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ON THE DIVERGENCE OF REARRANGED WALSH SERIES

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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K. Tandori [4] proved the following

THEOREM A. *If $\{\rho(n)\}$ is a sequence of positive numbers with*

$$\rho(n) = o(\sqrt{\log \log n}) , \tag{1}$$

then there exists a sequence of real numbers $\{a_n\}$ with

$$\sum_{n=1}^{\infty} a_n^2 \rho^2(n) < \infty \tag{2}$$

such that the Walsh series

$$\sum_{n=1}^{\infty} a_n w_n(x) \tag{3}$$

can be rearranged into an almost everywhere divergent series.

F. Móricz [1] proved a trigonometric series analogue of Theorem A, and later he sharpened it in [2]. The author proved in [3] the following theorem which includes F. Móricz's results.

THEOREM B. *If $\{\rho(n)\}$ is a sequence of positive numbers with*

$$\rho(n) = o(\sqrt[4]{\log n}) , \tag{4}$$

then there exists a sequence of real numbers $\{a_n, b_n\}$ with

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\rho^2(n) < \infty \tag{5}$$

such that the trigonometric series

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{6}$$

can be rearranged into an everywhere divergent series.

The aim of the present paper is to sharpen Theorem A or to prove a Walsh series analogue of Theorem B.

THEOREM. *If $\{\rho(n)\}$ is a sequence of positive numbers with* (4), *then* *there exists a sequence of real numbers* $\{a_n\}$ *with* (2) *such that the Walsh series* (3) *can be rearranged into an almost everywhere divergent series.*

COROLLARY. *Suppose* (4), *then there exists a sequence of real numbers* $\{a_n\} \in l_2$ *such that the Walsh series* (3) *can be rearranged to satisfy the condition*

$$\limsup_{N\to\infty} \frac{\left| \sum_{j=1}^{N} a_{n(j)} w_{n(j)}(x) \right|}{\rho(N)} > 0$$

almost everywhere.

1. Lemmas. First of all let us introduce a function $n = n(k, j)$ $(j = 1, \cdots, 2^k;\ k = 0, 1, \cdots)$. For each k, consider all integers

$$2^{\nu_0} + 2^{\nu_1} + \cdots + 2^{\nu_i} \tag{7}$$

satisfying $(1/2)k(k+1) + 1 \leqq \nu_0 < \nu_1 < \cdots < \nu_i = (1/2)(k+1)(k+2)$. On account of $(1/2)(k+1)(k+2) - \{(1/2)k(k+1) + 1\} = k$, the number of the integers (7) is $\sum_{i=0}^{k} \binom{k}{i} = 2^k$. And we label (7) as $n(k, j)$ $(j = 1, \cdots, 2^k)$. Thus we obtain

$$\begin{aligned} 2^{(1/2)(k+1)(k+2)} &= n(k, 1) < n(k, 2) < \cdots < n(k, 2^k) \\ &= 2^{(1/2)(k+1)(k+2)+1} - 2^{(1/2)k(k+1)+1} \end{aligned} \tag{8}$$

LEMMA 1. *Set* $\psi_1(0; x) = w_0(x) = 1$ *and*

$$\begin{aligned} \psi_{2j-1}(k+1; x) &= \frac{1}{2}\psi_j(k; x)(1 + w_{n(k,j)}(x)) , \\ \psi_{2j}(k+1; x) &= \frac{1}{2}\psi_j(k; x)(1 - w_{n(k,j)}(x)) \end{aligned} \tag{9}$$

$(j = 1, \cdots, 2^k;\ k = 0, 1, \cdots)$; *then the following* (i) – (iii) *hold.*

(i) *Each* $\psi_j(k; x)$ $(j = 1, \cdots, 2^k)$ *is a linear combination of Walsh functions with even indices* $2p$ $(< 2^{(1/2)k(k+1)+1})$.

(ii) $\psi_j(k; x) = 0$ or 1 $(x \in E)$

where E *denotes the set of all dyadic irrational numbers.*

(iii) *Set*

$$E_j^{(k)} = \{x \in E(0, 1/2);\ \psi_j(k; x) = 1\}\ (j = 1, \cdots, 2^k;\ k = 0, 1, \cdots) ,$$ [1]

then

$$E_{2j-1}^{(k+1)} \cup E_{2j}^{(k+1)} = E_j^{(k)}\ \ (j = 1, \cdots, 2^k;\ k = 0, 1, \cdots) ,$$

$$E_j^{(k)} \cap E_{j'}^{(k)} = \phi\ \ (1 \leqq j < j' \leqq 2^k),\ \bigcup_{j=1}^{2^k} E_j^{(k)} = E(0, 1/2) ,$$

$$\text{mes}\, E_j^{(k)} = 1/2^{k+1}\ \ (j = 1, \cdots, 2^k;\ k = 0, 1, \cdots) .$$

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[1] $E(0, 1/2)$ means $E \cap (0, 1/2)$.

