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TENSOR PRODUCTS OF BANACH ALGEBRAS AND HARMONIC ANALYSIS

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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In this paper we shall introduce the notion of the S-algebra induced from a given sequence of semi-simple (commutative, complex) Banach algebras with unit. Such an algebra will become a new semi-simple Banach algebra with a certain norm. We shall obtain some fundamental properties of S-algebras, and consider two problems; one is the problem of operating functions, and the other is that of spectral synthesis. Next we shall apply some of our results on S-algebras to the theory of restriction algebras of Fourier algebras. We shall construct, by a certain rule, compact subsets of a given locally compact abelian group G, and homomorphisms of restriction algebras of the Fourier algebra $A(G)$ on them. Such a restriction algebra will be isomorphic to an S-algebra induced from other restriction algebras of $A(G)$. Further, we shall explicitly construct a function g in $A(T)$ such that the closed ideals in $A(T)$ generated by g^m $(m = 1, 2, \cdots)$ are all distinct (see Example 6 at the end of this paper).

We begin with introducing some notations and definitions. Let $(A_n)_{n=1}^{\infty}$ be a sequence of semi-simple (commutative) Banach algebras with unit. We shall regard each A_n as a subalgebra of $C(E_n)$ in a trivial way, where E_n denotes the maximal ideal space of A_n, and assume that $\|1\|_{A_n} = 1$ for all n. Let N be a natural number, and let

$$A_1 \otimes A_2 \otimes \cdots \otimes A_N \quad \text{and} \quad A_1 \hat{\otimes} A_2 \hat{\otimes} \cdots \hat{\otimes} A_N$$

be the algebraic tensor product of $(A_n)_{n=1}^{N}$ and its completion with the projective norm, respectively; and put $E^{(N)} = E_1 \times E_2 \times E_N$, the product space of $(E_n)_{n=1}^{N}$. Let us also denote by

$$A^{(N)} = \bigodot_{n=1}^{N} A_n = A_1 \odot A_2 \odot \cdots \odot A_N$$

the subalgebra of $C(E^{(N)})$ consisting of those functions f that have an expansion of the form

$$\text{(I)} \qquad f(x_1, x_2, \cdots, x_n) = \sum_{k=1}^{\infty} f_{1k}(x_1)f_{2k}(x_2) \cdots f_{Nk}(x_N)$$

where the functions f_{nk} are in A_n and

$$\text{(II)} \qquad M = \sum_{k=1}^{\infty} \| f_{1k} \|_{A_1} \cdot \| f_{2k} \|_{A_2} \cdots \| f_{Nk} \|_{A_N} < \infty .$$

When (I) and (II) hold, let us agree to say that the series in the right-hand side of (I) absolutely converges to f in norm, and to write

$$f = \sum_{k=1}^{\infty} f_{1k} \odot f_{2k} \odot \cdots \odot f_{Nk} .$$

We denote by $\| f \|_S = \| f \|_{S(A_1, A_2, \cdots, A_N)}$ the infimum of the M's as in (II), and call it the S-norm of f. It is a routine matter to verify that, with this norm, $A^{(N)}$ is a Banach algebra whose maximal ideal space can be naturally identified with the product space $E^{(N)}$. It is also easy to prove that $A^{(N)}$ is isometrically isomorphic to the Banach algebra

$$(A_1 \hat{\otimes} A_2 \hat{\otimes} \cdots \hat{\otimes} A_N)/R_N$$

with the quotient norm, where R_N denotes the radical of the algebra $A_1 \hat{\otimes} A_2 \hat{\otimes} \cdots \hat{\otimes} A_N$ (cf. Tomiyama [11]). We call $A^{(N)}$ the S-algebra induced from $(A_n)_{n=1}^{N}$. Let now $E = E_1 \times E_2 \times \cdots$ be the product space of $(E_n)_{n=1}^{\infty}$, and consider the subalgebra $A = \odot_{n=1}^{\infty} A_n$ of $C(E)$ that consists of all functions f having an expansion of the form

$$\text{(I}'\text{)} \qquad f(x) = \sum_{k=1}^{\infty} f_{1k}(x_1)f_{2k}(x_2) \cdots f_{N_k k}(x_{N_k})$$

for all points $x = (x_n)_{n=1}^{\infty}$ of E, where the functions f_{nk} are in A_n and

$$\text{(II}'\text{)} \qquad M = \sum_{k=1}^{\infty} \| f_{1k} \|_{A_1} \cdot \| f_{2k} \|_{A_2} \cdots \| f_{N_k k} \|_{A_{N_k}} < \infty .$$

When (I′) and (II′) hold, let us again agree to say that the series in (I′) absolutely converges to f in norm, and to write

$$f = \sum_{k=1}^{\infty} f_{1k} \odot f_{2k} \odot \cdots \odot f_{N_k k} .$$

The infimum of the M's as in (II′) is called the S-norm of f, and is denoted by $\| f \|_S = \| f \|_{S(A_1, A_2, \cdots)}$. With this norm, A becomes a Banach algebra, and its maximal ideal space can be natural identified with the product space E. We call A the S-algebra induced from the sequence $(A_n)_{n=1}^{\infty}$. In a trivial way, we then have the sequence of isometrical and algebraical imbeddings:

$$A^{(1)} = A_1 \subset A^{(2)} \subset \cdots \subset A^{(N)} \subset \cdots \subset A .$$

Note that the union of all $A^{(N)}$ is a dense subalgebra of A. Of course we can also define, in a similar way, the S-algebra induced from an arbitrary family of semi-simple Banach algebra with unit.

Let now $O = (O_n)_{n=1}^{\infty}$ be any fixed point of E, and let

$$\mathfrak{J}_N = \mathfrak{J}_N[O]: A \to A^{(N)} \subset A$$

be the natural norm-decreasing homomorphism defined by

$$\text{(III)} \qquad (\mathfrak{J}_N f)(x) = f(x_1, x_2, \cdots, x_N, O_{N+1}, O_{N+2}, \cdots) .$$

It is then trivial that we have

$$\text{(IV)} \qquad \| \mathfrak{J}_N \| = 1 \ (N = 1, 2, \cdots), \quad \text{and} \quad \lim_N \|\mathfrak{J}_N f - f \|_S = 0 \ (f \in A) .$$

Finally observe that, if $(B_n)_{n=1}^{\infty}$ is a permutation of $(A_n)_{n=1}^{\infty}$ and if B is the S-algebra induced from $(B_n)_{n=1}^{\infty}$, then A and B are isometrically isomorphic.

Hereafter, we fix two sequences $(A_n)_{n=1}^{\infty}$ and $(B_n)_{n=1}^{\infty}$ of semi-simple Banach algebras with unit, and associate with them A and B (the S-algebras induced from them), the product spaces $E = \prod_{n=1}^{\infty} E_n$ and $F = \prod_{n=1}^{\infty} F_n$, etc.

