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# THE ABSOLUTE CESÀRO SUMMABILITY AND THE LITTLEWOOD-PALEY FUNCTION

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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**Introduction.** We denote by $H^\lambda$ the Hardy-class. That is the totality of all the functions $F(z)=\sum_{n=0}^{\infty}c_nz^n$ which are holomorphic in the disk $|z|<1$ in complex plane and

$$\|F\|_\lambda=\sup_{0<r<1}\left(\int_{-\pi}^{\pi}|F(re^{i\theta})|^\lambda d\theta\right)^{1/\lambda}=\left(\int_{-\pi}^{\pi}|F(\theta)|^\lambda d\theta\right)^{1/\lambda}<+\infty\ ,$$

where $F(\theta)$ is the boundary function of $F(re^{i\theta})$. By $\sigma_n^\alpha(\theta)=\sigma_n^\alpha(\theta;F)$, we denote the $n$-th Cesàro mean of order $\alpha$ of the series $\sum_{n=0}^{\infty}c_ne^{in\theta}$. The behavior of $h_\alpha(F)(\theta)=(\sum_{n=1}^{\infty}|\sigma_n^\alpha(\theta)-\sigma_n^{\alpha-1}(\theta)|^2/n)^{1/2}$ has been investigated by many authors. G. Sunouchi and A. Zygmund introduced

$$g_\alpha^*(F)(\theta)=\left[\int_0^1(1-r)^{2\alpha}\left(\int_{-\pi}^{\pi}\frac{|F'(re^{i(\theta+\phi)})|^2}{|1-re^{i\phi}|^{2\alpha}}d\phi\right)dr\right]^{1/2}$$

in [5] and [6], respectively. Between $h_\alpha(F)$ and $g_\alpha^*(F)$, there are the relations $C_\alpha h_\alpha(F)(\theta)\leqq g_\alpha^*(F)(\theta)\leqq C'_\alpha h_\alpha(F)(\theta)$ for some constants $C_\alpha$ and $C'_\alpha$. The following results have been known.

THEOREM A. (i) *If* $\alpha>\mathrm{Max}\,(1/2,1/\lambda)$ *and* $0<\lambda<\infty$, *then* $\|h_\alpha(F)\|_\lambda\leqq C_{\alpha,\lambda}\|F\|_\lambda$ *and* $\|g_\alpha(F)\|_\lambda\leqq C_{\alpha,\lambda}\|F\|_\lambda$. (ii) *If* $\alpha=1/\lambda$ *and* $0<\lambda\leqq 1$, *then* $\|h_\alpha(F)\|_\lambda\leqq C_\lambda\int_{-\pi}^{\pi}|F(\theta)|^\lambda\log^+|F(\theta)|d\theta$ *and* $\|h_\alpha(F)\|_{\mu\lambda}\leqq C_{\lambda,\mu}\|F\|_\lambda$ *for any* $0<\mu<1$ *and the same results hold for* $g_\alpha^*(F)$.

The case when $\alpha=1/\lambda$ and $1<\lambda<2$ is recently obtained by C. Fefferman [2].

Meanwhile, T. M. Flett proved the following theorem for $h_{\alpha,q}(F)(\theta)=(\sum_{n=1}^{\infty}|\sigma_n^\alpha(\theta)-\sigma_n^{\alpha-1}(\theta)|^q/n)^{1/q}$ and

$$g_{\alpha,q}^*(F)(\theta)=\left[\int_0^1(1-r)^{\alpha q}\left(\int_{-\pi}^{\pi}\frac{|F'(re^{i(\theta+\phi)})|^p}{|1-re^{i\phi}|^{\alpha p}}d\right)^{q/p}dr\right]^{1/q}\ ,$$

where $(1/p)+(1/q)=1$ in [3].

THEOREM B. *If* $2\leqq q<\infty$, $0<\lambda<\infty$ *and* $\alpha>\mathrm{Max}\,(1/\lambda,1-(1/q))$, *then* $\|h_{\alpha,q}(F)\|_\lambda\leqq C_{\alpha,\lambda,q}\|F\|_\lambda$ *and* $\|g_{\alpha,q}^*(F)\|_\lambda\leqq C_{\alpha,\lambda,q}\|F\|_\lambda$.

We use the letter $C$ as a positive constant, not always the same in different occasions. The constants with indices may depend on them.

The purpose of this paper is to study a critical case $\alpha = 1/\lambda$ in theorem $B$ with the recipe used by C. Fefferman in [2].

