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CONTRACTIONS OF FOURIER TRANSFORMS IN R_k.

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Introduction. In the present paper, we shall characterize some functions, those which satisfy a Lipschitz condition, as Fourier transforms of a certain sub-class of $L^p(R_k)$, and we shall give a contraction theorem of L^p-Fourier transforms.

A complex valued function $f(x_1, x_2, \cdots, x_k)$ on R_k, the k-dim. Euclidean space, is denoted by $f(x)$.

When f has the following property (i) or (ii), we say f is (p)-normalized:

(i) if $1 < p \leqq 2$, then $\lim_{|y|\to\infty}\int_{I+y} |f(x)|^{p'} dx = 0$, for any finite interval I, where $1/p + 1/p' = 1$;

(ii) if $p = 1$, then f is continuous and $\lim_{|x|\to\infty} f(x) = 0$.

We denote the j-th difference of $f(x)$, with respect to $h \in R_k$, by $\Delta_h^j(f(x))$, that is,

$$\Delta_h^j(f(x)) = \sum_{m=0}^{j}(-1)^{j+m}\binom{j}{m} f(x + mh) .$$

We say $g(x)$ is a normalized j-contraction of $f(x)$ if g is normalized and $|\Delta_h^j(g(x))| \leqq |\Delta_h^j(f(x))|$ for any x and $h \in R_k$.

Let X be a sub-space of $L^p(R_k)$ with norm $\|*\|_X$ and $\hat{X}$ be the space of Fourier transforms of functions in X. We say an element $\hat{f}$ of $\hat{X}$ is j-contractible in $\hat{X}$, if every normalized j-contraction of $\hat{f}$ is also in $\hat{X}$. And we say $\hat{f} \in \hat{X}$ is uniformly j-contractible in $\hat{X}$, if $\hat{f}$ is j-contractible in $\hat{X}$ and if $\lim_{n\to\infty} \|g_n\|_X = 0$ for any sequence $\hat{g}_n(x)$ of normalized j-contractions of $\hat{f}$ such that $\lim_{n\to\infty} \hat{g}_n(x) = 0$ on R_k.

Our main result is as follows:

THEOREM 1. *Let $1 \leqq p < 2$ and $k/p - k/2 < j$. Suppose that $w(r)$ is a monotone decreasing function on $(0, \infty)$ such that*

$$\int_0^\infty r^{k-1} w^p(r)dr < \infty .$$

If $|f(x)| \leqq w(|x|)$, then $\hat{f}$ is uniformly j-contractible in $L^p(R_k)$.

Theorem 1 can be proved by a way of characterizing some Lipschitz classes of functions by means of Fourier transforms. There are several results on characterizing Lipschitz classes (cf. for example, Herz [4].). Our Theorems 3 and 4 (in § 5 and 6) concerned with the above problem are very similar to Theorem 1 in Herz [4]. However, our proof of Theorems 3 and 4 adopted in this paper is quite elementary. That is, we shall discuss the problem along the line set by Beurling [1], Boas [2], Sunouchi [6], and Kinukawa [5].

The one dimensional case of Theorem 1 is refered to Kinukawa [5].

2. **Notations.** We shall use the following notations:

$$(t, x) = \sum_{m=1}^{k} t_m x_m$$

$$\tilde{Y}_{a,j}(t; F) = \tilde{Y}_{a,j}(t) = \tilde{Y}(t) = \left[\int_{R_k} |F(x)|^a |\sin(t, x)/2|^{aj} dx\right]^{1/a}$$

$$Y_{a,j}(t; f) = \left[\int_{R_k} |\Delta_t^j(f(x))|^a dx\right]^{1/a}$$

$${}_a\tilde{A}_{p,j,\alpha}(F) = \left\{\int_{R_k} [|t|^{-\alpha} \tilde{Y}_{a,j}(t; F)]^p |t|^{-k} dt\right\}^{1/p}$$

$${}_aA_{p,j,\alpha}(f) = \left\{\int_{R_k} [|t|^{-\alpha} Y_{a,j}(t; f)]^p |t|^{-k} dt\right\}^{1/p}$$

$${}_aB_{p,\alpha}(F) = \left\{\int_{R_k} \left[|t|^{a\alpha} \int_{|x|>|t|} |F(x)|^a dx\right]^{p/a} |t|^{-k} dt\right\}^{1/p}$$

$${}_aC_{p,j,\alpha}(F) = \left\{\int_{R_k} \left[|t|^{a(\alpha-j)} \int_{|x|<|t|} |F(x)|^a |x|^{aj} dx\right]^{p/a} |t|^{-k} dt\right\}^{1/p}$$

$${}_aC^*_{p,j,\alpha}(F) = \left\{\int_{R_k} \left[|t|^{-a\alpha} \int_{|x|<1/|t|} |F(x)|^a |(t, x)|^{aj} dx\right]^{p/a} |t|^{-k} dt\right\}^{1/p}$$

$K =$ Constant numbers which may be different from one occurrence to another.