PROPOSITION 1. (cf. Hewitt and Ross [3: (42.7)]). (a) *For every natural number N, we have*

$$\text{(i)} \qquad \|f_1 \odot f_2 \odot \cdots \odot f_N \|_S = \prod_{n=1}^{N} \| f_n \|_{A_n} \quad (f_n \in A_n;\ n = 1, 2, \cdots, N) .$$

(b) *Let $(H_n: A_n \to B_n)_{n=1}^{N}$ be N bounded linear operators, then there exists a unique bounded linear operator $A^{(N)} \to B^{(N)}$, denoted by $H^{(N)} = \bigodot_{n=1}^{N} H_n$, such that*

$$\text{(ii)} \qquad H^{(N)}(f_1 \odot f_2 \odot \cdots \odot f_N) = H_1(f_1) \odot H_2(f_2) \odot \cdots \odot H_N(f_N)$$

for all functions f_n in A_n ($n = 1, 2, \cdots, N$). Further, th eoperator norm of $H^{(N)}$ is given by

$$\text{(iii)} \qquad \| H^{(N)} \| = \prod_{n=1}^{N} \| H_n \| .$$

PROOF. The first statement in part (*b*) is well-known and is contained in Hewitt and Ross [3: (42.7)]. Taking as B_n the field of complex numbers ($n = 1, 2, \cdots, N$), and applying the Hahn-Banach theorem, we obtain (i). Finally, (iii) is an easy consequence of (i). We omit the details.

PROPOSITION 2. *Let $(H_n: A_n \to B_n)_{n=1}^{\infty}$ be a sequence of bounded linear operators such that $H_n(1) = 1$ for all n and $\prod_{n=1}^{\infty} \| H_n \|$ converges. Then* *there exists a unique bounded linear operator* $A \to B$, *denoted by* $H = \bigodot_{n=1}^{\infty} H_n$, *such that*

$$\text{(i)} \qquad H(f_1 \odot f_2 \odot \cdots \odot f_N) = H_1(f_1) \odot H_2(f_2) \odot \cdots \odot H_N(f_N)$$

for all functions f_n *in* A_n $(n = 1, 2, \cdots, N;\ N = 1, 2, \cdots)$. *Further, the operator norm of* H *is given by*

$$\text{(iii)} \qquad \| H \| = \prod_{n=1}^{\infty} \| H_n \| .$$

PROOF. For each $N \geqq 1$, let us denote by $\tilde{H}^N\colon A \to B$ the composition of the three operators

$$A \xrightarrow{\mathfrak{J}_N} A^{(N)} \xrightarrow{H^{(N)}} B^{(N)} \xrightarrow{\mathfrak{d}_N} B ,$$

where $\mathfrak{J}_N$ is the operator defined by (III) for any fixed point O of E, $H^{(N)} = \bigodot_{n=1}^{N} H_n$, and $\mathfrak{d}_N$ the canonical imbedding. It is a routine matter to verify that $\| \tilde{H}^N \| = \prod_{n=1}^{N} \| H_n \|$ and that the sequence $(\tilde{H}^N f)_{N=1}^{\infty}$ converges in B for every f in $\bigcup_{N=1}^{\infty} A^{(N)}$. Therefore we can immediately prove the existence of H with the required property. The identity (ii) follows from Proposition 1, which completes the proof.

PROPOSITION 3. *Let* $(H_n\colon A_n \to B_n)_{n=1}^{\infty}$ *be a sequence of norm-decreasing linear operators with* $H_n(1) = 1$ *for all* n, *and suppose that each* H_n *has an approximating inverse in the sense of Varopoulos* [13]. *Then* $H = \bigodot_{n=1}^{\infty} H_n\colon A \to B$ *is an isometry.*

PROOF. For each $N \geqq 1$, the restriction of H to the closed linear subspace $A^{(N)}$ of A can be identified with the operator $H^{(N)}\colon A^{(N)} \to B^{(N)}$. It is then easy to see from Proposition 1 that each $H^{(N)}$ has an approximating inverse under our hypothesis, from which our assertion immediately follows.

We now consider any sequence $(H_n\colon A_n \to B_n)_{n=1}^{\infty}$ of norm-decreasing homomorphisms that satisfies the two requirements in Proposition 3. Let $(q_n\colon F_n \to E_n)_{n=1}^{\infty}$ be the sequence of the continuous mappings naturally induced by $(H_n)_{n=1}^{\infty}$, and denote by

$$q^{(N)} = q_1 \times q_2 \times \cdots \times q_N\colon F^{(N)} \to E^{(N)} ,$$

$$q = q_1 \times q_2 \times q_3 \times \cdots\colon F \to E ,$$

their product mappings. Observe then that we have

$$H^{(N)} f = f \circ q^{(N)} \ (f \in A^{(N)});\ Hf = f \circ q \ (f \in A) .$$

Using the operators $(\mathfrak{J}_N)_{N=1}^{\infty}$ defined as in (III) for a fixed point of F and the fact that H is an isometry, we have the following, which we do not prove.

PROPOSITION 4. *Suppose that we have*

(i) $\mathrm{Im}(H^{(N)}) = \{g \in B^{(N)}: g = f\circ q^{(N)}$ *for some* f *in* $C(E^{(N)})\}$ *for all* $N = 1, 2, \cdots$, *then*

(ii) $\mathrm{Im}(H) = \{g \in B: g = f\circ q$ *for some* f *in* $C(E)\}$

EXAMPLE 1. Suppose here that $A_n = C(E_n)$ and $B_n = C(F_n)$ for all n. Then the condition (i) of Proposition 4 is satisfied if every q_n is a continuous mapping of F_n onto E_n (see Saeki [9]). In particular, taking as B_n the Banach algebra consisting of all bounded complex-valued functions on E_n, we have: let f be a continuous function on E that has an expansion of the form

$$f(x) = \sum_{k=1}^{\infty} f_{1k}(x_1) f_{2k}(x_2) \cdots f_{N_k k}(x_{N_k}) \quad (x = (x_n)_{n=1}^{\infty} \in E) ,$$

where each f_{nk} is a bounded function on E_n and

$$\sum_{k=1}^{\infty} \| f_{1k} \|_\infty \cdot \| f_{2k} \|_\infty \cdots \| f_{N_k k} \|_\infty < \infty .$$

Then f is a function in the space $\textcircled{\bullet}_{n=1}^{\infty} C(E_n)$.

EXAMPLE 2. Suppose here that each E_n is a compact abelian group and $A_n = A(E_n)$, the Fourier algebra on E_n. Then we can identify the S-algebra A with the Fourier algebra $A(E)$ on the compact abelian group E. Suppose that $F_n = E_n \times E_n$ and $B_n = C(E_n) \textcircled{\bullet} C(E_n)$, and that

$$q_n(x, y) = x + y \quad (x, y \in E_n) \qquad \text{for all } n .$$

Then the condition (i) of Proposition 3 is satisfied (see Herz [2]).

