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ON THE ANALYTICITY OF THE KERNEL OF A CLASS OF CONVOLUTION TRANSFORM

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. **Introduction.** In the previous papers [6], [7] we have studied the convergence properties and inversion theory of convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)d\alpha(t) , \tag{1}$$

for which the kernel $G(t)$ is of the form

$$G(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [F(s)]^{-1} e^{st} ds . \tag{2}$$

Here $F(s)$ is the meromorphic function with only real zeros and poles, and is of the form

$$F(s) = e^{bs} \prod_{k=1}^{\infty} (1 - s/a_k)e^{s/a_k}/(1 - s/c_k)e^{s/c_k} ,$$

where b, $\{a_k\}_1^\infty$, $\{c_k\}_1^\infty$ are constants such that $0 \leqq a_k/c_k < 1$, $\sum_{k=1}^\infty a_k^{-2} < \infty$ and c_k may be equal to $\pm\infty$.

In these papers we assumed the order of $[F(s)]^{-1}$ as $|\tau| \to \infty (s = \sigma + i\tau)$, however, this order should be determined originally by the correlation of zeros a_k and poles c_k of $F(s)$.

From this point of view, Z. Ditzian and A. Jakimovski [1], [2], [3] showed that for all integer $n \leqq N$ $(N \equiv N(\{a_k\}, \{c_k\})$

$$|F(s)|^{-1} = O(|\tau|^{-n}) \qquad |\tau| \to \infty$$

uniformly in the strip $|\sigma| \leqq R$ for every R and they obtained the inversion formula of the transform (1) which differs from that of ours where it was constructed by repeated integro-differential operators and our formula consisted of integral operator and differential operator separated from each other.

If the series $\sum_{k=1}^\infty (a_k^{-1} - c_k^{-1})$ converges then the kernel $G(t)$ becomes a special one called class III kernel and has the characteristic properties ([2], [7]).

In this paper we shall suppose that

$$(3)\qquad \sum_{k=1}^{\infty}(a_k^{-1}-c_k^{-1})=\infty$$

and for some positive number δ

$$(4)\qquad a_{k+1}-a_k>\delta \qquad (k=1,2,\cdots)\,.$$

The condition (3) means that $G(t)$ is a kernel of class II, and in § 4 we shall show that this condition implies $N=\infty$ necessarily and the infinite differentiability of $G(t)$ can be obtained. However, this condition (3) is not enough to become clear the construction of $G(t)$ completely.

If we assume the condition (4) we shall see that the class of kernel $G(t)$ can be considered as a class of Dirichlet series and that if we replace the variable t of $G(t)$ by the complex variable z the analyticity of $G(z)$ can be obtained. This class of Dirichlet series is a generalization of the class of entire function generated by the given sequence which is studied by J. Mikusiński ([4]). Though the assumption (4) seems to be severe, it causes no inconvenience for practical use.

2. Function $H(t)$ as a Dirichlet series. For brevity, we assume hereafter that $b=0$ and the constants a_k, c_k are positive and increasing.

Let us define

$$g_k(t)=\begin{cases}a_ke^{a_kt-1} & (-\infty<t<a_k^{-1})\\ 0 & (a_k^{-1}<t<\infty)\,,\end{cases}$$

$$h_k(t)=\int_{-\infty}^{\infty}(1-a_k/c_k)g_k(u+c_k^{-1})du+\frac{a_k}{c_k}j[t-(a_k^{-1}-c_k^{-1})]\,,$$

where $j(t)$ is the standard jump function, that is, $j(t)=1$ for $t>0$, $1/2$ for $t=0$, and 0 for $t<0$. By a simple calculation it is easily verified that $h_k(t)$ is a normalized distribution function and that

$$h_k(t)=(1-a_k/c_k)\exp\{a_k[t-(a_k^{-1}-c_k^{-1})]\}\qquad (t<a_k^{-1}-c_k^{-1})$$
$$h_k(a_k^{-1}-c_k^{-1})=1-a_k/2c_k$$
$$\int_{-\infty}^{\infty}e^{-st}dh_k(t)=(1-s/c_k)e^{s/c_k}/(1-s/a_k)e^{s/a_k}\,,$$

the bilateral Laplace transform converging absolutely for $\Re s<a_k$.

THEOREM 2.1. *If we set for n ※ $(n\geqq 2)$*

$$H_n(t)=h_1 ※ h_2 ※ \cdots ※ h_n(t)\,,$$

where the operation ※ denotes the Stieltjes convolution for distribution functions, then:

A. *$H_n(t)$ is a normalized distribution function;*

B.
$$H_n(t) = \begin{cases} \sum_{j=1}^{n} B_j^{(n)} \exp\left\{a_j\left[t - \sum_{k=1}^{n}(a_k^{-1} - c_k^{-1})\right]\right\} & t < \sum_{k=1}^{n}(a_k^{-1} - c_k^{-1}) \\ 1 - \dfrac{a_1 a_2 \cdots a_n}{2c_1 c_2 \cdots c_n} & t = \sum_{k=1}^{n}(a_k^{-1} - c_k^{-1}) \\ 1 & t > \sum_{k=1}^{n}(a_k^{-1} - c_k^{-1}) \ , \end{cases}$$

where

$$B_j^{(n)} = \prod_{k=1}^{n}(1 - a_j/c_k)/\prod_{k=1}^{n}{}'(1 - a_j/a_k) \qquad (j = 1, 2, \cdots, n)$$

and $\prod'$ denotes the product excluding the factor for $k = j$;

C.
$$\int_{-\infty}^{\infty} e^{-st} dH_n(t) = \prod_{k=1}^{n}(1 - s/c_k)e^{s/c_k}/(1 - s/a_k)e^{s/a_k} \ ,$$

the bilateral Laplace transform converging absolutely for $\Re s < a_1$.