Let W be the class of radial functions $w(x) \in L^1(R_k)$ such that

$$w(|x|) = w(r) \geqq 0$$

is decreasing on $(0, \infty)$. For each $w \in W$, we define

$${}_a\|F\|_{p,w} = \left\{\int_{R_k} |F(x)|^a w^{1-a/p}(x) dx\right\}^{1/a}$$

and

$${}_a\|F\|_p = \inf_{w \in W} \{{}_a\|F\|_{p,w} \cdot \|w\|_1^{1/p-1/a}\} .$$

${}_aL_p$ is defined by a class of F with ${}_a\|F\|_p < \infty$. For the case $0 < p \leqq a$, we see ${}_aL_p \subseteqq L^p$, and ${}_aL_a = L^a$.

3. Lemmas.

LEMMA 1. *For* $t \in R_k$, *there is an orthogonal transformation* $y_m = \sum_{l=1}^k a_{ml}x_l$ *from* $x \in R_k$ *to* $y \in R_k$ *with the determinant* 1, *in which* $|x| = |y|$ *and* $(t, x) = \sum_{l=1}^k t_l x_l = |t| y_1$. (cf. Bochner [3], p. 70.)

LEMMA 2. *Let* $w(r)$ *be non-negative and decreasing on* $(0, \infty)$. *Then, for given constants* ε $(0 < \varepsilon < 1)$ *and* δ $(k < \delta)$, *there exists a non-negative function* $w^*(r)$ *such that* (i) $w(r) \leqq w^*(r)$, (ii) $r^\varepsilon w^*(r)$ *is decreasing on* $(0, \infty)$, (iii) $r^\delta w^*(r)$ *is increasing on* $(0, \infty)$, *and* (iv) $\int_0^\infty w^*(r) r^{k-1} dr = K\int_0^\infty w(r)\, r^{k-1}\, dr$. (Cf. Herz [4], Lemma 2.5.)

LEMMA 3. *Suppose* $0 < p < a$ *and* $0 < \alpha < j$. *Then*

$$ {}_a\| F(x)\, |x|^{\alpha - k/p + k/a} \|_p \leqq K\, {}_a\widetilde{A}_{p,j,\alpha}(F) \,. $$

PROOF. (Cf. Beurling [1].) Suppose ${}_a\widetilde{A}_{p,j,\alpha}(F) < \infty$. We shall prove that there is $w(x) \in W$ such that ${}_a\| F(x)\, |x|^{\alpha-k/p+k/a} \|_{p,w}^a \leqq K\ {}_a\widetilde{A}_{p,j,\alpha}^p(F)$. Put $w(x) = \int_{|t|<1/|x|} |t|^{-p\alpha} \widetilde{Y}^p(t) dt$. Then we have $\|w\|_1 = \int_{R_k} w(x)dx = K\, {}_a\widetilde{A}_{p,j,\alpha}^p(F) < \infty$. Therefore $w(x) = w(|x|) \in L^1(R_k)$ and $w(|x|)$ is decreasing on $(0, \infty)$. That is, $w \in W$.

We have

$$ \begin{aligned} {}_a\widetilde{A}_{p,j,\alpha}^p(F) &= \int_{R_k} |t|^{-k-p\alpha} \widetilde{Y}^{p-a}(t)\, \widetilde{Y}^a(t) dt \\ &= \int_{R_k} |F(x)|^a \left[\int_{R_k} |t|^{-k-p\alpha} \widetilde{Y}^{p-a}(t)\, |\sin(t,x)/2|^{aj} dt \right] dx \\ &= \int_{R_k} |F(x)|^a [M(x)]^{-1} dx, \text{ say} \,. \end{aligned} $$