THEOREM 1. *Suppose that every* E_n *contains at least two distinct points, and that every* A_n *satisfies the following two conditions*:

(a) *If* $f \in A_n$, *then* $\bar{f} \in A_n$ *and* $\|\bar{f}\|_{A_n} = \|f\|_{A_n}$;

(b) *With any* $\varepsilon > 0$ *and any two distinct points* O_n *and* x_n *of* E_n *there corresponds a function* u_n *in* A_n *such that*

$$\| u_n \|_{A_n} \leqq 1 + \varepsilon,\ u_n(O_n) = 0 , \quad \text{and} \quad u_n(x_n) = 1 .$$

Suppose also that $\Phi(t)$ *is a function defined on the interval* $[-1, 1]$ *of the real line* R, *and that* $\Phi(t)$ *operates in* A. *Then* $\Phi(t)$ *is analytic on the interval* $[-1, 1]$.

PROOF. We first prove our statement under the additional assumption that every E_n contains precisely two distinct points O_n and x_n. Let $\mathfrak{J}_N: A \to A$ be the operator defined by (III) for the point $O = (O_n)_{n=1}^{\infty}$ of

E, let A' be the Banach space dual of A, and take any functional P in A'. Then it is easy to see from (III) and (IV) that every $\mathfrak{J}_N^*(P)$ is a discrete measure in $M(E)$, and the sequence $(\mathfrak{J}_N^*(P))_{N=1}^\infty$ converges to P in the weak-star topology of A'. Since every $\mathfrak{J}_N$ has norm 1, it follows that

$$(1) \qquad \| f \|_S = \sup\left\{\left|\int_E f d\mu\right| : \mu \in M_d(E),\ \| \mu \|_{A'} \leqq 1\right\}$$

for all functions f in A. Suppose now that $\Phi(t)$ is as in our Theorem, and define for each r with $0 < r < 1$

$$\Phi_r(t) = \Phi(r \cdot \sin t) \qquad (-\infty < t < \infty)\,.$$

Using (b), we can easily prove that $\Phi(t)$ is continuous. It also follows from (b) and (1) that there are two positive numbers r and C such that

$$(2) \qquad \| \Phi_r(f + t) \|_S \leqq C \qquad (-\infty < t < \infty)\,.$$

for all functions f in $A_R = A \cap C_R(E)$ with $\| f \|_S \leqq \pi$ (see Rudin [7; 6.6.3]). Therefore, in order to prove that $\Phi(t)$ is analytic at $t = 0$, it suffices to find a positive number a such that

$$(3) \qquad \sup\{\| e^{ikf} \|_S : f \in A_R,\ \| f \|_S \leqq \pi\} \geqq e^{a|k|} \quad (k = 0, \pm 1, \pm 2, \cdots)\,.$$

For each n, let u_n be the function in A_n defined by $u_n(O_n) = 0$ and $u_n(x_n) = 1$. Then, by (b), $\| u_n \|_{A_n} = 1$; further, we have

$$\begin{aligned} \| \exp(i\pi u_{2n-1} \odot u_{2n}) \|_S &= \| \exp(i\pi\, u_{2n-1} \odot u_{2n}) \|_{A_{2n-1} \odot A_{2n}} \\ &\geqq \| \exp(i\pi\, u_{2n-1} \odot u_{2n}) \|_{C(E_{2n-1}) \odot C(E_{2n})} \geqq 2^{1/2}\,; \end{aligned}$$

the last inequality following from Lemma 2.1 in Saeki [10]. Therefore, setting

$$f_k = k^{-1}\pi \sum_{n=1}^{k} u_{2n-1} \odot u_{2n} \qquad (k = 1, 2, \cdots)\,,$$

we have $\| f_k \|_S \leqq \pi$, and

$$\begin{aligned} \| \exp(ikf_k) \|_S &= \| \exp(-ikf_k) \|_S \\ &= \prod_{n=1}^{k} \| \exp(i\pi\, u_{2n-1} \odot u_{2n}) \|_S \geqq 2^{k/2} \quad (k = 1, 2, \cdots) \end{aligned}$$

by Proposition 1. Thus (3) holds for $a = 2^{-1} \log 2$. This completes the proof of our statement in the case that $\mathrm{Card}(E_n) = 2$ for all n.

Suppose now that $\mathrm{Card}\,(E_n) \geqq 2$ for all n. We take any two distinct points O_n and x_n of E_n, and put $F_n = \{O_n, x_n\}$. Let B_n be the restriction algebra of A_n on the set F_n endowed with the natural quotient norm; it is easy to see that the maximal ideal space of B_n is F_n, and that the restriction algebra B of A on the set $F = F_1 \times F_2 \times \cdots$ can be identified with the S-algebra induced from the sequence $(B_n)_{n=1}^{\infty}$ in a trivial way. Since A is self-adjoint by (a), every function, that is defined on the real line and operates in A, operates in B. This fact, combined with the result in the preceding paragraph, establishes our Theorem.

REMARK. Under the same assumption, we can prove that: if $\Phi(z)$ is a function defined on the square $L = \{z;\ |\operatorname{Re}(z)| \leqq 1, \text{ and } |\operatorname{Im}(z)| \leqq 1\}$ of the complex plane, and if $\Phi(z)$ operates in A, then $\Phi(z)$ is real-analytic on L.

THEOREM 2. *Suppose that, for each n, there exist a function u_n in A_n and two points O_n and x_n of E_n such that*

$$\|u_n\|_{A_n} \leqq C,\ u_n(O_n) = 0\ , \quad \text{and} \quad u_n(x_n) = 1\ ,$$

where C is a constant independent of n. Then there exists a function g in A such that the closed ideals in A which are generated by $g^m (m = 1, 2, \cdots)$ are all distinct.

PROOF. By considering some restriction algebra of A, we may assume that $E_n = \{O_n, x_n\}$ for all n. We regard each E_n as a "compact" abelian group, and E as the product group of $(E_n)_{n=1}^{\infty}$. We then define μ to be the Haar measure on E normalized so that $\mu(E) = 1$. Let u_n be as in our theorem and write

$$f = \sum_{n=1}^{\infty} 4n^{-2} u_{2n-1} \odot u_{2n}\ , \tag{1}$$

which absolutely converges in norm by hypothesis. We then assert that, for some real number a, the function $g = f - a$ has the required property. To prove this, let $m < n$ be two natural numbers, and s an arbitrary real number. We then have

(2)
$$\begin{aligned} &\sup\left\{\left|\int_E (f_m \odot f_n)\cdot \exp(isu_m \odot u_n)d\mu\right| : f_j \in A_j,\ \|f_j\|_{A_j} \leqq 1\ (j = m, n)\right\} \\ &\leqq \sup\left\{\left|\int_E (f_m \odot f_n)\cdot \exp(isu_m \odot u_n)d\mu\right| : f_j \in C(E_j),\ |f_j| \leqq 1\ (j = m, n)\right\} \\ &\leqq 4^{-1}\sup\{|z+1| + |z+e^{is}| : |z| \leqq 1\} = \max\{|\cos(s/4)|, |\sin(s/4)|\}\ . \end{aligned}$$