PROOF. Since the conclusion A and C are familiar results [2], [6], [7], let us prove only conclusion B by the induction.

For the sake of simplicity we shall write s_n for $\sum_{k=1}^{n}(a_k^{-1} - c_k^{-1})$.

In the case $n = 2$, if $t > s_2$, then $h_1(t - u) \equiv 1$ for $u < a_2^{-1} - c_2^{-1}$ and we have $h_1 \divideontimes h_2(t) = \int_{-\infty}^{a_2^{-1}-c_2^{-1}} dh_2(u) + a_2/c_2 = 1$, and if $t < s_2$ then

$$\begin{aligned} h_1 \divideontimes h_2(t) &= \int_{-\infty}^{a_2^{-1}-c_2^{-1}} h_1(t-u)dh_2(u) + h_1(t - (a_2^{-1} - c_2^{-1}))\frac{a_2}{c_2} \\ &= \int_{-\infty}^{t-s_1} h_1(t-u)dh_2(u) + \int_{t-s_1}^{a_2^{-1}-c_2^{-1}} h_1(t-u)dh_2(u) + h_1(t - (a_2^{-1} - c_2^{-1}))\frac{a_2}{c_2} \\ &= (1 - a_2/c_2)e^{a_2/c_2-1}e^{a_2(t-s_1)} \\ &\quad + (1 - a_1/c_1)(1 - a_2/c_2)a_2 e^{a_1/c_1-1}e^{a_2/c_2-1}e^{a_1 t}\frac{1}{a_2 - a_1}[e^{(a_2-a_1)(a_2^{-1}-c_2^{-1})} - e^{(a_2-a_1)(t-s_1)}] \\ &\quad + a_2/c_2(1 - a_1/c_1)e^{a_1/c_1-1}e^{a_1(t-s_1)} \\ &= \left\{a_2/c_2 + \frac{a_2}{a_2 - a_1}(1 - a_2/c_2)\right\}(1 - a_1/c_1)e^{a_1/c_1-1}e^{a_1(t-s_1)} \\ &\quad + \left\{1 - \frac{a_2}{a_2 - a_1}(1 - a_1/c_1)\right\}(1 - a_2/c_2)e^{a_2/c_2-1}e^{a_2(t-s_1)} \\ &= \frac{a_2(c_1 - a_1)(c_2 - a_1)}{c_1c_2(a_2 - a_1)}e^{a_1(t-s_2)} + \frac{a_1(c_1 - a_2)(c_2 - a_2)}{c_1c_2(a_1 - a_2)}e^{a_2(t-s_2)} \\ &= B_1^{(2)}e^{a_1(t-s_2)} + B_2^{(2)}e^{a_2(t-s_2)} \end{aligned}$$

and it is clear that $h_1 \divideontimes h_2(s_2) = 1 - a_1a_2/(2c_1c_2)$. Thus we have proved the conclusion C for $n = 2$.

Now, we assume that the conclusion B is valid for $n = m$. For $t < s_{m+1}$, $t - s_m < a_{m+1}^{-1} - c_{m+1}^{-1}$ and we have

$$\begin{aligned}
H_{m+1}(t) &= \int_{-\infty}^{\infty} H_m(t-u)dh_{m+1}(u) \\
&= \int_{-\infty}^{t-s_m} H_m(t-u)dh_{m+1}(u) + \int_{t-s_m}^{a_{m+1}^{-1}-c_{m+1}^{-1}} H_m(t-u)dh_{m+1}(u) \\
&\quad + H_m(t-(a_{m+1}^{-1}-c_{m+1}^{-1}))\frac{a_{m+1}}{c_{m+1}} \\
&= (1-a_{m+1}/c_{m+1})e^{a_{m+1}(t-s_{m+1})} \\
&\quad + (1-a_{m+1}/c_{m+1})a_{m+1}\int_{t-s_m}^{a_{m+1}^{-1}-c_{m+1}^{-1}}\left\{\sum_{j=1}^{m} B_j^{(m)} e^{a_j(t-u-s_m)}\right\}e^{a_{m+1}[u-(a_{m+1}^{-1}-c_{m+1}^{-1})]}du \\
&\quad + (a_{m+1}/c_{m+1})\sum_{j=1}^{m} B_j^{(m)a_j(t-s_{m+1})} \\
&= (1-a_{m+1}/c_{m+1})e^{a_{m+1}(t-s_{m+1})} + (1-a_{m+1}/c_{m+1})\sum_{j=1}^{m}\frac{a_{m+1}B_j^{(m)}}{a_{m+1}-a_j}e^{a_j(t-s_{m+1})} \\
&\quad - (1-a_{m+1}/c_{m+1})\sum_{j=1}^{m}\frac{a_{m+1}B_j^{(m)}}{a_{m+1}-a_j}e^{a_{m+1}(t-s_{m+1})} + a_{m+1}/c_{m+1}\sum_{j=1}^{m} B_j^{(m)}e^{a_j(t-s_{m+1})} \\
&= \sum_{j=1}^{m}\left\{(1-a_{m+1}/c_{m+1})\frac{a_{m+1}B_j^{(m)}}{a_{m+1}-a_j} + a_{m+1}B_j^{(m)}/c_{m+1}\right\}e^{a_j(t-s_{m+1})} \\
&\quad + \left\{(1-a_{m+1}/c_{m+1}) - (1-a_{m+1}/c_{m+1})\sum_{j=1}^{m}\frac{a_{m+1}B_j^{(m)}}{a_{m+1}-a_j}\right\}e^{a_{m+1}(t-s_{m+1})} .
\end{aligned}$$