Let $P = a/p$ and $Q = a/(a-p)$. Then, by the Hölder inequality, we get

$$ \begin{aligned} V &= w^{1/Q}(x)\, M^{-1/p}(x) \\ &\geqq \int_{|t|<1/|x|} |t|^{\gamma_1} \widetilde{Y}^{\gamma_2}\, |\sin(t,x)/2|^{pj} dt \,, \end{aligned} $$

where $\gamma_1 = (-k-p\alpha)/P + (-p\alpha)/Q = -p\alpha - kp/a$ and $\gamma_2 = p/Q + (p-a)/P = 0$. So we have the following inequality

$$ V \geqq \int_{|t|<1/|x|} |t|^{\gamma_1} |\sin(t,x)/2|^{pj} dt = S, \text{ say} \,. $$

For the case $k \geqq 2$, apply Lemma 1 to the above integral S. Then (cf. Bochner [3], p. 70),

$$\begin{aligned} S &= \int_{|y|\leqq 1/|x|} |\, y\,|^{\gamma_1}\, |\sin(y_1\, |\, x\,|)/2\,|^{pj} dy \\ &= \int_0^{1/|x|} r^{k-1+\gamma_1} dr \int_0^{\pi} |\sin(r\,|\,x\,|\cos\theta)/2\,|^{pj} \sin^{k-2}\theta\, d\theta \\ &= |\,x\,|^{-(k-1+\gamma_1)-1} \int_0^1 \beta^{(k-1+\gamma_1)} d\beta \int_0^{\pi} |\sin(\beta\cos\theta)/2\,|^{pj} (\sin\theta)^{k-2} d\theta\ . \end{aligned}$$

Since $\alpha < j$ and $0 < p < a$, we have

$$k - 1 + \gamma_1 + pj = k(1 - p/a) + p(j - \alpha) - 1 > -1\ .$$

Hence we have

$$S = K\,|\,x\,|^{-k+p\alpha+kp/a}\ .$$

For the case $k = 1$, transform the variable t by $\beta/|\,x\,|$ in the original form of S, then we have also the above equality. Now we have a lower bound for $(M(x))^{-1}$ such that

$$\begin{aligned} (M(x))^{-1} &\geqq K(|\,x\,|^{-k+p\alpha+kp/a})^{P}\, (w(x))^{-P/Q} \\ &= K\,|\,x\,|^{a(\alpha-k/p+k/a)}\, w(x)^{1-a/p}\ . \end{aligned}$$

Finally we have

$${}_a\widetilde{A}^{p}_{p,j,\alpha}(F) \geqq K \int_{R_k} |\,F(x)\,|^{a}\, |\,x\,|^{a(\alpha-k/p+k/a)}\, w(x)^{1-a/p}\, dx\ ,$$

which completes the proof of Lemma 3, because of $\|\,w\,\|_1 = {}_a\widetilde{A}^{p}_{p,j,\alpha}(F)$. (For the case $a = p$, we have directly

$${}_a\widetilde{A}^{a}_{a,j,\alpha}(F) \geqq K \int_{R_k} |\,F(x)\,|^{a} S\, dx = K\,\|\,F(x)\,|\,x\,|^{\alpha}\,\|_a^a .)$$

LEMMA 4. *Let* $0 < p < a$, $0 < \alpha < j$ *and* $(k-1)(1/p - 1/a) < \alpha$. *Then we have*

$${}_a\widetilde{A}_{p,j,\alpha}(F) \leqq K\, {}_a\|\,F(x)\,|\,x\,|^{\alpha-k/p+k/a}\,\|_p\ .$$

PROOF. Suppose that there exists $w \in W$ such that

$${}_a\|\,F(x)\,|\,x\,|^{\alpha-k/p+k/a}\,\|_{p,w} < \infty\ .$$

For this w, we find $w^*(x)$ which has the properties of Lemma 2. Let $P = a/p$, $Q = a/(a-p)$, $\alpha_1 = -(p\alpha + k - 2k/Q)$ and $\alpha_2 = -2k/Q$. By the Hölder inequality, we have

$$\begin{aligned} {}_a\widetilde{A}^{p}_{p,j,\alpha}(F) &= \int_{R_k} \{\widetilde{Y}^{p}(t) w^*(1/|\,t\,|)^{-1+p/a} |\,t\,|^{\alpha_1}\} \times \{w^*(1/|\,t\,|)^{1-p/a}\, |\,t\,|^{\alpha_2}\} dt \\ &\leqq \left\{\int_{R_k} \widetilde{Y}(t)^{pP} w^*(1/|\,t\,|)^{P(p/a-1)} |\,t\,|^{\alpha_1 P} dt\right\}^{1/P} \\ &\quad \times \left\{\int_{R_k} w^*(1/|\,t\,|)^{Q(1-p/a)}\, |\,t\,|^{\alpha_2 Q} dt\right\}^{1/Q}\ . \end{aligned}$$