Let now N be any natural number, and take any function f_n in A_n with $\|f_n\|_{A_n} \leqq 1$, $n = 1, 2, \cdots, 2N$. Then, setting $f_n = 1$ for all n larger than $2N$, we observe that the functions

$$g_n = (f_{2n-1} \odot f_{2n})\cdot \exp(i4tn^{-2}u_{2n-1} \odot u_{2n}), \qquad (n = 1, 2, \cdots)$$

are independent random variables on the probability space (E, μ). It follows from (2) that

$$\left|\int_E (f_1 \odot f_2 \odot \cdots \odot f_{2N}) \cdot \exp(itf)d\mu\right|$$
$$= \prod_{n=1}^{\infty} \left|\int_E g_n d\mu\right| \leqq \prod_{n=1}^{\infty} \max\{|\cos(n^{-2}t)|, |\sin(n^{-2}t)|\} \qquad (-\infty < t < \infty) .$$

Consequently we have

$$\sup\left\{\left|\int_E h \cdot \exp(itf)d\mu\right| : h \in A, \|h\|_S \leqq 1\right\}$$
$$\leqq \prod_{n=1}^{\infty} \max\{|\cos(n^{-2}t)|, |\sin(n^{-2}t)|\} \qquad (-\infty < t < \infty) . \tag{3}$$

Therefore, our assertion will follow from a theorem of *P.* Malliavin [5] (see also Rudin [7: 7.6.3]) as soon as we have proved that

$$\prod_{n=1}^{\infty} \max\{|\cos(n^{-2}t)|, |\sin(n^{-2}t)|\} \leqq b \cdot \exp(-c|t|^{1/2}) \qquad (-\infty<t<\infty) . \tag{4}$$

for some positive numbers b and c. For a given $t > 8\pi$, let $N = N_t$ be the smallest positive integer such that $t \leqq (\pi/4)N^2$. Since

$$\cos s \leqq 1 - 4^{-1}s^2 \leqq \exp(-4^{-1}s^2) \qquad (-\pi/2 \leqq s \leqq \pi/2) ,$$

we then have

$$\prod_{n=1}^{\infty} \max\{|\cos(n^{-2}t)|, |\sin(n^{-2}t)|\} \leqq \prod_{n=N}^{\infty} |\cos(n^{-2}t)|$$
$$\leqq \exp\left(-4^{-1} \sum_{n=N}^{\infty} n^{-4}t^2\right)$$
$$\leqq \exp(-(12)^{-1}N^{-3}t^2) .$$

But it is clear that $N^2 \leqq 8t/\pi$, and hence (4) follows. This completes the proof.

REMARKS. Let E_n, u_n, and μ be as in the proof of Theorem 2.

(a) We can determine the range of the values of a with the required property as follows. Let

$$f_1 = 4 \sum_{n=1}^{\infty} (2n - 1)^{-2} u_{4n-3} \odot u_{4n-2} ,$$
$$f_2 = 4 \sum_{n=1}^{\infty} (2n)^{-2} u_{4n-1} \odot u_{4n} ,$$

and let $F_1(t)$, $F_2(t)$, $F(t)$ be the distribution functions of f_1, f_2, f when they are regarded as random variables on the probability space (E, μ). It is easy to see that these distribution functions are all infinitely differentiable. Further, since f_1 and f_2 are independent, $w(t)$ is the convolution of $w_1(t)$ and $w_2(t)$, where $w_1(t)$, $w_2(t)$ and $w(t)$ are the derivatives of $F_1(t)$, $F_2(t)$, and $F(t)$. Since $\sum_{n=1}^{\infty}(2n-1)^{-2} = 8^{-1}\pi^2$ and $\sum_{n=1}^{\infty} n^{-2} = 6^{-1}\pi^2$, it is easy to prove that

$$\operatorname{supp}(w_1) = [0, 2^{-1}\pi^2 - 4] \cup [4, 2^{-1}\pi^2] ;$$
$$\operatorname{supp}(w_2) = [0, 6^{-1}\pi^2 - 1] \cup [1, 6^{-1}\pi^2] .$$

But $w_1(t)$ and $w_2(t)$ are both non-negative, and so we have

$$L = \{a \in R: w(a) \neq 0\} = (0, 3^{-1}2\pi^2 - 4) \cup (4, 3^{-1}2\pi^2) .$$

Therefore, for every a in L, the closed ideals in A generated by each $(f-a)^m$ $(m = 1, 2, \cdots)$ are all distinct. Note also that, for every b in $R\backslash L$, the set $f^{-1}(b)$ is empty or consists of a single point. Hence the range of the values of a with the required property is precisely L.

Another example may be given by

$$(*) \qquad h = 6 \sum_{n=1}^{\infty} n^{-2}(u_{4n-3} \odot u_{4n-2} - u_{4n-1} \odot u_{4n}) .$$

Then the range of the required a's is the open interval $(-\pi^2, \pi^2)$.

(b) Let $(Z_p)_{p=1}^{\infty}$ be any countable family of countable disjoint subsets of the index set $\{1, 2, 3, \cdots\}$, and let S_p be the S-algebra induced from the family $\{A_n: n \in Z_p\}$. We shall identify each S_p with a closed subalgebra of A. Let h_p be the function in S_p defined quite similarly as in (*). Then the closed ideals in A generated by each

$$h_1^{q_1}h_2^{q_2} \cdots h_m^{q_m}(q_j = 0, 1, 2, \cdots; j = 1, 2, \cdots, m; m = 1, 2, \cdots)$$

are all distinct. The same conclution is true for the sequence $(f_p)_{p=1}^{\infty}$, where $f_{2p-1} = h_{2p-1} + ih_{2p}$ and $f_{2p} = h_{2p-1} - ih_{2p}$ $(p = 1, 2, \cdots)$.

Let now G be a locally compact abelian group, and $\hat{G}$ its dual. Let also $(E_n)_{n=1}^{\infty}$ be a sequence of compact subsets of $G_n = G$, and put

$$E = \prod_{n=1}^{\infty} E_n \subset G^{\infty} = \prod_{n=1}^{\infty} G_n .$$

We require the sequence $(E_n)_{n=1}^{\infty}$ to satisfy the following condition.

(R) For every point $x = (x_n)_{n=1}^{\infty}$ of E, the series $p(x) = p_E(x) = \sum_{n=1}^{\infty} x_n$ converges in G, and the mapping p: $E \to G$ so obtained is continuous.

Under this condition, we put $\tilde{E} = p(E)$, which is a compact subset of G. Observe then that, for every character γ in $\hat{G}$, the product

$$\gamma \circ p(x) = \prod_{n=1}^{\infty} \gamma(x_n) \qquad (x = (x_n)_{n=1}^{\infty} \in E)$$

uniformly converges on E. We now proceed to obtain a sufficient condition for the restriction algebra $A(\tilde{E})$ of the Fourier algebra $A(G)$ to be isomorphic to the S-algebra induced from the sequence $(A(E_n))_{n=1}^{\infty}$. We begin with proving the following.