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1. **Conclusions.** We set as usual for holomorphic function $F(z) = F(re^{i\theta}) = \sum_{n=0}^{\infty} c_n z^n$ in $|z| < 1$,

$$\tau_n^\alpha(\theta) = \tau_n^\alpha(\theta; F) = \sum_{\nu=0}^{n} \frac{A_{n-\nu}^{\alpha-1}}{A_n^\alpha} \nu c_\nu e^{i\nu\theta} ,$$

where $A_n^\alpha = \binom{n+\alpha}{n}$. If we denote $\sigma_n^\alpha(\theta) = \sigma_n^\alpha(\theta; F)$ the $n$-th Cesàro mean of degree $\alpha$ of the series $\sum_{n=0}^{\infty} c_n e^{in\theta}$ and if $\alpha > 0$, then we have

$$\tau_n^\alpha(\theta) = \alpha\{\sigma_n^{\alpha-1}(\theta) - \sigma_n^\alpha(\theta)\} .$$

If we define a function $h_{\alpha,q}(F)$ on $(-\pi, \pi)$ with respect to the above function $F$ by

$$h_{\alpha,q}(F)(\theta) = \left(\sum_{n=1}^{\infty} \frac{|\tau_n^\alpha(\theta; F)|^q}{n}\right)^{1/q} . \tag{1.1}$$

Then we get the following theorem.

THEOREM 1. *When $2 \leqq q < \infty$, $0 < \lambda < 2$, $1/\lambda + 1/q > 1$ and $\alpha = 1/\lambda$, then*

$$|\{\theta \in (-\pi, \pi); h_{\alpha,q}(F)(\theta) > \kappa\}| \leqq \left(\frac{C_{\lambda,q}}{\kappa}\|F\|_\lambda\right)^\lambda$$

*for any $\kappa > 0$ and $F \in H^\lambda$.*

In the sequel, we denote by $|E|$ the Lebesgue measure of a set $E$ in $(-\infty, \infty)$. For $E \subset (-\pi, \pi)$, $\complement E$ will be the set $(-\pi, \pi)\backslash E$.

To prove theorem 1, we introduce an auxiliary function

$$g_{\alpha,q}^*(F)(\theta) = \left[\int_0^1 (1 - r)^{\alpha q}\left(\int_{-\pi}^{\pi} \frac{|F'(re^{i(\theta+\phi)})|^p}{|1 - re^{i\phi}|^{\alpha p}} d\phi\right)^{q/p} dr\right]^{1/q} , \tag{1.2}$$

where, now and in the sequel, the relation $1/p + 1/q = 1$ is always assumed. T. M. Flett ([3; Theorem 13]) proved that if $q \geqq 2$ and $\alpha > (-1/q)$, then

$$h_{\alpha,q}(F)(\theta) \leqq C_{\alpha,q} g_{\alpha,q}^*(F)(\theta) \tag{1.3}$$

for every point $\theta$. We prove in §4 that

THEOREM 2. *If $\alpha$, $q$ and $\lambda$ satisfy the conditions of theorem 1, then*

$$|\{\theta \in (-\pi, \pi);\, g^*_{\alpha,q}(F)(\theta) > \kappa\}| \leqq \left(\frac{C_{\lambda,q}}{\kappa}\|F\|_\lambda\right)^\lambda$$

*for any* $\kappa > 0$ *and* $F \in H^\lambda$.

Theorem 1 follows easily from Theorem 2 and (1.3).

The proof of Theorem 2 proceeds in two steps:

$$1 < \lambda < 2,\ 2 \leqq q < \lambda/(\lambda - 1) \text{ and } \alpha = 1/\lambda\,, \tag{1.4}$$

$$0 < \lambda \leqq 1,\ 2 \leqq q < +\infty \quad \text{ and } \alpha = 1/\lambda\,. \tag{1.5}$$

We use also the following notations. For a function $\phi(z) = \phi(re^{i\theta})$ in $|z| < 1$, we denote

$$\nabla\phi(z) = \left(\frac{\partial\phi}{\partial r}(z), \frac{\partial\phi}{\partial\theta}(z)\right),\ |\nabla\phi(z)|^2 = \left|\frac{\partial\phi}{\partial r}(z)\right|^2 + \left|\frac{\partial\phi}{\partial\theta}(z)\right|^2.$$

For $f \in L(-\pi, \pi)$, we introduce

$$S_{\alpha,q}(f)(\theta) = \left[\int_0^1 (1-r)^{\alpha q}\left(\int_{-\pi}^{\pi} \frac{|\nabla f(re^{i(\theta+\phi)})|^p}{|1 - re^{i\phi}|^{\alpha p}}d\phi\right)^{q/p} dr\right]^{1/q}, \tag{1.6}$$

where $f(z) = f(re^{i\theta})$ is the Poisson integral of $f$. We prove in §3 the following theorem for $S_{\alpha,q}(f)$, which is equivalent to Theorem 2.