The second part of the above is equal to $K\|w\|_1^{1/Q}$ and is finite. (Cf. Lemma 2 – (iv).) Note that $pP = a$. We have

$$\|w\|_1^{-P/Q}\cdot {}_a\widetilde{A}^a_{p,j,\alpha}(F) \leqq K\int_{R_k} \widetilde{Y}^a(t)w^*(1/|t|)^{1-a/p}|t|^{\alpha_1 P}dt$$
$$= K\int_{R_k} |F(x)|^a dx \int_{R_k} |\sin(t,x)/2|^{aj} w^*(1/|t|)^{1-a/p}|t|^{\alpha_1 P}dt\ .$$

We have to estimate the second integral, say S^*, in the above. We have

$$S^* \leqq \int_{|t|<1/|x|} |x|^{aj} w^*(1/|t|)^{1-a/p}|t|^{\alpha_1 P+aj}dt + \int_{|t|>1/|x|} w^*(1/|t|)^{1-a/p}|t|^{\alpha_1 P}dt$$
$$= I_1 + I_2\ , \quad \text{say}\ .$$

The integrands of I_1 and I_2 are radial. Therefore, we have

$$I_1 = K|x|^{aj}\int_0^{1/|x|} w^*(1/r)^{1-a/p}\, r^{\alpha_1 P+aj+k-1}dr$$
$$= K|x|^{-\alpha_1 P-k}\int_0^1 w^*(|x|/r)^{1-a/p}\, r^{\alpha_1 P+aj+k-1}\, dr = K|x|^{-\alpha_1 P-k} I_{11}\ ,$$

say, and

$$I_2 = K\int_{1/|x|}^{\infty} w^*(1/r)^{1-a/p}\, r^{\alpha_1 P+k-1}dr$$
$$= K|x|^{-\alpha_1 P-k}\int_1^{\infty} w^*(|x|/r)^{1-a/p} r^{\alpha_1 P+k-1}dr = K|x|^{-\alpha_1 P-k} I_{21}\ ,$$

say. By the properties (ii) and (iii) of Lemma 2, we have the following inequalities: If $0 < r < 1$, then $w^*(|x|/r)^{1-a/p} \leqq [r^\delta\, w^*(|x|)]^{1-a/p}$, and if $1 \leqq r$, then $w^*(|x|/r)^{1-a/p} \leqq [r^\varepsilon\, w^*(|x|)]^{1-a/p}$, where $k < \delta$ and $0 < \varepsilon < 1$. Now we have

$$I_{11} \leqq w^*(|x|)^{1-a/p}\int_0^1 r^{\Delta_1}\, dr$$

and

$$I_{21} \leqq w^*(|x|)^{1-a/p}\int_1^{\infty} r^{\Delta_2}\, dr\ ,$$

where $\Delta_1 = \alpha_1 P + aj + k - 1 + \delta(1 - a/p)$ and $\Delta_2 = \alpha_1 P + k - 1 + \varepsilon(1 - a/p)$. Since $0 < \alpha < j$ and $(k-1)(1/p - 1/a) < \alpha$, we can choose constants δ and ε such that $k < \delta < k + pa(j-\alpha)/(a-p)$ and $k - ap\alpha/(a-p) < \varepsilon < 1$. The choice of δ and ε makes $\Delta_1 > -1$ and $\Delta_2 < -1$. Therefore, $\int_0^1 r^{\Delta_1} dr$ and $\int_1^{\infty} r^{\Delta_2} dr$ are finite constants, and we have

$$I_1 + I_2 \leqq K|x|^{-\alpha_1 P-k} w^*(|x|)^{1-a/p} \leqq K|x|^{-\alpha_1 P-k} w(|x|)^{1-a/p}\ .$$

Summarizing the above results, we have

$$ {}_a\tilde{A}^a_{p,j,\alpha}(F) \leqq K\left\{\int_{R_k} |F(x)|^a |x|^{-\alpha_1 P-k} w(x)^{1-a/p} dx\right\} \cdot \|w\|_1^{P/Q} $$
$$ = K_a \| F(x)|x|^{\alpha-k/p+k/a} \|^a_{p,w} \cdot \|w\|^{P/Q} $$

which completes the proof of Lemma 4.