LEMMA 1 (cf. Varopoulos [12]). (a) *For every real number d with $0 < d < \pi$, we have*

$$\eta(d) = \| e^{is} - 1 \|_{A(d)} < \{(\pi + d)/(\pi - d)\}^{1/2} d ,$$

where $A(d)$ denotes the the restriction algebra of $A(T)$ on the interval $[-d, d]$.

(b) *Let A be a simi-simple Banach algebra represented as a function algebra on some space, and let f_1 and f_2 be two functions in A such that*

$$|f_j| \equiv 1 , \quad \text{and} \quad \| f_j^k \|_A \leqq M_j \qquad (j = 1, 2; k = 0, \pm 1, \pm 2, \cdots) .$$

Then $|arg(f_1 \cdot \bar{f}_2)| \leqq d < \pi$ implies $\| f_1 - f_2 \|_A \leqq \eta(d) M_1 M_2$.

PROOF. Let g_1 and g_2 be the characteristic functions of the intervals $[-(\pi + d)/2, (\pi + d)/2]$ and $[-(\pi - d)/2, (\pi - d)/2]$ of the real line R. Writing $w = (\pi - d)^{-1} g_1 * g_2$, observe that

$$\| w \|_{A(R)} < \{(\pi + d)/(\pi - d)\}^{1/2} , \qquad w = 1 \text{ on } [-d, d] ,$$

and

$$\operatorname{supp}(w) = [-\pi, \pi] .$$

Let v be the odd function in $B(R)$ with period $4d$ defined by the requirements $v(s) = s$ $(0 \leqq s \leqq d)$ and $v(s) = 2d - s$ $(d \leqq s \leqq 2d)$. It is clear that $v(s - d)$ is positive-definite, and hence $\| v \|_{B(R)} = d$. Define

$$u(e^{is}) = iw(s)v(s)\int_0^1 e^{ist} dt \qquad (-\pi \leqq s \leqq \pi) .$$

It is then trivial that $u(e^{is}) = e^{is} - 1$ on $[-d, d]$. Further,

$$\begin{aligned} \hat{u}(k) &= \frac{i}{2\pi} \int_0^1 \left\{ \int_{-\pi}^{\pi} w(s)v(s) e^{i(t-k)s} ds \right\} dt \\ &= \frac{i}{2\pi} \int_0^1 \widehat{w \cdot v}(k - t) dt \qquad (k = 0, \pm 1, \pm 2, \cdots) \end{aligned}$$

and hence the $A(T)$-norm of u is smaller than the $A(R)$-norm of wv, which establishes part (a).

Suppose now that f_1 and f_2 are functions in A as in part (b), and let u be any function in $A(T)$ such that $u(e^{is}) = e^{is} - 1$ on $[-d, d]$. Then, if $|arg(f_1 \cdot \bar{f}_2)| \leqq d$, we have

$$f_1 - f_2 = f_2 \cdot u(f_1 \cdot \bar{f}_2) = \sum_{k=-\infty}^{\infty} \hat{u}(k) f_1^k f_2^{1-k} ,$$

and hence

$$\| f_1 - f_2 \|_A \leqq \sum_{k=-\infty}^{\infty} | \hat{u}(k) | M_1 M_2 = \| u \|_{A(T)} M_1 M_2 ,$$

which, combined with part (a), establishes part (b).

Throughout the remainder part of this paper, we denote by d_0 the positive solution of the equation $\{(\pi + d)/(\pi - d)\}^{1/2} d = 1$. Then note that $d_0 = 0.77 \cdots$, and that $0 < d \leqq d_0$ implies $\eta(d) < 1$.

LEMMA 2 (cf. Hewitt and Ross [3: (40.17)]). *Let K be any compact subset of a locally compact abelian group G, and let f be any function in $A(K)$. Then, for every positive real number C larger than the $A(K)$-norm of f, there are a sequence $(a_n)_{n=1}^{\infty}$ of complex numbers and a sequence $(\gamma_n)_{n=1}^{\infty}$ of characters in $\hat{G}$ such that*

$$\sum_{n=1}^{\infty} | a_n | \leqq C , \quad \text{and} \quad f = \sum_{n=1}^{\infty} a_n \gamma_n \text{ on } K .$$

PROOF. It suffices to note that the set

$$\left\{ \sum_{n=1}^{\infty} a_n \gamma_n \in A(K) : \sum_{n=1}^{\infty} | a_n | \leqq 1, \gamma_n \in \hat{G} \ (n = 1, 2, \cdots) \right\}$$

is norm-dense in the closed unit ball of $A(K)$, which is an easy consequence of the Hahn-Banach theorem.

LEMMA 3. *Let $(E_n)_{n=1}^{\infty}$ be a sequence of compact subsets of a locally compact abelian group G.*

(a) *If G is compact, then the restriction algebra $A(E)$ of $A(G^{\infty})$ is isometrically isomorphic to the S-algebra A_E induced from the sequence $(A(E_n))_{n=1}^{\infty}$.*

(b) *If the sequence $(E_n)_{n=1}^{\infty}$ satisfies Condition (R), then the operator $P = P_E$ defined by*

$$P(f) = f \circ p_E \qquad (f \in A(\tilde{E}))$$

is a norm-decreasing homomorphism of $A(\bar{E})$ into A_E.

PROOF. Part (a) is a direct consequence of the definition of an S-algebra and the fact that $A(G^{\infty})$ is the S-algebra induced from the sequence $(A(G_n))_{n=1}^{\infty}$ if G is compact.

We now prove part (b). By Lemma 2, it suffices to verify that, for every character γ in $\hat{G}$, the function $\chi = \gamma \circ p_E$ is in A_E and $\| \chi \|_S = 1$. Define

$$\chi_N(x) = \prod_{n=1}^{N} \gamma(x_n) \qquad (x = (x_n)_{n=1}^{\infty} \in E;\ N = 1, 2, \cdots) .$$

Then each χ_N is in A_E and its S-norm is 1 by Proposition 1. Since $(\chi_N)_{N=1}^{\infty}$ uniformly converges to χ, it follows from Lemma 1 that χ is in A_E and its S-norm is 1. This completes the proof.

THEOREM 3. *Let $(E_n)_{n=1}^{\infty}$ be a sequence of compact subsets of a locally compact abelian group G that satisfies Condition* (R). *Suppose, in addition, that there exists a constant d, $0 < d \leqq d_0$, such that:*

(S, d) For any characters $(\gamma_n)_{n=1}^{N}$ in $\hat{G}$, we can find a character γ in $\hat{G}$ such that

$$|\, arg[\overline{(\gamma \circ p)} \cdot (\gamma_1 \odot \gamma_2 \odot \cdots \odot \gamma_N)]\,| \leqq d \text{ on } E .$$

Then the homomorphism $P = P_E$ defined in Lemma 3, is an isomorphism of $A(\tilde{E})$ onto A_E, and $\|P^{-1}\| \leqq (1 - \eta(d))^{-1}$. In particular, if Condition (S, d) holds for every $d > 0$, then P is an isometry.