THEOREM 3. *If* $\alpha$, $q$ *and* $\lambda$ *have the relation* (1.4), *then*

$$|\{\theta \in (-\pi, \pi);\, S_{\alpha,q}(f)(\theta) > \kappa\}| \leqq \left(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\right)^\lambda$$

*for any* $\kappa > 0$ *and* $f \in L^\lambda(-\pi, \pi)$.

2. **Preparations.** We collect up in this section the lemmas which are used in the proofs. Some of them are well known and others are proved completely.

LEMMA 1. $\int_{-\pi}^{\pi} 1/|1 - re^{i\theta}|^\beta d\theta \leqq C_\beta(1 - r)^{1-\beta}$ *for any* $\beta > 1$ *and* $0 < r < 1$.

LEMMA 2. *When* $m$ *is an integer greater than* 1 *and* $\beta > -1$,

$$\int_0^1 (1 - r)^\beta r^{m-1} dr = \frac{1}{mA_m^\beta}\,.$$

LEMMA 3. (e.g. [4; VI. 2]) *If* $\Omega \subset (-\infty, \infty)$ *is an open set which does not contain any infinite interval, then there exists a disjoint sequence* $\{I_j\}$ *of intervals such that* $\Omega = \bigcup_{j=1}^\infty I_j$ *and they have the following properties*:

$$\text{dis.}\,(I_j, \complement\Omega) = \inf\{|x - y|;\, x \in I_j,\, y \in \complement\Omega\} = |I_j|\,. \tag{2.1}$$

$$\bar{I}_j \cap \bar{I}_i \neq \varnothing \text{ implies } |I_j| \leqq 2|I_i|\,. \tag{2.2}$$

(2.3) *We set* $I_j^* = \cup\{I_i; \bar{I}_i \cap \bar{I}_j \neq \emptyset\}$. *If* $x \notin I_j^*$ *and* $y_1, y_2 \in I_j$, *then* $|x - y_1| \leqq 3|x - y_2|$.

LEMMA 4. *For* $f \in L^\lambda(-\pi, \pi)$ *and* $\kappa > 0$, *we denote*

$$\Omega = \left\{\theta \in (-\pi, \pi];\ \sup_{0<\xi,\eta\leqq\pi}\left(\frac{1}{\xi + \eta}\int_{\theta-\xi}^{\theta+\eta}|f(\tau)|^\lambda d\tau\right) > \kappa^\lambda\right\}. \tag{2.4}$$

*If* $\Omega \neq (-\pi, \pi]$, *then we have* $\Omega = \bigcup_{j=1}^\infty I_j$ *by applying lemma* 3 *and*

$$|\Omega| = \sum_{j=1}^\infty |I_j| \leqq \left(\frac{C_\lambda}{\kappa}\|f\|_\lambda\right)^\lambda. \tag{2.5}$$

$$|f(\theta)| \leqq \kappa \text{ a.e. in } \complement\Omega. \tag{2.6}$$

$$\int_{I_j}|f(\theta)|^\lambda d\theta \leqq C\kappa^\lambda|I_j|. \tag{2.7}$$

*Furthermore, if we decompose* $f$ *as*

$$g(\theta) = \begin{cases} \dfrac{1}{|I_j|}\displaystyle\int_{I_j} f(\tau)d\tau & (\theta \in I_j,\ j = 1, 2, \cdots) \\ f(\theta) & (\theta \notin \Omega) \end{cases} \tag{2.8}$$

$$f(\theta) = g(\theta) + h(\theta),\ h_j(\theta) = h(\theta)\chi_{I_j}(\theta) \qquad (j = 1, 2, \cdots)$$

*where* $\chi_{I_j}$ *means the characteristic function of* $I_j$, *then*

$$|g(\theta)| \leqq C_\lambda\kappa \text{ a.e. and } \|g\|_\lambda \leqq \|f\|_\lambda. \tag{2.9}$$

$$\int_{I_j}|h_j(\theta)|\,d\theta \leqq C_\lambda\kappa\,|I_j| \text{ and } \|h_j\|_\lambda \leqq C_\lambda\kappa\,|I_j|^{1/\lambda}. \tag{2.10}$$

$$\int_{I_j} h_j(\theta)d\theta = 0. \tag{2.11}$$

LEMMA 5. *Let* $P(z) = P(re^{i\theta}) = \text{Re.}\,(1 + z)/(1 - z)$ *be the Poisson kernel,* $R_r(\theta) = \nabla P(re^{i\theta})$ *and* $H_1(r, \theta) = \sum_{(j:\theta\notin I_j^*)}(R_r * h_j)(\theta)$, *then*