LEMMA 5. *Let $0 < p < a$ and $0 < \alpha$, then*

$$ {}_aB_{p,\alpha}(F) \leqq K\, {}_aC_{p,j,\alpha}(F) . $$

PROOF. (Cf. Sunouchi [6]) Note that

$$ {}_aB^p_{p,\alpha}(F) = K\int_0^\infty r^{p\alpha-1}\left[\int_{|x|>r} |F(x)|^a dx\right]^{p/a} dr , $$

and

$$ {}_aC^p_{p,j,\alpha}(F) = K\int_0^\infty r^{p\alpha-1-pj}\left[\int_{|x|<r} |F(x)|^a |x|^{aj} dx\right]^{p/a} dr . $$

We have

$$ \begin{aligned} {}_aB^p_{p,\alpha}(F) &\leqq K \sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} r^{p\alpha-1} dr\left[\sum_{l=m}^{\infty}\int_{2^l<|x|<2^{l+1}} |F(x)|^a dx\right]^{p/a} \\ &\leqq K \sum_{l=-\infty}^{\infty} (2^l)^{-pj}\left[\int_{2^l<|x|<2^{l+1}} |F(x)|^a |x|^{aj} dx^{p/a}\right] \times \sum_{m=-\infty}^{l} \int_{2^m}^{2^{m+1}} r^{p\alpha-1} dr \\ &= K \sum_{l=-\infty}^{\infty} (2^l)^{-pj+p\alpha}\left[\int_{2^l<|x|<2^{l+1}} |F(x)|^a |x|^{aj} dx\right]^{p/a} \\ &\leqq K\, {}_aC^p_{p,j,\alpha}(F) . \end{aligned} $$

LEMMA 6. *Let $0 < p < a$ and $0 < \alpha < j$, then*

$$ {}_aC_{p,j,\alpha}(F) \leqq K\, {}_aB_{p,\alpha}(F) . $$

PROOF. By the same way of Lemma 5, we have

$$ {}_aC^p_{p,j,\alpha}(F) \leqq K \sum_{l=-\infty}^{\infty} (2^l)^{pj}\left[\int_{2^l<|x|<2^{l+1}} |F(x)|^a dx\right]^{p/a} \times \sum_{m=l}^{\infty} \int_{2^m}^{2^{m+1}} r^{p\alpha-pj-1} dr . $$

Since, by the assumtion $\alpha < j$, $p\alpha - pj - 1 < -1$, we have the conclusion.

LEMMA 7. (i) *We have*

$$ {}_aC^*_{p,j,\alpha}(F) \leqq {}_aC_{p,j,\alpha}(F) . $$

(ii) *If $F(x)$ is radial, then*

$$ {}_aC^*_{p,j,\alpha}(F) = K\, {}_aC_{p,j,\alpha}(F) . $$

PROOF. (i) is trivial, since $|(t, x)| \leqq |t||x|$. We need to prove (ii) for the case $k \geqq 2$. By the same argument of Lemma 3, we have

$$\begin{aligned}\int_{|x|<1/|t|} |F(x)|^a (t, x)|^{aj} dx &= K|t|^{aj}\int_0^{1/|t|} |F(r)|^a r^{aj+k-1} dr \\ &\quad \times \int_0^{\pi} |\cos\theta|^{aj} (\sin\theta)^{k-2} d\theta \\ &= K|t|^{aj}\int_{|x|<1/|t|} |F(x)|^a |x|^{aj} dx ,\end{aligned}$$

which shows (ii).

LEMMA 8. *Let $0 < p < a$ and $0 < \alpha < j$, then*

$${}_a\tilde{A}_{p,j,\alpha}(F) \leqq K\, {}_aB_{p,\alpha}(F) .$$

PROOF. Split the domain of the integral in $\tilde{Y}_{a,j}(t\colon F)$ into two parts; $D_1 = \{x \in R_k;\ |x| < 1/|t|\}$ and $D_2 = \{x \in R_k;\ |x| > 1/|t|\}$. In $\tilde{Y}_{a,j}(t)$, replace $|\sin(t, x)/2|$ by $(|t||x|/2)$ on D_1, and by 1 on D_2, then we see that ${}_a\tilde{A}^p_{p,j,\alpha}(F) \leqq K\{{}_aB^p_{p,\alpha}(F) + {}_aC^p_{p,j,\alpha}(F)\}$. Combining with Lemma 6, we have the conclusion.