PROOF. (i): The statement for $k=0$ holds trivially. Supposing the case $k=k$, we see by (8) and (9) that each $\psi_j(k+1; x)$ $(j=1, \cdots, 2^{k+1})$ is a linear combination of Walsh functions with even indices $2p$ such that

$$2p < 2^{(1/2)k(k+1)+1} + n(k, 2^k) = 2^{(1/2)(k+1)(k+2)+1} .$$

(ii) is obvious by the definition of $\psi_j(k; x)$.

(iii): (9) and (ii) yield

$$E_{2j-1}^{(k+1)} = \{x \in E(0, 1/2); \psi_j(k; x) = 1, w_{n(k,j)}(x) = 1\}$$

and

$$E_{2j}^{(k+1)} = \{x \in E(0, 1/2); \psi_j(k; x) = 1, w_{n(k,j)}(x) = -1\} .$$

Hence we get

$$E_{2j-1}^{(k+1)} \cup E_{2j}^{(k+1)} = E_j^{(k)},\ E_{2j-1}^{(k+1)} \cap E_{2j}^{(k+1)} = \phi ;$$

$$\bigcup_{j=1}^{2^k} E_j^{(k)} = E_1^{(0)} = E(0, 1/2) , \quad E_j^{(k)} \cap E_{j'}^{(k)} = \phi \ (1 \leqq j < j' \leqq 2^k) .$$

In virtue of

$$\left.\begin{aligned} \text{mes } E_{2j-1}^{(k+1)} &= \int_{E(0,1/2)} \psi_{2j-1}(k+1; x)dx \\ \text{mes } E_{2j}^{(k+1)} &= \int_{E(0,1/2)} \psi_{2j}(k+1; x)dx \end{aligned}\right\} = \frac{1}{2} \text{mes } E_j^{(k)} ,$$

we get

$$\text{mes } E_j^{(k)} = \frac{1}{2^k} \text{mes } E_1^{(0)} = 1/2^{k+1} \ (j = 1, \cdots, 2^k; k = 0, 1, \cdots) .$$

This completes the proof of Lemma 1.

LEMMA 2. *Set* $\Psi_1(0; x) = w_1(x)$ *and*

$$\begin{aligned} \Psi_{2j-1}(k+1; x) &= \psi_j(k; x) w_{n(k,j)}(x) , \\ \Psi_{2j}(k+1; x) &= -w_1(x)\psi_j(k; x) w_{n(k,j)}(x) \end{aligned} \tag{10}$$

$(j = 1, \cdots, 2^k; k = 0, 1, \cdots)$; *then the following* (iv) – (vii) *hold.*

(iv) *Each* $\Psi_j(k; x)$ $(j = 1, \cdots, 2^k)$ *is a linear combination of Walsh functions with indices* p *such that*

$$2^{(1/2)k(k+1)} \leqq p < 2^{(1/2)k(k+1)+1}$$

(v) *If* $1 \leqq j < j' \leqq 2^{k+1}$, *then* $\Psi_{j'}(k+1; x)$ *and* $\Psi_j(k+1; x)$ *have no term of the same index.*

(vi) $\Psi_{2j-1}(k+1; x) = 1 \qquad (x \in E_{2j-1}^{(k+1)})$,
$\Psi_{2j-1}(k+1; x) + 2\Psi_{2j}(k+1; x) = 1 \quad (x \in E_{2j}^{(k+1)})$,
$\Psi_{2j-1}(k+1; x) = \Psi_{2j}(k+1; x) = 0 \quad (x \in E(0, 1/2) - E_j^{(k)})$.

(vii) $$\int_0^1 \Psi_j^2(k; x)dx = \begin{cases} 1 & (j = 1;\ k = 0) , \\ 1/2^{k-1} & (j = 1, \cdots, 2^k;\ k = 1, 2, \cdots) . \end{cases}$$

PROOF. (iv): The statement can be checked directly for $k = 0$, and by using (i), (8) and (10) for $k \geqq 1$.

(v): By the definition (10), the indices p_j of the Walsh functions in $\Psi_{2j-1}(k+1; x)$ are all even, and those in $\Psi_{2j}(k+1; x)$ are all odd $(= 1 + p_j)$. Moreover we have

$$n(k, j) \leqq p_j < n(k, j) + 2^{(1/2)k(k+1)+1} \ (\leqq n(k, j+1))$$

for $1 \leqq j \leqq 2^k$, so we get the statement (v).