$$|H_1(r, \theta)| \leqq C_\lambda\frac{\kappa}{1 - r}.$$

$I_j^*$ *and* $h_j$ *are defined by* (2.3) *and* (2.8), *respectively.*

PROOF. If we set $\Phi(z) = (1 + z)/(1 - z)$, then $|R_r(\theta)| \leqq \sqrt{2}\,|\Phi'(re^{i\theta})| = 2\sqrt{2}/|1 - re^{i\theta}|^2$. For $\theta \notin I_j^*$ and $\tau_1, \tau_2 \in I_j$, we have $|1 - re^{i(\theta-\tau_1)}| \leqq (1 - r) + |\theta - \tau_1| \leqq (1 - r) + 3|\theta - \tau_2| \leqq C|1 - re^{i(\theta-\tau_2)}|$ by (2.3). Hence

$$|I_j|\sup_{\tau\in I_j}|1 - re^{i(\theta-\tau)}|^{-2} \leqq C\int_{I_j}|1 - re^{i(\theta-\tau)}|^{-2}d\tau.$$

By (2.10), we have the estimation that

$$|(R_r*h_j)(\theta)| \leqq \sup_{\tau\in I_j}|R_r(\theta-\tau)|\int_{I_j}|h_j(\tau)|d\tau \leqq C_\lambda\kappa|I_j|\sup_{\tau\in I_j}|1-re^{i(\theta-\tau)}|^{-2}$$
$$\leqq C_\lambda\kappa\int_{I_j}|1-re^{i(\theta-\tau)}|^{-2}d\tau$$

for $\theta\notin I_j^*$. Therefore, we have

$$|H_1(r,\theta)| \leqq \sum_{\{j:\theta\notin I_j^*\}}|(R_r*h_j)(\theta)| \leqq C_\lambda\kappa\int_{-\pi}^{\pi}\frac{d\tau}{|1-re^{i\theta}|^2} \leqq \frac{C_\lambda\kappa}{1-r}\ ,$$

by lemma 1.

LEMMA 6. $\displaystyle\int_0^1\int_{\pi\geqq|\phi|\geqq 2|\tau|}|R_r(\phi-\tau)-R_r(\phi)|d\phi dr \leqq C\ .$

PROOF. A simple calculation shows that

$$|R_r(\phi-\tau)-R_r(\phi)| \leqq 2\sqrt{2}|e^{-i\tau}(1-re^{i(\phi-\tau)})^{-2}-(1-re^{i\phi})^{-2}|\ .$$

If $2|\tau|\leqq|\phi|\leqq\pi$, then

$$|e^{-i\tau}(1-re^{i(\phi-\tau)})^{-2}-(1-re^{i\phi})^{-2}| \leqq |(1-re^{i(\phi-\tau)})^{-2}-(1-re^{i\phi})^{-2}|$$
$$+|(e^{-i\tau}-1)(1-re^{i\phi})^{-2}| \leqq C\{|\tau|[(1-r)+|\phi|]^{-3}+|\tau||\phi|^{-2}\}\ .$$

Hence

$$\int_0^1\int_{\pi\geqq|\phi|\geqq 2|\tau|}|R_r(\phi-\tau)-R_r(\phi)|d\phi dr \leqq C\int_{\pi\geqq|\phi|\geqq 2|\tau|}\left(C'\frac{|\tau|}{|\phi|^2}+\frac{|\tau|}{|\phi|^2}\right)d\phi \leqq C\ .$$

LEMMA 7. $\displaystyle\int_0^1\int_{\complement I_j^*}|(R_r*h_j)(\phi)|d\phi dr \leqq C_\lambda\kappa|I_j| \qquad (j=1,2,\cdots)$

PROOF. Let $\tau_j$ be the center of $I_j$. Using the properties (2.11), (2.2), lemma 6 and (2.10) step by step

$$\int_0^1\int_{\complement I_j^*}|(R_r*h_j)(\phi)|d\phi dr$$
$$\leqq \int_{I_j}|h_j(\tau)|d\tau\int_0^1\int_{\complement I_j^*}|R_r(\phi-\tau)-R_r(\phi-\tau_j)|d\phi dr$$
$$\leqq \int_{I_j}|h_j(\tau)|d\tau\int_0^1\int_{|\phi-\tau_j|\geqq 2|\tau-\tau_j|}|R_r(\phi-\tau)-R_r(\phi-\tau_j)|d\phi dr$$
$$\leqq C\int_{I_j}|h_j(\tau)|d\tau \leqq C_\lambda\kappa|I_j|\ .$$