It is easily seen that

$$(1-a_{m+1}/c_{m+1})\frac{a_{m+1}B_j^{(m)}}{a_{m+1}-a_j} + a_{m+1}B_j^{(m)}/c_{m+1} = B_j^{(m+1)} \qquad (j = 1, 2, \cdots, m) ,$$

$$(1-a_{m+1}/c_{m+1})\left\{1-\sum_{j=1}^{m}\frac{a_{m+1}B_j^{(m)}}{a_{m+1}-a_j}\right\} = B_{m+1}^{(m+1)} ,$$

and hence for $t < s_{m+1}$

$$H_{m+1}(t) = \sum_{j=1}^{m+1} B_j^{(m+1)}e^{a_j(t-s_{m+1})} .$$

On the other hand, it is also clear that $H_{m+1}(s_{m+1}) = 1 - a_1a_2\cdots a_{m+1}/(2c_1c_2\cdots c_{m+1})$ and $H_{m+1}(t) = 1$ for $t > s_{m+1}$. Thus the conclusion B is valid for $n = m+1$ and this completes the proof.

COROLLARY 2.1. *If $H_n(t)$ is defined as in Theorem 2.1, then for n $(n \geqq 2)$*

$$(1)\qquad H_n(t) = \begin{cases} \sum_{j=1}^{n} A_j^{(n)} e^{a_j t} & t < s_n \\ 1 - a_1 a_2 \cdots a_n/(2c_1 c_2 \cdots c_n) & t = s_n \\ \ 1 & t > s_n\,, \end{cases}$$

where

$$(2)\qquad A_j^{(n)} = \prod_{k=1}^{n} (1 - a_j/c_k) e^{a_j/c_k} / e \prod_{k=1}^{n}{}' (1 - a_j/a_k) e^{a_j/a_k}\,.$$

This is an immediate rewriting of the conclusion B of the preceding theorem.

THEOREM 2.2. *Let*

$$A_j = \prod_{k=1}^{\infty} (1 - a_j/c_k) e^{a_j/c_k} / e \prod_{k=1}^{\infty}{}' (1 - a_j/a_k) e^{a_j/a_k}$$

$(j = 1, 2, \cdots)$. *Then the series* $H(t) = \sum_{j=1}^{\infty} A_j e^{a_j t}$ *converges absolutely and uniformly in any interval* $(-\infty, t_0]$ $(-\infty < t_0 < \infty)$.

PROOF. Since $\sum_{k=1}^{\infty} a_k^{-2} < \infty$ and $\sum_{k=1}^{\infty} c_k^{-2} < \infty$ the infinite products $\prod_k (1 - s/a_k) e^{s/a_k}$ and $\prod_k (1 - s/c_k) e^{s/c_k}$ converge absolutely and uniformly in any closed bounded set of the s-plane that contains none of the points a_k and c_k. Thus A_j are well-defined for every j. If all c_k equal $+\infty$, then $H(t)$ is a slight modification of entire fuction which treated by Mikusiński [4, p. 388] and the most part of the proof of this theorem due to him.

Now, for any fixed j we denote by $k(j)$ the largest k such that $c_k < a_j$. Evidently, $k(j) < j$ and $k(j) \uparrow \infty$ as $j \uparrow \infty$.