PROOF. We fix any function f in A_E, and take any positive number C larger than $\|f\|_S$. It is easy to see from Lemma 2 that f has an expansion of the form

$$f = \sum_{k=1}^{\infty} a_k(\gamma_{1k} \odot \gamma_{2k} \odot \cdots \odot \gamma_{N_k k}) \text{ on } E ,$$

where the γ_{nk} are characters in $\hat{G}$ regarded as functions on E_n, and $\sum_{k=1}^{\infty} |a_k| < C$. By condition (S, d), we can choose a sequence $(\gamma_k)_{k=1}^{\infty}$ of characters so that

$$|\, arg[\bar{\chi}_k \cdot (\gamma_{1k} \odot \gamma_{2k} \odot \cdots \odot \gamma_{N_k k})]\,| \leqq d \quad \text{on } E ,$$

where $\chi_k = \gamma_k \circ p_E$. Putting $g_0 = \sum_{k=1}^{\infty} a_k \gamma_k$, we see that g_0 is in $A(\tilde{E})$ and $\|g_0\|_{A(\tilde{E})} < C$. It also follows from part (b) of Lemma 1 that

$$\begin{aligned} \|f - P(g_0)\|_S &\leqq \sum_{k=1}^{\infty} |a_k| \cdot \|\gamma_{1k} \odot \cdots \odot \gamma_{N_k k} - \chi_k\|_S \\ &\leqq \sum_{k=1}^{\infty} |a_k|\, \eta(d) < C \cdot \eta(d) . \end{aligned}$$

Repeating the same argument for $f - P(g_0)$ and $C \cdot \eta(d)$, and so on, we can find a sequence $(g_j)_{j=0}^{\infty}$ of functions in $A(\tilde{E})$ such that

$$\|g_j\|_{A(\tilde{E})} < C \cdot \eta(d)^j , \quad \text{and} \quad \|f - P(\textstyle\sum_{k=0}^{j} g_k)\|_S < C \cdot \eta(d)^{j+1}$$

for all $j = 1, 2, \cdots$. Since $\eta(d) < 1$ by hypothesis, the series $g = \sum_{j=0}^{\infty} g_j$ converges in $A(\tilde{E})$, and we have

$$\|g\|_{A(\tilde{E})} < C \cdot (1 - \eta(d))^{-1} , \quad \text{and} \quad f = P(g) .$$

But, since P is a monomorphism and C was an arbitrary number larger than $\| f \|_S$, we have $\| g \|_{A(\tilde{E})} \leqq (1 - \eta(d))^{-1} \| f \|_S$. This implies that P is an isomorphism and $\| P^{-1} \| \leqq (1 - \eta(d))^{-1}$. Finally, the last statement in our theorem is now trivial since P is a norm-decreasing operator. This completes the proof.

COROLLARY 3.1. *Let G_1 and G_2 be two locally compact abelian groups, let $(E_n \subset G_1)_{n=1}^\infty$ and $(F_n \subset G_2)_{n=1}^\infty$ be two sequences of compact sets, and put $E = \prod_{n=1}^\infty E_n$ and $F = \prod_{n=1}^\infty F_n$. Let also $(H_n;\ A(E_n) \to A(F_n))_{n=1}^\infty$ be a sequence of homomorphisms with $H_n(1) = 1$, and let $(q_n\colon F_n \to E_n)_{n=1}^\infty$ be the sequence of the continuous mapping naturally induced by $(H_n)_{n=1}^\infty$. Suppose, in addition, that the product $\prod_{n=1}^\infty \| H_n \|$ converges, and that E satisfies Condition* (R) *while F satisfies both Conditions* (R) *and* (S, d) *for some d with $0 < d \leqq d_0$. If we define*

$$\tilde{q}\left(\sum_{n=1}^\infty y_n\right) = \sum_{n=1}^\infty q_n(y_n) \in \tilde{E} \qquad (y_n \in F_n;\ n = 1, 2, \cdots)\ ,$$

and $\tilde{H}(f) = f \circ \tilde{q}$ $(f \in A(\tilde{E}))$, then $\tilde{H}$ is a homomorphism of $A(\tilde{E})$ into $A(\tilde{F})$, and $\| \tilde{H} \| \leqq (1 - \eta(d))^{-1} \prod_{n=1}^\infty \| H_n \|$; further, the diagram

$$\begin{array}{ccc}
A(\tilde{E}) & \xrightarrow{\ P_E\ } & A_E = \bigodot_{n=1}^\infty A(E_n) \\
\tilde{H}\Big\downarrow & & \Big\downarrow H \\
A(\tilde{F}) & \xrightarrow{\ P_F\ } & A_F = \bigodot_{n=1}^\infty A(F_n)
\end{array}$$

is commutative, where H denotes the homomorphism naturally induced by the sequence $(H_n)_{n=1}^\infty$.

PROOF. Put

$$p_E(x) = \sum_{n=1}^\infty x_n\ , \quad \text{and} \quad p_F(y) = \sum_{n=1}^\infty y_n \qquad (x \in E,\ y \in F)\ ,$$

and let $q\colon F \to E$ be the product mapping of $(q_n)_{n=1}^\infty$. Note that p_F is a homeomorphism since P_F is an isomorphism by Theorem 3. It is trivial that $\tilde{q} = p_E \circ q \circ p_F^{-1}$, and hence $\tilde{H} = P_F^{-1} \circ H \circ P_E$, which, together with Lemma 3, Proposition 2, and Theorem 3, yields the desired conclusions.

Theorem 1 and Theorem 3 yield the following Helson-Kahane-Katznelson-Rudin theorem [1], which is a special case of Theorem 9.3.4 of Varopoulos [13].

COROLLARY 3.2. *Let $(E_n)_{n=1}^\infty$ be a sequence of compact subsets of a locally compact abelian group G. Suppose that $Card(E_n) \geqq 2$ for all n, and that $(E_n)_{n=1}^\infty$ satisfies both Conditions* (R) *and* (S, d) *for some d with $0 < d \leqq d_0$. Under these conditions, if $\Phi(t)$ is a function defined on the* *interval* $[-1, 1]$ *of the real line, and if* $\Phi(t)$ *operates in* $A(\tilde{E})$, *then* $\Phi(t)$ *is analytic on the interval* $[-1, 1]$.

Theorem 2 and Theorem 3 yield the following Malliavin theorem [5].

COROLLARY 3.3. *Let* $(E_n)_{n=1}^{\infty}$ *be as in Corollary* 3.2. *Then there exists a sequence* $(h_n)_{n=1}^{\infty}$ *of real-valued functions in* $A(\tilde{E})$ *for which we have*:

(a) *The closed ideals in* $A(\tilde{E})$ *generated by each function*

$$h_1^{q_1} h_2^{q_2} \cdots h_m^{q_m} (q_j = 0, 1, 2, \cdots; j = 1, 2, \cdots, m; m = 1, 2, \cdots)$$

are all distinct.