LEMMA 8. (T. M. Flett [3; Theorem 7]) *If $F\in H^\mu$, $\mu>0$, and $k\geqq 2$, then*

$$\left\{\int_{-\pi}^{\pi}\left(\int_0^1(1-r)^{k-1}|F'(re^{i\theta})|^k dr\right)^{\mu/k}d\theta\right\}^{1/\mu} \leqq C_{\mu,k}||F||_\mu\ .$$

LEMMA 9. *We define $G_{\alpha,q}(f)(\theta)$ for a real-valued function $f$ in $L$ $(-\pi, \pi)$ by*

$$G_{\alpha,q}(f)(\theta) = \left(\int_0^1 (1-r)^{\alpha q} |\nabla f(re^{i\theta})|^q dr\right)^{1/q} = \left(\int_0^1 (1-r)^{\alpha q} |(R_r * f)(\theta)|^q dr\right)^{1/q}.$$

*If $\alpha$, $q$ and $\lambda$ satisfy the condition* (1.4), *then*

$$\| G_{\alpha,q}(f) \|_p \leqq C_{\lambda,q} \| f \|_\lambda .$$

PROOF. For $f$ in $L^1(-\pi, \pi)$, we define a function $T(f)$ on the product measure space $\{1, 2\} \times (0, 1) \times (-\pi, \pi)$ by

$$T(f)(\nu, r, \theta) = \begin{cases} (1-r)^\alpha \dfrac{\partial}{\partial r} f(re^{i\theta}) & (\nu = 1) \\ (1-r)^\alpha \dfrac{\partial}{\partial \theta} f(re^{i\theta}) & (\nu = 2) . \end{cases}$$

Then we have

$$\| T(f) \|_{2,2,2} \leqq C_\lambda \| f \|_\lambda \tag{2.12}$$

$$\| T(f) \|_{2,\lambda',\lambda} \leqq C_\lambda \| f \|_\lambda \qquad (1/\lambda + 1/\lambda' = 1) , \tag{2.13}$$

where $\| T(f) \|_{r,s,t} = \left\{\int_{-\pi}^{\pi} \left[\int_0^1 (\sum_{\nu=1}^2 | T(f)(\nu, \rho, \theta) |^r)^{s/r} d\rho\right]^{t/s} d\theta\right\}^{1/t}$. To prove the inequality (2.12), we use the Parseval's theorem and fractional integral theorem. Let $\{c_n; n = o, \pm 1, \pm 2, \cdots\}$ be the Fourier coefficients of $f$. We may assume $c_0 = 0$ without loss of generality. Then, by virtue of lemma 2 and the relation $\alpha - 1/2 = 1/\lambda - 1/2$, we have

$$\begin{aligned} \| T(f) \|_{2,2,2} &= \left(\int_{-\pi}^{\pi} \int_0^1 (1-r)^{2\alpha} |\nabla f(re^{i\theta})|^2 dr d\theta\right)^{1/2} \\ &\leqq C\left(\sum_{n \neq 0} n^2 |c_n|^2 \int_0^1 (1-r)^{2\alpha} r^{(2|n|-1)-1} dr\right)^{1/2} \\ &= C\left(\sum_{n \neq 0} \frac{n^2 |c_n|^2}{(2|n|-1) A_{2|n|-1}^{2\alpha}}\right)^{1/2} \leqq C\left(\sum_{n \neq 0} \left| \frac{c_n}{n^{\alpha-(1/2)}} \right|^2\right)^{1/2} \\ &\leqq C_\lambda \| f \|_\lambda . \end{aligned}$$

To prove (2.13), consider $F(z) = f(z) + i\tilde{f}(z)$ where $\tilde{f}(z)$ is the conjugate harmonic function of $f(z)$ with $\tilde{f}(0) = 0$. Then $F \in H^\lambda$ and $|\nabla f| \leqq \sqrt{2} |F'|$. Hence

$$\| T(f) \|_{2,\lambda',\lambda} \leqq \sqrt{2} \left\{\int_{-\pi}^{\pi} \left(\int_0^1 (1-r)^{\lambda'-1} |F'(re^{i\theta})|^{\lambda'} dr\right)^{\lambda/\lambda'} d\theta\right\}^{1/\lambda} \leqq C_\lambda \| F \|_\lambda \leqq C_\lambda \| f \|_\lambda$$

by lemma 8 and M. Riesz theorem.

We apply the interpolation theorem of A. Benedek and R. Panzone ([1; Theorem 2, p. 316]) to the relations (2.12) and (2.13). Since $1/q =$

$(1-t)/2+t/\lambda'$ and $1/p=(1-t)/2+t/\lambda$ for $0\leqq t<1$, we have $\|T(f)\|_{2,q,p}\leqq C_{\lambda,q}\|f\|_\lambda$. The equation $\|T(f)\|_{2,q,p}=\|G_{\alpha,q}(f)\|_q$ conclude the lemma 9.