LEMMA 9. ${}_aC^*_{p,j,\alpha}(F) \leqq K\, {}_a\tilde{A}_{p,j,\alpha}(F)$.

PROOF. Since

$$\begin{aligned}\int_{|x|<1/|t|} |F(x)|^a |(t, x)|^{aj} dx &\leqq K\int_{|x|<1/|t|} |F(x)|^a |\sin(t, x)/2|^{aj} dx \\ &\leqq K\,\tilde{Y}^a_{a,j}(t) ,\end{aligned}$$

we have the conclusion.

LEMMA 10. *Let $0 < p < a$ and $0 < \alpha$. If a non-negative function $w(x)$ is radial and $w(|x|)$ is decreasing on $(0, \infty)$, then*

$${}_aB_{p,\alpha}(w) \leqq K\, \| w(x)\, |x|^{\alpha - k(1/p - 1/a)} \|_p .$$

PROOF.

$$\begin{aligned}{}_aB^p_{p,\alpha}(w) &\leqq \sum_{m=-\infty}^{\infty} \int_{2^m}^{2^{m+1}} r^{p\alpha-1} dr \sum_{l=m}^{\infty} \left[\int_{2^l}^{2^{l+1}} |w(\xi)|^a \xi^{k-1} d\xi\right]^{p/a} \\ &\leqq K\sum_{l=-\infty}^{\infty} [w(2^l)]^p (2^l)^{kp/a} \sum_{m=-\infty}^{l} \int_{2^m}^{2^{m+1}} r^{p\alpha-1} dr \\ &\leqq K\sum_{l=-\infty}^{\infty} [w(2^l)]^p (2^l)^{kp/a+p\alpha} \\ &\leqq K\, \| w(x)\, |x|^{\alpha-k(1/p-1/a)} \|_p^p .\end{aligned}$$

4. THEOREM 2. *Let $0 < p < a$ and $(k-1)(1/p - 1/a) < \alpha$. Suppose $F(x)$ is radial. Then ${}_a\| F(x)\, |x|^{\alpha-k(1/p-1/a)} \|_p < \infty$, if and only if ${}_aB_{p,\alpha}(F) < \infty$.*

PROOF. By Lemmas 3 to 9, we have the result.

5. THEOREM 3. *Let* $1 < a \leqq 2$, $1/a + 1/a' = 1$, $1 \leqq p < 2$, $\alpha = k/p - k/a'$ *and* $\alpha < j$. *If* $f(x)$ *is* (p)-*normalized and* ${}_aA_{p,j,\alpha}(f) < \infty$, *then there exists* $F \in {}_{a'}L_p$ *such that its Fourier transform* $\hat{F}$ (in L^p) *is equal to* f *and* ${}_{a'}\| F \|_p \leqq K\, {}_aA_{p,j,\alpha}(f)$.

PROOF. By the assumption ${}_aA_{p,j,\alpha}(f) < \infty$, we have $\Delta_t^j f(x) \in L^a(R_k)$ for almost all $t \in R_k$. Therefore, we have the Fourier transform

$$(\Delta_t^j f)^\wedge = \underset{N\to\infty}{\overset{(a')}{\text{l.i.m.}}}\ K \int_{|u|<N} e^{-i(u,x)} \Delta_t^j f(u)\, du\ ,$$

which we write $[e^{i(t,x)} - 1]^j F_t(x)$. Since $\Delta_t^j(\Delta_s^j f) = \Delta_s^j(\Delta_t^j f)$, the Fourier transforms of the both sides are equal, which means $F_t = F_s = F$, say.