We have

$$\begin{aligned} \log |A_j| &= \sum_{k=1}^{\infty} (\log |1 - a_j/c_k| + a_j/c_k) - \sum_{k=1}^{j-1} (\log(a_j/a_k - 1) + a_j/a_k) \\ &\quad - \sum_{k=j+1}^{\infty} (\log(1 - a_j/a_k) + a_j/a_k) - 1 \\ &= \sum_{k=1}^{k(j)} (\log(a_j/c_k - 1) + a_j/c_k - a_j/a_k) + \sum_{k=k(j)+1}^{\infty} (\log(1 - a_j/c_k) + a_j/c_k) \\ &\quad - \sum_{k=k(j)+1}^{j} a_j/a_k - \sum_{k=1}^{j-1} \log(a_j/a_k - 1) - \sum_{k=j+1}^{\infty} (\log(1 - a_j/a_k) + a_j/a_k)\,. \end{aligned}$$

It is clear that

$$\sum_{k=k(j)+1}^{\infty} (\log(1 - a_j/c_k) + a_j/c_k) < 0\,,$$

and, using the fact that the function $\log(a_j/x - 1)$ is decreasing for

$0 < x < a_j$, it is also clear that

$$\sum_{k=1}^{k(j)} \log(a_j/c_k - 1) < \sum_{k=1}^{k(j)} \log(a_j/a_k - 1) .$$

Then we have

$$\begin{aligned}\frac{1}{a_j}\log|A_j| &< -\sum_{k=1}^{k(j)}(a_k^{-1} - c_k^{-1}) - \frac{1}{a_j}\sum_{k=k(j)+1}^{j-1}\log(a_j/a_k - 1)\\ &\quad - \frac{1}{a_j}\sum_{k=j+1}^{\infty}(\log(1 - a_j/a_k) + a_j/a_k)\\ &\equiv -\sum_{k=1}^{k(j)}(a_k^{-1} - c_k^{-1}) + S_1 + S_2 , \quad \text{say} .\end{aligned}$$

If $j = 1$ or $k(j) = j - 1$, then the second term vanishes. We have by (4) of §1, using Mikusiński's method,

$$\begin{aligned} S_1 &= \frac{1}{a_j}\sum_{k=k(j)+1}^{j-1}\log\frac{a_k}{a_j - a_k} < \frac{1}{a_j}\sum_{k=k(j)+1}^{j-1}\log\frac{a_j - (j-k)\delta}{(j-k)\delta}\\ &= \frac{1}{a_j}\sum_{k=1}^{j-k(j)-1}\log\frac{a_j - k\delta}{k\delta} < \frac{1}{a_j}\int_0^{j-k(j)-1}\log\frac{a_j - \delta x}{\delta x}dx\\ &= 1/\delta\int_0^{\delta(j-k(j)-1)/a_j}\log(1/t - 1)dt < \log 2/\delta ,\end{aligned}$$

and

$$S_2 < 1/\delta .$$

Thus we have

$$\frac{1}{a_j}\log|A_j| < (1 + \log 2)/\delta - \sum_{k=1}^{k(j)}(a_k^{-1} - c_k^{-1}) ,$$

so that $\lim_{j\to\infty} k(j) = \infty$ and (3) of § 1 implies that

$$\lim_{j\to\infty}|A_j|^{1/a_j} = 0 .$$

Now, let t_0 be an arbitrary fixed real number, then, when $t \leqq t_0$

$$|A_j e^{a_j t}| \leqq |A_j| e^{a_j t_0} = (|A_j|^{1/a_j} e^{t_0})^{a_j} .$$

Therefore, for sufficiently large j, we have

$$|A_j e^{a_j t}| < (1/2)^{a_j} = (j)^{-\frac{a_j}{\log j}\log 2} .$$

But, by the assumption (4) it follows that $\lim_{j\to\infty}\dfrac{a_j}{\log j} = \infty$, so we have

$$|A_j e^{a_j t}| < j^{-2}$$

for sufficiently large j. This implies that the Dirichlet series $H(t) =$

$\sum_{j=1}^{\infty} A_j e^{a_j t}$ converges absolutely and uniformly in $-\infty < t \leqq t_0$ and completes the proof.

REMARK. In Theorem 2.2, it should be noted that real number t can be replaced by complex number z and the Dirichlet series $H(z) = \sum_{j=1}^{\infty} A_j e^{a_j z}$ converges absolutely and uniformly in any half plane $\Re z < t_0$.

If all c_k equal to $+\infty$, this class of Dirichlet series reduces to a slight modification that is called the class of entire function generated by the given sequence $\{a_k\}$ of exponents. Thus, our class of Dirichlet series is a generalization of it.

THEOREM 2.3. *If $H(t)$ is defined as in Theorem 2.2, then $H(t)$ is infinitely differentiable term by term in $(-\infty, \infty)$, that is, for any positive integer n, $H^{(n)}(t) = \sum_{j=1}^{\infty} A_j a_j^n e^{a_j t}$ converges absolutely for every t and uniformly in any interval $-\infty < t \leqq t_0$ $(-\infty < t_0 < +\infty)$.*

PROOF. In the preceding Theorem 2.2 we had

$$|A_j e^{a_j t}| < (1/2)^{a_j}$$

for sufficiently large j. From this fact, for sufficiently large j, we have

$$\begin{aligned} |A_j a_j^n e^{a_j t}| &< a_j^n (1/2)^{a_j} = 2^{n \log_2 a_j - a_j} \\ &\leqq 2^{-a_j/2} < 2^{-\frac{j-1}{2}\delta} = (2^{-\frac{2}{\delta}})^{j-1} . \end{aligned}$$

Thus we obtain our desired results.