(b) *The same conclusion is true for the sequence* $(f_n)_{n=1}^{\infty}$, *where*

$$f_{2n-1} = h_{2n-1} + i h_{2n}, \quad \text{and} \quad f_{2n} = \bar{f}_{2n-1} \qquad \text{for all } n .$$

Let now G be a locally compact, metric, abelian group with a translation-invariant metric $d(x, y)$, and let $(\varepsilon_n)_{n=1}^{\infty}$ be a sequence of positive real numbes such that $\sum_{n=1}^{\infty} n\varepsilon_n < \infty$. Let also $(E_n)_{n=1}^{\infty}$ be a sequence of compact subsets of G such that

$$\text{(A)} \qquad \sum_{n=1}^{\infty} \sup\{d(x, 0) : x \in E_n\} < \infty .$$

Then it is easy to see that $(E_n)_{n=1}^{\infty}$ satisfies Condition (R). We assume that there exists a sequence $(\Gamma_n)_{n=1}^{\infty}$ of subsets of $\hat{G}$ such that:

(B) For every natural number n, we have

$$\chi \in \Gamma_n \Longrightarrow |1 - \chi| < \varepsilon_N \text{ on } \sum_{k=N}^{\infty} E_k \qquad (N = n+1, n+2, \cdots) ;$$

(C) For every natural number n and every character γ in $\hat{G}$, we can find a character χ in Γ_n such that $|\gamma - \chi| < \varepsilon_n$ on E_n.

Under these conditions we assert that the sequence $(E_n)_{n=1}^{\infty}$ satisfies Condition (S, d) for some $0 < d \leqq d_0$, provided that the sum $\sum_{n=1}^{\infty} n\varepsilon_n$ is smaller than a certain constant. In fact, let $(\gamma_n)_{n=1}^{N}$ be given N characters in $\hat{G}$. By (C), there exists a χ_N in Γ_N such that $|\gamma_N - \chi_N| < \varepsilon_N$ on E_N. Again by (C), there exists a character χ_{N-1} in Γ_{N-1} such that

$$|\gamma_{N-1} - \chi_{N-1} \cdot \chi_N| < \varepsilon_{N-1} \text{ on } E_{N-1} .$$

Repeating this process, we obtain N characters $(\chi_n \in \Gamma_n)_{n=1}^{N}$ such that

$$\left|\gamma_n(x_n) - \prod_{j=n}^{N} \chi_j(x_n)\right| < \varepsilon_n \qquad (x_n \in E_n;\ n = 1, 2, \cdots, N) .$$

Put $\chi = \chi_1 \cdot \chi_2 \cdots \chi_N \in \hat{G}$; then, for any points $(x_n \in E_n)_{n=1}^{N}$, we have by (B)

$$\begin{aligned}\left|\prod_{n=1}^{N}\gamma_n(x_n)-\prod_{n=1}^{N}\chi(x_n)\right| &\leqq \sum_{n=1}^{N}|\gamma_n(x_n)-\chi(x_n)| \\ &\leqq \sum_{n=1}^{N}\left\{\left|\gamma_n(x_n)-\prod_{j=n}^{N}\chi_j(x_n)\right|+\left|1-\prod_{j=1}^{n-1}\chi_j(x_n)\right|\right\} \\ &\leqq \sum_{n=1}^{N}\{\varepsilon_n+(n-1)\varepsilon_n\}=\sum_{n=1}^{N}n\varepsilon_n\,.\end{aligned}$$

Therefore, for any point $x=(x_n)_{n=1}^{\infty}$ of $E=\prod_{n=1}^{\infty}E_n$, we have

$$\begin{aligned}&|(\gamma_1\circledast\gamma_2\circledast\cdots\circledast\gamma_N)(x)-(\chi\circ p_E)(x)| \\ &\leqq\left|\prod_{n=1}^{N}\gamma_n(x_n)-\prod_{n=1}^{N}\chi(x_n)\right|+\left|\prod_{n=N+1}^{\infty}\prod_{j=1}^{N}\chi_j(x_n)-1\right| \\ &<\sum_{n=1}^{N}n\varepsilon_n+N\varepsilon_{N+1}<\sum_{n=1}^{\infty}n\varepsilon_n\,.\end{aligned}$$

Consequently we conclude from Theorem 3 that $A(\widetilde{E})$ is isomorphic to the S-algebra induced from the sequence $(A(E_n))_{n=1}^{\infty}$ if the sum $\sum_{n=1}^{\infty}n\varepsilon_n$ is smaller than a certain constant, say, $2\sin(d_0/2)$. Thus we can now prove the following.

THEOREM 4. *Let G be any non-discrete locally compact abelian group.*

(a) *Suppose that G contains a closed subgroup which is an I-group. Then, for every $\varepsilon>0$, there exists a Cantor subset K of G such that the restriction algebra $A(K)$ is isomorphic to the S-algebra $S(K)$ induced from countable replicas of $C(K)$ and such that*

$$\|f\|_{S(K)}\leqq\|f\|_{A(K)}\leqq(1+\varepsilon)\|f\|_{S(K)}\qquad(f\in A(K))$$

when we identify $A(K)$ and $S(K)$ in a natural way.

(b) *Suppose that G does not contain any I-subgroup, then G contains a compact subgroup K isomorphic to D_q for some $q\geqq 2$. In this case, $A(K)$ is isometrically isomorphic to the S-algebra induced from countable replicas of $A(D_q)$.*

PROOF. The first statement in part (b) is well-known (see Rudin [7; 2.5.5]), and the second one is trivial.

In order to prove part (a), we may assume that G is itself an I-group having a translation-invariant metric compatible with its topology. Thus, for any given sequence $(r_n)_{n=1}^{\infty}$ of natural numbers and any given sequence $(\varepsilon_n)_{n=1}^{\infty}$ of positive real numbers, it is easy to construct a sequence $(E_n)_{n=1}^{\infty}$ of subsets of G so that: every E_n consists of r_n independent elements and $(E_n)_{n=1}^{\infty}$ satisfies all the Conditions (A), (B) and (C) (cf [7: 5.2.4]). In particular, it follows from the above observations that, for any $\varepsilon>0$, G contains a compact subset $\widetilde{E}$ such that $A(\widetilde{E})$ can be identified with

$S(E) = \odot_{n=1}^{\infty} C(E_n)$, where each E_n is a compact space consisting of two distinct points, and such that

$$\| f \|_{S(E)} \leqq \| f \|_{A(\tilde{E})} \leqq (1 + \varepsilon) \| f \|_{S(E)} .$$

But it is easy to see that $E = \prod_{n=1}^{\infty} E_n$ contains a Cantor set K such that the restriction algebra of $S(E)$ on K is isometrically isomorphic to $C(K)$. Further, $S(E)$ may be regarded as the S-algebra induced from countable replicas of itself. These facts establish part (a), and the proof is complete.