By the Hausdorff-Young inequality, we have

$$\| (e^{i(t,x)} - 1)^j F(x) \|_{a'} \leqq K \| \Delta_t^j f(u) \|_a\ ,$$

that is,

$$\tilde{Y}_{a',j}(t; F) \leqq K Y_{a,j}(t; f)\ .$$

Hence we have

$${}_{a'}\tilde{A}_{p,j,\alpha}(F) \leqq K\, {}_aA_{p,j,\alpha}(f) < \infty\ .$$

By Lemma 3, ${}_{a'}\| F(x) \|_p \leqq K\, {}_{a'}\tilde{A}_{p,j,\alpha}(F)$. Therefore, we have

$$F(x) \in {}_{a'}L_p \subseteqq L^p\ .$$

We have to show that f is the Fourier transform of F. Denote the Fourier transform of F by $\hat{F}$, then

$$\Delta_t^j \hat{F}(u) = \underset{N\to\infty}{\overset{(p')}{\text{l.i.m.}}}\ K \int_{|x|<N} F(x)(e^{i(t,x)} - 1)^j e^{i(u,x)} dx\ .$$

By the inversion argument, we get $\Delta_t^j \hat{F}(u) = \Delta_t^j f(u)$, that is,

$$f(u) - \hat{F}(u) = \sum_{\nu=1}^{j} (-1)^\nu \binom{j}{\nu} [f(u + \nu t) - \hat{F}(u + \nu t)]\ .$$

Consider the case $1 < p < 2$. Then, for any finite interval I,

$$\int_I | f(u) - \hat{F}(u) |^{p'} du \leqq K \sum_{\nu=1}^{j} \binom{j}{\nu} \int_{I+\nu t} | f(u) - \hat{F}(u) |^{p'} du\ ,$$

which converges to zero when $|t| \to \infty$ because of (p)-normalization of f and of $\hat{F} \in L^{p'}$. (When $p = 1$, we do not need to integrate $| f(u) - \hat{F}(u) |$ in order to get the conclusion.) Therefore, we have $f(u) = \hat{F}(u)$, a.e..

REMARK. If we start from assuming that f is the Fourier transform of $F \in L^p (1 \leqq p < 2)$, then a direct implication of Lemma 3 is

$$ {}_{a'}\| F(x) \mid x \mid^{\alpha - k(1/p - 1/a')} \|_p \leqq K {}_aA_{p,j,\alpha}(f) . $$

6. THEOREM 4. *Let* $1 < a \leqq 2$, $1/a + 1/a' = 1$, $1 \leqq p < a$, $\alpha = k/p - k/a$ *and* $\alpha < j$. *If* $F \in {}_aL_p$, *then the Fourier transform* $\hat{F} = f$ *satisfies* ${}_{a'}A_{p,j,\alpha}(f) \leqq K {}_a\| F \|_p$.

PROOF. Since $[(e^{i(x,t)} - 1)^j F(x)]^\wedge = \varDelta_t^j f$, by the Hausdorff-Young inequality, we have

$$ \| \varDelta_t^j f \|_{a'} \leqq K \| (e^{i(x,t)} - 1)^j F(x) \|_a , $$

that is, $Y_{a',j}(f) \leqq K \tilde{Y}_{a,j}(F)$. Hence we have

$$ {}_{a'}A_{p,j,\alpha}(f) \leqq K {}_a\tilde{A}_{p,j,\alpha}(F) . $$

By Lemma 4, we have the conclusion.

REMARK. A full use of Lemma 4 is as follows: If $F \in L^p$ and

$$ (k - 1)(1/p - 1/a) < \alpha < j , $$

then the Fourier transform $\hat{F} = f$ satisfies

$$ {}_{a'}A_{p,j,\alpha}(f) \leqq K {}_a\| F(x) \mid x \mid^{\alpha - k(1/p - 1/a)} \|_p . $$

7. THEOREM 5. *Suppose* $1 \leqq p \leqq 2$ *and* $\alpha = k/p - k/2 < j$. *A* (p)-*normalized function* $f(x)$ *is the Fourier transform of* $F \in {}_2L_p$, *if and only if* ${}_2A_{p,j,\alpha}(f) < \infty$. *And* ${}_2\| F \|_p \leqq K {}_2A_{p,j,\alpha}(f) \leqq K {}_2\| F \|_p$.

This is a corollary of Theorems 3 and 4.

THEOREM 6. *Let* $1 \leqq p \leqq 2$, $\alpha = k/p - k/2$ *and* $\alpha < j$. *Suppose that* $f(x)$ *is* (p)-*normalized, and that* $f(x)$ *and* $F(x)$ *are radial. Then,* $f(x)$ *is the Fourier transform of* $F(x)$ *with* ${}_aB_{p,\alpha}(F) < \infty$, *if and only if* ${}_2A_{p,j,\alpha}(f) < \infty$.

This is a result from Theorems 2 and 5.