THEOREM 2.4. *If $H_n(t)$ and $H(t)$ are defined as in Theorem 2.1 and Theorem 2.2, respectively, then for any t $(-\infty < t < \infty)$*

$$\lim_{n\to\infty} H_n(t) = H(t) .$$

PROOF. By the estimations similar to that of Theorem 2.2, we have for $n > j$

$$\begin{aligned} \frac{1}{a_j} \log |A_j^{(n)}| &< -\sum_{k=1}^{k(j)} (a_k^{-1} - c_k^{-1}) - \frac{1}{a_j} \sum_{k=k(j)+1}^{j-1} \log(a_j/a_k - 1) \\ &\quad - \frac{1}{a_j} \sum_{k=j+1}^{n} \{\log(1 - a_j/a_k) + a_j/a_k\} \\ &< -\sum_{k=1}^{k(j)} (a_k^{-1} - c_k^{-1}) + \frac{1 + \log 2}{\delta} , \end{aligned}$$

and for arbitrary fixed t we have

$$|A_j^{(n)} e^{a_j t}| < j^{-2}$$

for sufficiently large j. Thus, this estimation together with the same of A_j implies that, given $\varepsilon > 0$, we can find j_0 such that

$$\left|\sum_{j=j_0+1}^{n} A_j^{(n)} e^{a_j t}\right| < \varepsilon/2 \text{ and } \left|\sum_{j=j_0+1}^{\infty} A_j e^{a_j t}\right| < \varepsilon/2 .$$

Then, we have

$$|H_n(t) - H(t)| < \sum_{j=1}^{j_0} |A_j^{(n)} - A_j| e^{a_j t} + \varepsilon .$$

However, it is clear that

$$\lim_{n\to\infty} A_j^{(n)} = A_j \qquad (j = 1, 2, \cdots, j_0) ,$$

and consequently

$$\overline{\lim_{n\to\infty}} |H_n(t) - H(t)| \leqq \varepsilon ,$$

which is the required result.

THEOREM 2.5. *If $H(t)$ is defined as in Theorem 2.2, then $H(t)$ is a distribution function.*

PROOF. By Theorem 2.1, $H_n(t)$ is the normalized distribution function and

$$\int_{-\infty}^{\infty} e^{i\tau t} dH_n(t) = \sum_{j=1}^{n} (1 + i\tau/c_j) e^{i\tau/c_j} / (1 + i\tau/a_j) e^{i\tau/a_j} .$$

For any $\tau(-\infty < \tau < \infty)$, it is clear that

$$\prod_{j=1}^{n} (1 + i\tau/c_j) e^{i\tau/c_j} / (1 + i\tau/a_j) e^{i\tau/a_j} \longrightarrow 1/F(i\tau) \quad \text{as} \quad n \longrightarrow \infty ,$$

and that $1/F(i\tau)$ is continuous at $\tau = 0$ and $1/F(0) = 1 = H_n(\infty) - H_n(-\infty)$. By the theory of distribution function [9, vol. II p. 262], there exists a non-decreasing function $H^*(t)$ such that

$$H^*(\infty) = 1, H^*(-\infty) = 0 , \quad \lim_{n\to\infty} H_n(t) = H^*(t)$$

at all points of continuity of $H^*(t)$ and $1/F(i\tau) = \int_{-\infty}^{\infty} e^{i\tau t} dH^*(t)$. Using Theorem 2.4, it is clear that this function equal to $H(t)$. This completes the proof.

3. Construction of the kernel. Now, we may construct the kernel $G(t)$ and can be established its analyticity in the whole plane of z.

THEOREM 3.1. *If $H(t)$ is defined as in Theorem 2.2, then $H(t)$ has its derivative $G(t) = H'(t)$ and*

(1) *$G(t)$ is a frequency function and $G(t) \in C^\infty(-\infty, \infty)$,*

$$(2) \qquad \int_{-\infty}^{\infty} e^{-st}G(t)dt = 1/F(s)\ ,$$

the bilateral Laplace transform converging absolutely for $\Re s < a_1$, *and*

$$(3) \qquad G(t) = 1/(2\pi i)\int_{-i\infty}^{i\infty} [F(s)]^{-1}e^{st}ds\ .$$

PROOF. It follows from Theorem 2.3 that $H(t)$ has its derivative $G(t) = H'(t)$ and (1) follows from Theorem 2.3 and Theorem 2.5.

Let s be an arbitrary real number $<a_1$. Then, by (3) and (4) of §1, it is clear that

$$(4) \qquad \sum_{k=1}^{\infty}\left[\frac{1}{a_k - s} - \frac{1}{c_k - s}\right] = \infty \text{ and } (a_{k+1} - s) - (a_k - s) > \delta > 0 \qquad (k = 1, 2, \cdots)\ .$$

Therefore, putting

$$B_j = \prod_{k=1}^{\infty}\left(1 - \frac{a_j - s}{c_k - s}\right)e^{\frac{a_j - s}{c_k - s}} \Big/ e \prod_{k=1}^{\infty}{}'\left(1 - \frac{a_j - s}{a_k - s}\right)e^{\frac{a_j - s}{a_k - s}}$$