(vi): If $x \in E_{2j-1}^{(k+1)}$, then

$$1 = \psi_{2j-1}(k+1; x) = \frac{1}{2}\psi_j(k; x)(1 + w_{n(k,j)}(x)) .$$

Hence

$$\psi_j(k; x) = 1 , \qquad w_{n(k,j)}(x) = -1$$

and

$$\Psi_{2j-1}(k+1; x) = 1 \quad (x \in E_{2j-1}^{(k+1)}) .$$

In the same way, if $x \in E_{2j}^{(k+1)}$, then

$$\Psi_{2j-1}(k+1; x) = -1,\ \Psi_{2j}(k+1; x) = 1 ,$$
$$\Psi_{2j-1}(k+1; x) + 2\Psi_{2j}(k+1; x) = 1 ;$$

and if $x \in E(0, 1/2) - E_j^{(k)}$, then $\psi_j(k; x) = 0$,

$$\Psi_{2j-1}(k+1; x) = \Psi_{2j}(k+1; x) = 0 .$$

(vii): $$\int_0^1 \Psi_{2j-1}^2(k+1; x)dx = \int_0^1 \Psi_{2j}^2(k+1; x)dx = \int_0^1 \psi_j^2(k; x)dx$$
$$= 2 \operatorname{mes} E_j^{(k)} = 1/2^k$$
$$(j = 1, \cdots, 2^k;\ k = 0, 1, \cdots) .$$

This completes the proof of Lemma 2.

2. **Proof of the theorem.** Set $S_0(x) = \Psi_1(0; x)$. When $S_k(x)$ has been determined, we define $S_{k+1}(x)$ by inserting $+ \Psi_{2j-1}(k+1; x) + 2\Psi_{2j}(k+1; x)$ after $((3+(-1)^j)/2)\Psi_j(k; x)$ in $S_k(x)$.[2] Then for each $x \in E(0, 1/2)$ and

[2] For example, $S_1(x) = \Psi_1(0; x) + \Psi_1(1; x) + 2\Psi_2(1; x)$, $S_2(x) = \Psi_1(0; x) + \Psi_1(1; x) + \Psi_1(2; x) + 2\Psi_2(2; x) + 2\Psi_2(1; x) + \Psi_3(2; x) + 2\Psi_4(2; x)$.

k, $S_k(x)$ has a partial sum which equals to $k+1$. For, taking $j = j(x, k)$ such that $x \in E_j^{(k)}$, we easily get

$$\Psi_1(0; x) + \cdots + \frac{3+(-1)^j}{2} \Psi_j(K; x) = k+1$$

with the aid of (vi).

Now define a sequence of integers $(0 \leqq)\, m_1 < m_2 < \cdots$ such that

$$\frac{\rho(n)}{\sqrt[4]{\log n}} \leqq \frac{1}{i} \quad \text{for} \quad n \geqq 2^{M_i} \left(M_i = \frac{1}{2} m_i(m_i+1) + 1;\ i = 1, 2, \cdots\right).$$

Define $\{a_n\}$ by setting $a_n = 0\ (1 \leqq n < 2^{M_1})$ and

$$\begin{aligned} T_i(x) &= \frac{1}{m_i+1} w_{2^{M_i}}\left(x - \frac{1+(-1)^i}{4}\right) S_{m_i}\left(x - \frac{1+(-1)^i}{4}\right) \\ &= \sum_{n=2^{M_i}}^{2^{M_i+1}} a_n w_n(x) = \sum_{n=2^{M_i}}^{2^{M_i+1}-1} a_n w_n(x) \end{aligned}$$

$(i = 1, 2, \cdots)$. Then it is obvious that the series (3) can be rearranged so as to diverge everywhere on $E(0, 1)$. And we get

$$\begin{aligned} &\int_0^1 T_i^2(x)dx \\ &= \frac{1}{(m_i+1)^2} \int_0^1 S_{m_i}^2(x)dx \\ &= \frac{1}{(m_i+1)^2} \left[\int_0^1 \Psi_1^2(0; x)dx + \sum_{k=1}^{m_i} \sum_{j=1}^{2^{k-1}} \left\{\int_0^1 \Psi_{2j-1}^2(k; x)dx + 4\int_0^1 \Psi_{2j}^2(k; x)dx\right\}\right] \\ &= \frac{1}{(m_i+1)^2} \left[1 + m_i \cdot 2^{k-1} \cdot \frac{5}{2^{k-1}}\right] \leqq \frac{5}{m_i+1} \qquad (i = 1, 2, \cdots), \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} a_n^2 \rho^2(n) &\leqq \sum_{i=1}^{\infty} \frac{\sqrt{M_i+1}}{i^2} \sum_{n=2^{M_i}}^{2^{M_i+1}} a_n^2 \\ &\leqq \sum_{i=1}^{\infty} \frac{\sqrt{M_i+1}}{i^2} \int_0^1 T_i^2(x)dx \leqq \sum_{i=1}^{\infty} \frac{10}{i^2} < \infty . \end{aligned}$$

Thus the assertion of our theorem has been proved. The proof of Corollary can be carried out analogically to that of Theorem 2 in [1].

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