O(c^x)$, $z = x + iy$, $x \to \infty$.

We finitely remark that our result contains a well known theorem of Titchmarsh [6] p. 166 stating that $h(x) = \int_0^x f(x-t)g(t)dt$, $x \geqq 0, f, g$ continuous, vanishes identically if and only if $f(x)$ or $g(x)$ vanishes identically. For let $d_n^{(1)}$, $d_n^{(2)}$ be Hausdorff moments with orders ρ_1 and ρ_2 then by theorem 2 the order of the product sequence $d_n = d_n^{(1)}d_n^{(2)}$ is $\rho = \rho_1\rho_2$, which gives a generalized form of Titchmarsh's theorem. Clearly this can also be derived from Pòlya's theorem.

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REFERENCES

[1] Bieberbach, Analytische Fortsetzung, Springer 1953.
[2] Hardy, Divergent Series, Oxford 1956.
[3] Gelfand, Differenzenrechnung, Berlin 1958.
[4] Lorentz, Bernstein Polynomials, Toronto 1953.
[5] Pòlya, Untersuchungen über Lücken und Singularitäten von Potenzreihen, II. Mitteilung, Ann. Math. II, 34, 731-777, (1953).
[6] Yosida, Functional Analysis, Springer 1968.

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