**3. Proof of theorem 3.** It suffices to prove for real-valued functions. Let $f\in L^\lambda(-\pi,\pi)$ be a real-valued function and define $\Omega$ by (2.4). First we consider the case $\Omega\neq(-\pi,\pi]$. Decomposing $f$ as $f=g+h$ by (2.8), we have

$$S_{\alpha,q}(f)(\theta)\leqq S_{\alpha,q}(g)(\theta)+S_{\alpha,q}(h)(\theta)\,. \tag{3.1}$$

Further we define $H_1(r,\theta)=\sum_{\{j:\theta\notin I_j^*\}}(R_r*h_j)(\theta)$ and $H_2(r,\theta)=\sum_{\{j:\theta\in I_j^*\}}(R_r*h_j)(\theta)=\nabla h(re^{i\theta})-H_1(r,\theta)$. By definition (1.6),

$$\begin{aligned}S_{\alpha,q}(h)(\theta)&\leqq\left[\int_0^1(1-r)^{\alpha q}\left(\int_{-\pi}^{\pi}\frac{|H_1(r,\theta+\phi)|^p}{|1-re^{i\phi}|^{\alpha p}}d\phi\right)^{q/p}\right]^{1/q}\\&\quad+\left[\int_0^1(1-r)^{\alpha q}\left(\int_{-\pi}^{\pi}\frac{|H_2(r,\theta+\phi)|^p}{|1-re^{i\phi}|^{\alpha p}}d\phi\right)^{q/p}dr\right]^{1/q}\equiv Q_1(\theta)+Q_2(\theta),\ \text{say}\,.\end{aligned}$$

Hence, replacing $S_{\alpha,q}(h)(\theta)$ by $Q_1(\theta)+Q_2(\theta)$ in (3.1), we get

$$S_{\alpha,q}(f)(\theta)\leqq S_{\alpha,q}(g)(\theta)+Q_1(\theta)+Q_2(\theta)\,. \tag{3.2}$$

We estimate the each terms in the right hand side.

The estimation for $S_{\alpha,q}(g)$. For any $\mu$, $\mu>\lambda$, we see that $g\in L^\mu(-\pi,\pi)$ and $\|g\|_\mu^\mu\leqq C_{\lambda,\mu}\kappa^{\mu-\lambda}\|g\|_\lambda^\lambda\leqq C_{\lambda,\mu}\kappa^{\mu-\lambda}\|f\|_\lambda^\lambda$ by (2.9). In particular, if we take $\mu$, $\lambda<\mu<p$, then $\|S_{\alpha,q}(g)\|_\mu^\mu\leqq C_{\alpha,\mu,q}\|g\|_\mu^\mu\leqq C_{\lambda,\mu,q}\kappa^{\mu-\lambda}\|f\|_\lambda^\lambda$ by theorem $B$ in the introduction. Therefore

$$\left|\left\{\theta\in(-\pi,\pi);\,S_{\alpha,q}(g)(\theta)>\frac{\kappa}{3}\right\}\right|\leqq\left(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\right)^\lambda\,. \tag{3.3}$$

The estimation for $Q_1$. Note that $p\leqq q$. Then we have

$$\begin{aligned}\int_{-\pi}^{\pi}[Q_1(\theta)]^q d\theta&=\int_{-\pi}^{\pi}\int_0^1\left(\int_{-\pi}^{\pi}\frac{(1-r)^{\alpha p}}{|1-re^{i\phi}|^{\alpha p}}|H_1(r,\theta+\phi)|^p d\phi\right)^{q/p}drd\theta\\&\leqq\int_0^1\left[\int_{-\pi}^{\pi}\frac{(1-r)^{\alpha p}}{|1-re^{i\phi}|^{\alpha p}}\left(\int_{-\pi}^{\pi}|H_1(r,\theta+\phi)|^q d\theta\right)^{p/q}d\phi dr\right]^{q/p}\\&\qquad\qquad\text{(by Minkowski's inequality)}\\&\leqq C_{\alpha,p}\int_0^1(1-r)^{q/p}\left(\int_{-\pi}^{\pi}|H_1(r,\theta)|^q d\theta\right)dr\qquad\text{(by lemma 1)}\\&\leqq C_{\alpha,p}\int_0^1(1-r)^{q-1}\int_{-\pi}^{\pi}\left(\frac{C_\lambda\kappa}{1-r}\right)^{q-1}|H_1(r,\theta)|\,d\theta dr\qquad\text{(by lemma 5)}\\&\leqq C_{\lambda,q}\kappa^{q-1}\int_0^1\int_{-\pi}^{\pi}\sum_{\{j:\theta\notin I_j^*\}}|(R_r*h_j)(\theta)|\,d\theta dr\end{aligned}$$

$$= C_{\lambda,q}\kappa^{q-1}\sum_{j=1}^{\infty}\int_0^1\int_{\complement I_j^*}|(R_r*h_j)(\theta)|\,d\theta dr$$

$$\leqq C_{\lambda,q}\kappa^q\sum_{j=1}^{\infty}|I_j| \leqq \kappa^q\Big(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\Big)^\lambda \qquad \text{(by lemma 7 and (2.5)).}$$