REMARK. For every sequence $(E_n)_{n=1}^{\infty}$ of compact spaces, the S-algebra induced from $(C(E_n))_{n=1}^{\infty}$ is isometrically isomorphic to a restriction algebra of the Fourier algebra of some compact abelian group. This follows from the fact that every compact space is homeomorphic to a Kronecker subset of a compact abelian group (see Saeki [8: Theorem 2]).

EXAMPLE 3. Let X_1 and X_2 be two perfect compact spaces, and

$$V(X) = C(X_1) \hat{\otimes} C(X_2) = C(X_1) \odot C(X_2) .$$

For simplicity, suppose that both X_1 and X_2 are totally disconected. Then there exists a continuous "onto" mapping q_j: $X_j \to D_2$ for $j = 1, 2$. We consider the diagram

$$A(D_2) \xrightarrow{M} V(D_2) = C(D_2) \hat{\otimes} C(D_2) \xrightarrow{Q} V(X) ,$$

where M is the isometric homomorphism defined by Herz [2], and Q is the isometric homomorphism naturally induced by the mappings q_1 and q_2. The operator Q has an approximating inverse consisting of norm-decreasing homomorphisms [9]. This property of Q, together with the well-known property of M [2] and Theorem 2, yields the following: there exists a sequence of real-valued functions in $V(X)$ that satisfies the conclusions (a) and (b) in Corollary 3.3.

EXAMPLE 4. Let $(E_n)_{n=1}^{\infty}$ be a sequence of finite subsets of R^N. Then we have isometrically $A(E_n) = A(tE_n)$ for every real positive number t, where $tE_n = \{tx\colon x \in E_n\}$. Thus, the observations preceding Theorem 4 assure that R^N contains a compact subset K such that $A(K)$ is isomorphic to $\odot_{n=1}^{\infty} A(E_n)$.

EXAMPLE 5. Let $(p_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be two sequence of positive integers and positive real numbers, respectively. Suppose that

$$\sum_{n=1}^{\infty} p_{n+1} t_{n+1}/t_n < \infty , \quad \text{and} \quad t_n > \sum_{k=n+1}^{\infty} p_k t_k \quad (n = 1, 2, \cdots) ,$$

and put

$$\tilde{F} = \left\{\sum_{n=1}^{\infty} r_n t_n : r_n = 0, 1, 2, \cdots, p_n \ (n = 1, 2, \cdots)\right\} \subset R .$$

Then, it is not difficult to prove that $A(\tilde{F})$ is isomorphic to the S-algebra $\odot_{n=1}^{\infty} A(F_n)$, where $F_n = \{rt_n : r = 0, 1, \cdots, p_n\}$ for all n (cf. the arguments preceding Theorem 4). Let now $(s_n)_{n=1}^{\infty}$ be any sequence of real numbers such that $\sum_{n=1}^{\infty} p_n \mid s_n \mid < \infty$, and put

$$\tilde{E} = \left\{\sum_{n=1}^{\infty} r_n s_n : r_n = 0, 1, 2, \cdots, p_n \ (n = 1, 2, \cdots)\right\} \subset R .$$

If we define $\tilde{q}$: $\tilde{F} \to \tilde{E}$ by setting

$$\tilde{q}\left(\sum_{n=1}^{\infty} r_n t_n\right) = \sum_{n=1}^{\infty} r_n s_n \quad (r_n = 0, 1, 2, \cdots, p_n;\ n = 1, 2, \cdots) ,$$

it follows from Corollary 3.1 that $\tilde{q}$ induces a homomorphism of $A(\tilde{E})$ into $A(\tilde{F})$. In particular, taking $p_n = 1$ for all n, we obtain a theorem of Y. Meyer [6].

EXAMPLE 6. Here we shall explicitly construct a function g in $A(T)$ such that the closed ideals in $A(T)$ which are generated by each g^m $(m = 1, 2, \cdots)$ are all distinct. To do this, we shall identify T with the interval $(-\pi, \pi]$ mod 2π. Let us fix any positive integer $p \geqq 3$, and let $w = w_p$ be any function in $A(T)$ such that: $w(t) = 0$ on the three intervals of length $2\pi/p^2(p-1)$ and with the left-end points 0, $2\pi/p^2$, $2\pi/p$; and $w(t) = 1$ on the interval $[2\pi/p + 2\pi/p^2, 2\pi/(p-1)]$. We put

$$f(t) = \sum_{n=1}^{\infty} n^{-2} w(p^{2n-2} t) ,$$

and assert that, for every real number a in the open set

$$M = (0, \pi^2/6 - 1) \cup (1, \pi^2/6) ,$$

the function $g = f - a$ has the required property. We consider the subsets of T

$$E_n = \{0, 2\pi/p^n\} , \quad \text{and} \quad \tilde{E} = \left\{\sum_{n=1}^{\infty} \varepsilon_n 2\pi/p^n : \varepsilon_n = 0 \text{ or } 1 \text{ for all } n\right\} ,$$

and define p_E: $E = \prod_{n=1}^{\infty} E_n \to \tilde{E}$ in a natural way. Then, by lemma 3, p_E induces a norm-decreasing homomorphism P of $A(\tilde{E})$ into $A_E = \odot_{n=1}^{\infty} A(E_n)$. Let u_n be the function in $A(E_n)$ defined by $u_n(0) = 0$ and $u_n(2\pi/p^n) = 1$. It is easy to see from the definition of f that we have

$$P(f|_{\tilde{E}}) = f \circ p_E = \sum_{n=1}^{\infty} n^{-2} u_{2n-1} \odot u_{2n} = f' .$$

The Remarks following the proof of Theorem 2 shows that the closed ideals in A_E which are generated by each $(f' - a)^m$ $(m = 1, 2, \cdots)$ are all distinct for each fixed a in M. But P is a norm-decreasing homomorphism, and so our assertion follows.

Another interesting example may be given by

$$h(t) = \sum_{n=1}^{\infty} n^{-2}\{w(p^{8n-8}t) - w(p^{8n-6}t)\} + i \sum_{n=1}^{\infty} n^{-2}\{w(p^{8n-4}t) - w(p^{8n-2}t)\} .$$

Then, for every complex number z with $|Re(z)| < \pi^2/6$ and $|Im(z)| < \pi^2/6$, the closed ideals in $A(T)$ which are generated by each function

$$(h - z)^m(\bar{h} - \bar{z})^n \quad (m, n = 0, 1, 2, \cdots)$$

are all distinct.

REMARKS. (a) An idea very like the one used in the proof of our Theorem 2 is due to Y. Katznelson [4: Chap. VIII].

(b) We can directly prove what was shown in Example 6 by applying the methods in the proof of Theorem 2.

(c) Professor O. C. McGehee kindly let me know that

$$\eta(d) = d + O(d^2) \quad \text{as} \quad d \to 0 .$$

My original estimate was $\eta(d) < 2^{1/2}d$.

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