$(j = 1, 2, \cdots)$, we can define an infinitely differentiable function $K(t)$ such as

$$K(t) = \sum_{j=1}^{\infty} B_j e^{(a_j - s)t}\ ,$$

where the infinite product defining B_j are convergent for all s, because

$$\sum_{k=1}^{\infty}(a_k - s)^{-2} < \infty \quad \text{for all} \quad s\ (s < a_1)\ .$$

It is trivial that $K(t)$ has the same properties analogous to $H(t)$.
By the definition A_j in Theorem 2.2, we have

$$\frac{B_j}{A_j} = \frac{a_j}{a_j - s}\prod_{k=1}^{\infty}(1 - s/a_k)\exp\left\{\frac{s}{a_k - s} + (a_j/a_k - a_j/(a_k - s)\right\}\Big/$$
$$(1 - s/c_k)\exp\left\{\frac{s}{c_k - s} + (a_j/c_k - a_j/(c_k - s))\right\}.$$

Then, putting

$$Q = \prod_{k=1}^{\infty}(1 - s/a_k)\exp\left(\frac{s}{a_k - s}\right)\Big/(1 - s/c_k)\exp\left(\frac{s}{c_k - s}\right)$$

and

$$q = \sum_{k=1}^{\infty}\left\{\frac{s}{a_k(a_k - s)} - \frac{s}{c_k(c_k - s)}\right\},$$

we have

$$B_j/A_j = \frac{a_j}{a_j - s} Q e^{-qa_j} \,, \tag{5}$$

where the convergence of the infinite product Q and the series q are easily verified by (4).

By Theorem 2.3, $H'(t)$ converges absolutely for any t and uniformly in any interval $-\infty < t \leqq u(-\infty < u < \infty)$. Thus, observing $H'(t) = O(e^{a_1 t})$ as $t \to -\infty$ and integrating by parts we have

$$\int_{-\infty}^{u} e^{-st}G(t)dt = \int_{-\infty}^{u} e^{-st}H'(t)dt = \sum_{j=1}^{\infty} \frac{A_j a_j}{a_j - s} e^{(a_j - s)u} = \frac{e^{qs}}{Q} \sum_{j=1}^{\infty} B_j e^{(a_j - s)(u+q)}$$
$$= \frac{e^{qs}}{Q} K(u + q)$$

by (5). However, the properties of $K(t)$ same as in Theorem 2.5 for $H(t)$ shows that $K(u + q) \to 1$ as $u \to \infty$.

Consequently, the integral $\int_{-\infty}^{\infty} e^{-st}G(t)dt$ exists for any real number $s(< a_1)$ and is equal to e^{qs}/Q, i.e.

$$\int_{-\infty}^{\infty} e^{-st}G(t)dt = \prod_{k=1}^{\infty} (1 - s/c_k)e^{s/c_k}/(1 - s/a_k)e^{s/a_k} = 1/F(s) \,. \tag{6}$$

If we take any complex number s whose real part less than a_1 instead of real number s such as $s < a_1$, it follows that the integral on the left side of (6) converges in any half-plane $\Re s \leqq \sigma_0 < a_1$ and it represents an analytic function in the half-plane $\Re s < a_1$.

On the other hand, it is familiar that the right side of (6) is analytic in the half-plane $\Re s < a_1$. Since the both sides of (6) coincides on the interval $(-\infty, a_1)$ of real axis, by the identity theorem, (6) is valid in the half-plane $\Re s < a_1$. Since that the bilateral Laplace transform converges absolutely for $\Re s < a_1$ is easily verified and (3) is an immediate consequence of familiar theorem of Laplace transform [8, p. 241], the proof is complete.

REMARK. As in Remark of §2, the real variable t of $G(t)$ can be replaced by the complex variable z and, on that occasion, the series $G(z) = \sum_{j=1}^{\infty} a_j A_j e^{a_j z}$ converges absolutely for every complex z and uniformly in any half-plane $\Re z \leqq t_0$ $(-\infty < t_0 < \infty)$. Thus $G(z)$ is an entire function.

4. Differentiability of $G(t)$. In this section, we shall show that if we take no notice of construction of $G(t)$ then its infinite differentiability can be obtained by the only condition (3) without the condition (4), provided that the sequences $\{a_k\}$ and $\{c_k\}$ satisfy the other conditions in §1.

We shall need some preliminary results.

Let $n_a(t) = n(\{a_k\}, t)$ be the number of a_k lying in the interval $(0, t)$, which is called the counting function of sequence $\{a_k\}$ ([1], [5]).

LEMMA [5, p. 25]. *For any λ such that $0 < \lambda < \infty$, we have*

$$\sum_{k=1}^{\infty} a_k^{-\lambda} = \int_0^\infty \frac{dn_a(t)}{t^\lambda} = \lambda \int_0^\infty \frac{n_a(t)}{t^{\lambda+1}} dt$$

in the sense that all three expressions are infinite, or all are finite and equal.

THEOREM 4.1. *If the sequences $\{a_k\}$ and $\{c_k\}$ are defined as in §2, then*

$$\sum_{k=1}^{\infty} (a_k^{-1} - c_k^{-1}) = \int_0^\infty \frac{n_a(t) - n_c(t)}{t^2} dt$$

in the sense that both sides are infinite, or both are finite and equal.