Therefore we can conclude that

$$\Big|\Big\{\theta\in(-\pi,\pi); Q_1(\theta) > \frac{\kappa}{3}\Big\}\Big| \leqq \Big(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\Big)^\lambda. \tag{3.4}$$

The estimation for $Q_2$. We set $h^i = \sum_{\{j:I_j\subset I_i^*\}}h_j$ for every $i$. Then $H_2(r,\phi) = (R_r*h^i)(\phi)$ for $\phi\in I_i$. $I_i^*$ contains only three intervals $I_j$ for each $i$. Therefore,

$$\|h^i\|_\lambda \leqq \sum_{\{j:I_j\subset I_i^*\}}C_\lambda\kappa|I_j|^{1/\lambda} \leqq C_\lambda\kappa\sum_{\{j:I_j\subset I_i^*\}}(2|I_i|)^{1/\lambda} = 3.2^{1/\lambda}C_\lambda\kappa|I_i|^{1/\lambda} = C_\lambda\kappa|I_i|^{1/\lambda}$$

by (2.10) and (2.2). We also have $|1-re^{i(\phi-\theta)}| \geqq C|\theta-\tau_i|$ for $\theta\notin\Omega$, $\phi\in I_i$ and the center $\tau_i$ of $I_i$ by the relation (2.3). Now we shall estimate $[Q_2(\theta)]^p$ at $\theta\notin\Omega$. Since $H_2(r,\phi)=0$ for $\phi\notin\Omega$,

$$[Q_2(\theta)]^p = \Big[\int_0^1\Big(\sum_{i=1}^{\infty}\int_{I_i}\frac{(1-r)^{\alpha p}}{|1-re^{i(\phi-\theta)}|^{\alpha p}}|H_2(r,\phi)|^p d\phi\Big)^{q/p}dr\Big]^{p/q}$$

$$\leqq C\Big[\int_0^1\Big(\sum_{i=1}^{\infty}\frac{1}{|\theta-\tau_i|^{\alpha p}}\int_{I_i}(1-r)^{\alpha p}|H_2(r,\phi)|^p d\phi\Big)^{q/p}dr\Big]^{p/q}$$

$$\leqq C\sum_{i=1}^{\infty}\frac{1}{|\theta-\tau_i|^{\alpha p}}\int_{I_i}\Big(\int_0^1(1-r)^{\alpha q}|H_2(r,\phi)|^q dr\Big)^{p/q}d\phi$$

(by Minkowski's inequality)

$$\leqq C\sum_{i=1}^{\infty}\frac{\|G_{\alpha,q}(h^i)\|_p^p}{|\theta-\tau_i|^{\alpha p}}$$

$$\leqq C_{\lambda,q}\sum_{i=1}^{\infty}\frac{\|h^i\|_\lambda^p}{|\theta-\tau_i|^{\alpha p}} \qquad \text{(by lemma 9)}$$

$$\leqq C_{\lambda,q}\sum_{i=1}^{\infty}\frac{\kappa^p|I_i|^{p/\lambda}}{|\theta-\tau_i|^{\alpha p}}.$$

Since $\int_{\complement\Omega}|\theta-\tau_i|^{-\alpha p}d\theta \leqq C_{\lambda,q}|I_i|^{-(p/\lambda)+1}$, we have

$$\int_{\complement\Omega}[Q_2(\theta)]^p d\theta \leqq C_{\lambda,q}\sum_{i=1}^{\infty}\kappa^p|I_i| \leqq \kappa^p\Big(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\Big)^\lambda.$$

Hence

$$\Big|\Big\{\theta\in\complement\Omega;\ Q_2(\theta) > \frac{\kappa}{3}\Big\}\Big| \leqq \Big(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\Big)^\lambda.$$

The measure $|\Omega|$ is estimated by (2.5). Therefore we get

$$\left|\left\{\theta \in (-\pi, \pi); Q_2(\theta) \geqq \frac{\kappa}{3}\right\}\right| \leqq \left(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\right)^\lambda . \tag{3.5}$$