8. THEOREM 7. *If* $\hat{f} \in {}_2\hat{L}_p$, *then* $\hat{f}$ *is uniformly* (j)-*contractible in* ${}_2\hat{L}_p$, *where* $1 \leqq p < 2$ *and* $k/p - k/2 < j$.

PROOF. Let $\hat{g}$ be a (p)-normalized (j)-contraction of $\hat{f} \in {}_2\hat{L}_p$. Then $| \varDelta_t^j \hat{g}(u) | \leqq | \varDelta_t^j \hat{f}(u) |$. Since $\hat{f} \in {}_2\hat{L}_p$, we have, by Theorem 5, ${}_2A_{p,j,\alpha}(\hat{f}) < \infty$. Hence ${}_2A_{p,j,\alpha}(\hat{g}) < \infty$. Again, by Theorem 5, we see that $\hat{g}$ is the Fourier transform of $g \in {}_2L_p$. This shows that $\hat{f} \in {}_2\hat{L}_p$ is (j)-contractible. Now we have to show the uniform contraction property. Suppose that $\hat{g}_n(t)$ is a sequence of (p)-normalized (j)-contractions of $\hat{f}$ such that $\lim_{n\to\infty} \hat{g}_n(t) = 0$ on R_k. Then, by the definition of norm and by Theorem 5,

$$ {}_2\| g_n \|_p \leqq K {}_2A_{p,j,\alpha}(\hat{g}_n) \leqq K {}_2A_{p,j,\alpha}(\hat{f}) < \infty . $$

Now apply the Lebesgue convergence theorem, then we have the conclusion, $\lim_{n\to\infty} {}_2\| g_n \|_p = 0$. (cf. Kinukawa [5].)

THEOREM 8. *Let* $1 \leqq p < 2$ *and* $\alpha = k/p - k/2 < j$. *Suppose that* $w(x)$ *is radial and* ${}_2B_{p,\alpha}(w) < \infty$. *If* $|f(x)| \leqq w(|x|)$, *then* $\hat{f}$ *is uniformly* (j)*-contractible in* ${}_2\hat{L}_p$.

PROOF. Since $w(x)$ is radial, by Theorem 2, $w \in {}_2L_p$. Hence $f \in {}_2L_p$. Apply Theorem 7, we have the result.

PROOF OF THEOREM 1. Theorem 1 is a corollary of Theorem 8 and Lemma 10.

9. THEOREM 9. *Let* $1 < a \leqq 2$, $1/a + 1/a' = 1$, $0 < p < a'$, $\alpha = k/p - k/a'$ and $\alpha < j$. *If* $F \in L^a(R_k)$ *and* ${}_aA_{p,j,\alpha}(F) < \infty$, *then* $\hat{F} \in {}_{a'}L_p$.

PROOF. Since $[\Delta_t^j F(x)]^\wedge = [e^{-i(u,t)} - 1]^j \hat{F}(u)$, the Hausdorff-Young theorem implies

$$\tilde{Y}_{a',j}(t; \hat{F}) \leqq K Y_{a,j}(t; F) ,$$

that is,

$${}_{a'}\tilde{A}_{p,j,\alpha}(F) \leqq K\, {}_aA_{p,j,\alpha}(F) .$$

By Lemma 3, we have the result.

COROLLARY. *Let* $1 < a \leqq 2$, $1/a + 1/a' = 1$, $k/p - k/a' < j$, *and* $ak/[a\beta + k(a-1)] < p < a/(a-1)$. *If*

$$Y_{a,j}(t; F) = \left[\int_{R_k} | \Delta_t^j F(x) |^a dx\right]^{1/a} \leqq K | t |^\beta ,$$

then $\hat{F} \in {}_{a'}L_p$. (Cf. Titchmarsh [7], p. 115.)

PROOF. It is enough to prove ${}_aA_{p,j,\alpha}(F) < \infty$. For this purpose, we divide the range of the integral in ${}_aA_{p,j,\alpha}(F)$ into two parts; $|t| \leqq 1$ and $|t| > 1$. In the first part, we have $Y_{a,j}(t; F) \leqq K\, |t|^\beta$, and in the last part, $Y_{a,j}(t; F) \leqq K$, because of $F \in L^a$, Therefore, we have

$${}_aA^p_{p,j,\alpha}(F) \leqq K\left[\int_0^1 r^{-p\alpha+p\beta-1} dr + \int_1^\infty r^{-p\alpha-1} dr\right] < \infty ,$$

since $-p\alpha + p\beta > 0$ by the assumption on p.

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