PROOF. For any $R > 0$, we have easily

$$\sum_{a_k < R} a_k^{-1} = \sum_{k=1}^{n_a(R)} a_k^{-1} = n_a(R)/R + \int_0^R \frac{n_a(t)}{t^2} dt .$$

Hence we have

$$\sum_{k=1}^{n_a(R)} (a_k^{-1} - c_k^{-1}) + \sum_{k=n_c(R)+1}^{n_a(R)} c_k^{-1} = \frac{n_a(R) - n_c(R)}{R} + \int_0^R \frac{n_a(t) - n_c(t)}{t^2} dt .$$

However, for all k such that $n_c(R) + 1 \leqq k \leqq n_a(R)$ it is trivial that $c_k^{-1} \leqq R^{-1}$. Therefore,

$$\sum_{k=n_c(R)+1}^{n_a(R)} c_k^{-1} \leqq \frac{n_a(R) - n_c(R)}{R} ,$$

and then we have

$$\sum_{k=1}^{n_a(R)} (a_k^{-1} - c_k^{-1}) \geqq \int_0^R \frac{n_a(t) - n_c(t)}{t^2} dt . \tag{1}$$

On the other hand, we have also easily

$$\sum_{k=1}^{n_c(R)} (a_k^{-1} - c_k^{-1}) + \sum_{k=n_c(R)+1}^{n_a(R)} a_k^{-1} = \frac{n_a(R) - n_c(R)}{R} + \int_0^R \frac{n_a(t) - n_c(t)}{t^2} dt$$

and

$$\sum_{k=n_c(R)+1}^{n_a(R)} a_k^{-1} \geqq \frac{n_a(R) - n_c(R)}{R} \quad \text{since} \quad a_k^{-1} \geqq R^{-1} \quad \text{for} \quad k \leqq n_a(R) .$$

Hence we have

$$(2)\qquad \sum_{k=1}^{n_c(R)} (a_k^{-1} - c_k^{-1}) \leqq \int_0^R \frac{n_a(t) - n_c(t)}{t^2} dt .$$

Combining (1) and (2) and appealing the fact that $n_a(R) \to \infty$ and $n_c(R) \to \infty$ as $R \to \infty$, we obtain our desired result.

The following two results are the immediate consequences of this theorem.

COROLLARY 4.1. *If $\{a_k\}$ and $\{c_k\}$ are defined as in Theorem 4.1 and $n_a(t) - n_c(t) = O(t^{1-\alpha})$ as $t \to \infty$ for some $\alpha > 0$, then the series $\sum_{k=1}^{\infty} (a_k^{-1} - c_k^{-1})$ converges, in particular, $\overline{\lim}_{t\to\infty} \{n_a(t) - n_c(t)\} < \infty$ implies that the series $\sum_{k=1}^{\infty} (a_k^{-1} - c_k^{-1})$ converges.*

COROLLARY 4.2. *If $\{a_k\}$ and $\{c_k\}$ are defined as in Theorem 4.1, then $\sum_{k=1}^{\infty} (a_k^{-1} - c_k^{-1}) = \infty$ implies that $\overline{\lim}_{t\to\infty} \{n_a(t) - n_c(t)\} = \infty$.*

THEOREM 4.2. *If $\{a_k\}$ and $\{c_k\}$ are defined as in §1 and $G(t)$ is defined by* (2) *of* §1, *then the kernel function $G(t)$ is infinitely differentiable.*

PROOF. By Corollary 4.2, using Theorem 2.2 of [1] and Corollary 3.2 and Theorem 3.5 of [2], we obtain our theorem.

5. Another class of Dirichlet series. In this section we suppose that the sequences $\{a_k\}$ and $\{c_k\}$ satisfy the following conditions:

$$(1)\qquad a_k > 0 , \quad c_k < 0 \quad (k = 1, 2, \cdots) ;$$

$$(2)\qquad \sum_{k=1}^{\infty} a_k^{-1} = \infty , \quad \sum_{k=1}^{\infty} c_k^{-2} < \infty ;$$

and for some positive number δ

$$(3)\qquad a_{k+1} - a_k > \delta \quad (k = 1, 2, \cdots) .$$

The following is an analogous result to Theorem 2.2.

THEOREM 5.1. *Let*

$$A_j = \prod_{k=1}^{\infty} (1 - a_j/c_k) e^{a_j/c_k} / e \prod_{k=1}^{\infty}{}' (1 - a_j/a_k) e^{a_j/a_k} \qquad (j = 1, 2, \cdots) ,$$

then the series $H(z) = \sum_{j=1}^{\infty} A_j e^{a_j z}$ converges absolutely and uniformly in any half plane $\Re z < t_0$ $(-\infty < t_0 < \infty)$.