From the estimates (3.3), (3.4) and (3.5), it follows the required estimate:

$$\left|\left\{\theta \in (-\pi, \pi); S_{\alpha,q}(f)(\theta) > \kappa\right\}\right| \leqq \left(\frac{C_{\lambda,q}}{\kappa}\|f\|_\lambda\right)^\lambda . \tag{3.6}$$

When $\Omega = (-\pi, \pi]$, we have easily

$$\left|\left\{\theta \in (-\pi, \pi); S_{\alpha,q}(f)(\theta) > \kappa\right\}\right| \leqq |\Omega| \leqq \left(\frac{C_\lambda}{\kappa}\|f\|_\lambda\right)^\lambda \tag{3.7}$$

by a property of the maximal function of Hardy-Littlewood. Consequently we have the theorem with a constant greater than $C_{\lambda,q}$ in (3.6) and $C_\lambda$ in (3.7).

4. **Proof of theorem 2.** In the case (1.4), theorem 2 is deduced from theorem 3, so that we prove the case (1.5). Without loss of generality, we may assume that $F \in H^\lambda$ is free from zero. We set $G(z) = [F(z)]^{\lambda/\mu}$ for some $1 < \mu < p$. Then $G \in H^\mu$ and the boundary function $G(\theta)$ is in $L^\mu(-\pi, \pi)$. Furthermore, if we set

$$G_\mu^*(\theta) = \sup_{0<|h|\leqq\pi}\left(\frac{1}{h}\int_0^h |G(\theta + \tau)|^\mu d\tau\right)^{1/\mu} ,$$

then $|G(re^{i(\theta+\phi)})| \leqq C_\mu G_\mu^*(\theta)[1 + |\phi|/(1 - r)]^{1/\mu}$. (See e.g. G. Sunouchi [4; Lemma 3]) Applying this relation and $F'(z) = (\mu/\lambda)[G(z)]^{(\mu/\lambda)-1}G'(z)$, we have

$$\begin{aligned} g_{\alpha,q}^*(F)(\theta) &= \frac{\mu}{\lambda}\left\{\int_0^1(1 - r)^{\alpha q}\left(\int_{-\pi}^{\pi} \frac{|[G(re^{i(\theta+\phi)})]^{(\mu/\lambda)-1}G'(re^{i(\theta+\phi)})|^p}{|1 - re^{i\phi}|^{\alpha p}}d\phi\right)^{q/p}dr\right\}^{1/q} \\ &\leqq C'_{\lambda,\mu}\{G_\mu^*(\theta)\}^{(\mu/\lambda)-1}\left\{\int_0^1(1 - r)^{q/\mu}\left(\int_{-\pi}^{\pi} \frac{|G'(re^{i(\theta+\phi)})|^p}{|1 - re^{i\phi}|^{p/\mu}}d\phi\right)^{q/p}dr\right\}^{1/q} \\ &= C'_{\lambda,\mu}\{G_\mu^*(\theta)\}^{(\mu/\lambda)-1}g_{\beta,q}^*(G)(\theta) , \end{aligned} \tag{4.1}$$

where $\beta = 1/\mu$. Since $\beta$, $q$ and $\mu$ satisfy the relation (1.4) replaced $\alpha$ and $\lambda$ by $\beta$ and $\mu$, respectively, we have from the above argument

$$\begin{aligned} &|\{\theta \in (-\pi, \pi); g_{\beta,q}^*(G)(\theta) > (C'_{\lambda,\mu}\kappa)^{\lambda/\mu}\}| \\ &\leqq \left(\frac{C_{\mu,q}}{(C'_{\lambda,\mu}\kappa)^{\lambda/\mu}}\|G\|_\mu\right)^\mu = \left(\frac{C_{\lambda,\mu,q}}{\kappa}\|F\|_\lambda\right)^\lambda . \end{aligned} \tag{4.2}$$

By a property of the maximal function of Hardy-Littlewood,

$$\begin{aligned} &|\{\theta \in (-\pi, \pi); G_\mu^*(\theta) > (C'_{\lambda,\mu}\kappa)^{\lambda/\mu}\}| \\ &\leqq \frac{C}{(C'_{\lambda,\mu}\kappa)^\lambda}\|G\|_\mu^\mu = \left(\frac{C_{\lambda,\mu}}{\kappa}\|F\|_\lambda\right)^\lambda . \end{aligned} \tag{4.3}$$

The relation (4.1) and the estimates (4.2) and (4.3) give

$$|\{\theta \in (-\pi, \pi);\ g^*_{\alpha,q}(F)(\theta) > \kappa\}| \leqq \left(\frac{C_{\lambda,q}}{\kappa}\|F\|_\lambda\right)^\lambda,$$

and this completes the proof.

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