PROOF. By the similar arguments to the proof of Theorem 2.2, we have from (1) and (3)

$$a_j^{-1} \log |A_j| = a_j^{-1} \sum_{k=1}^{\infty} \{\log(1 - a_j/c_k) + a_j/c_k\} - \sum_{k=1}^{j} a_k^{-1}$$
$$- a_j^{-1} \sum_{k=1}^{j-1} \log(a_j/a_k - 1) - a_j^{-1} \sum_{k=j+1}^{\infty} \{\log(1 - a_j/a_k) + a_j/a_k\}$$

$$< -\sum_{k=1}^{j} a_k^{-1} - a_j^{-1}\sum_{k=1}^{j-1}\log(a_j/a_k - 1)$$

$$- a_j^{-1}\sum_{k=j+1}^{\infty}\{\log(1 - a_j/a_k) + a_j/a_k\}$$

$$< -\sum_{k=1}^{j} a_k^{-1} + (1 + \log 2)/\delta\ ,$$

using the fact that

$$\log\{(1 - a_j/c_k) + a_j/c_k\} < 0 \qquad \text{for every } k\ .$$

Then, appealing the first condition of (2) we have

$$|A|^{1/a_j} \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty\ ,$$

and we obtain our theorem by the similar arguments to the one in §2.

From this theorem it follows that $H(z)$ is analytic in the whole plane of z and $H(z)$ is an entire function. This is also a generalization of the entire function generated by the sequence $\{a_k\}$ introduced by Mikusiński.

For example, the Fourier sine transform

$$F(X) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin XT \cdot \Phi(T)dT$$

becomes, after exponential change of variables, the convolution transform (1) of §1 with the kernel $G(t) = \sqrt{2/\pi}\ \sin e^t$ whose bilateral Laplace transform is

$$\int_{-\infty}^{\infty} e^{-st}G(t)dt = \sqrt{\frac{2}{\pi}}\sin\Big(-\frac{1}{2}\pi s\Big)\Gamma(-s)$$

$$= e^{\gamma s}\prod_{k=1}^{\infty}\Big(1 + \frac{s}{2k}\Big)e^{-s/2k}\Big/\Big(1 - \frac{s}{2k-1}\Big)e^{s/(2k-1)}$$

$$= 1/F(s)\ , \quad 0 < \Re s < 1\ .$$

The meromorphic function $F(s)$ has zeros $a_k = 2k - 1$ and poles $c_k = -2k$ $(k = 1, 2, \cdots)$ and the sequences $\{a_k\}$ and $\{c_k\}$ satisfy the conditions (1), (2) and (3) with δ (<2). In this case, A_j is defined as

$$A_j = \prod_{k=1}^{\infty}\Big(1 + \frac{2j-1}{2k}\Big)e^{-\frac{2j-1}{2k}}\Big/e^{1-(2j-1)\gamma}\prod_{k=1}^{\infty}{}'\Big(1 - \frac{2j-1}{2k-1}\Big)e^{\frac{2j-1}{2k-1}}$$

$$= \sqrt{\frac{2}{\pi}}\sin\Big(-\frac{1}{2}\pi(2j-1)\Big)\lim_{s\to 2j-1}\Big(1 - \frac{s}{2j-1}\Big)\Gamma(-s)$$

$$= \sqrt{\frac{2}{\pi}}\frac{1}{2j-1}\frac{(-1)^{j-1}}{(2j-1)!}\ ,$$

using the familiar formula

$$\lim_{s\to l}(-s+l)\Gamma(-s)=(-1)^l/l!\ .$$

The entire function generated by the sequences $\{a_k\}$ and $\{c_k\}$ is

$$H(z)=\sum_{j=1}^{\infty}A_j e^{a_j z}=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}\frac{1}{2j-1}\frac{(-1)^{j-1}}{(2j-1)!}e^{(2j-1)z}\ ,$$

and

$$H'(z)=\sqrt{\frac{2}{\pi}}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{(2j-1)!}e^{(2j-1)z}=\sqrt{\frac{2}{\pi}}\sin e^z\ .$$

This function $H'(z)$ is equal to the kernel $G(t)$ in which the real variable t replaced by the complex variable z.

Moreover, it is well known that the Laplace transform can be reduced to the convolution transform (1) with the kernel $G(t)=e^{-e^t}e^t$ and the meromorphic function

$$F(s)=1/\Gamma(1-s)=e^{-\gamma s}\prod_{k=1}^{\infty}(1-s/k)e^{s/k}$$

whose zeros are $a_k=k$ $(k=1,2,\cdots)$ and all c_k can be regarded as $+\infty$ or $-\infty$. In this case, A_j is defined as

$$\begin{aligned}A_j&=1/e^{1-\gamma j}\prod_{k=1}^{\infty}{}'(1-j/k)e^{j/k}\\&=\lim_{s\to j}(1-s/j)\Gamma(1-s)\\&=\frac{1}{j}\frac{(-1)^{j-1}}{(j-1)!}=\frac{(-1)^{j-1}}{j!}\ .\end{aligned}$$

The entire function generated by the sequence $\{a_k\}$ is

$$H(z)=\sum_{j=1}^{\infty}A_j e^{a_j z}=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}e^{jz}$$

and

$$H'(z)=\sum_{j=1}^{\infty}\frac{(-1)^{j}}{(j-1)!}e^{jz}=e^{-e^z}e^z\ .$$

This function $H'(z)$ coincides also with the kernel $G(t)$ whose variable replaced by the complex variable z.

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