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ON THE SPECTRA OF TENSOR PRODUCT OF LINEAR OPERATORS IN LOCALLY CONVEX SPACES

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1. Introduction. In [2], A. Brown and C. Pearcy have proved the following result about linear operators on a Hilbert space.

THEOREM. *Let T_1 and T_2 be bounded linear operators on a Hilbert space H, and $T_1 \otimes T_2$ be the tensor product of T_1 and T_2 on $H \otimes H$. Then we have*

$$\sigma(T_1 \otimes T_2) = \sigma(T_1)\sigma(T_2) = \{\lambda_1\lambda_2 \mid \lambda_1 \in \sigma(T_1), \lambda_2 \in \sigma(T_2)\}$$

where $\sigma(T)$ denotes the spectrum of T acting on H.

T. Ichinose [4], M. Schechter [8] and M. Reed, B. Simon [6] extended this result to the case of Banach spaces.

The purpose of this paper is to discuss the spectra of linear operators on certain locally convex space and the result of A. Brown and C. Pearcy analogous for locally convex spaces.

In § 2, we debate upon the algebra $L(X)$ of the continuous linear operators on a locally convex space X, and the spectrum of $T \in L(X)$, and show its property.

In § 3, we consider a quasi-complete commutative locally convex algebra and prove some results concerning the spectrum and the joint spectrum of its element.

In § 4, we shall prove the main theorem which is the result about the spectrum of tensor product of linear operators on nuclear Fréchet spaces and shall show the application of Theorem.

Throughout this paper, let X be a complete locally convex space over the complex numbers, C, and an operator means always a linear operator on a locally convex space.

We consider the simple convergence topology in $L(X)$ and we denote by $L_s(X)$ the linear space $L(X)$ with this topology. A multiplication $(TU)x = T(Ux)$ $(T, U \in L(X))$ induces a structure of algebra to $L_s(X)$, and the map $(T, U) \to TU$ of $L_s(X) \times L_s(X)$ into $L_s(X)$ is obviously separately continuous, hence the algebra $L_s(X)$ is a locally convex algebra in the sense of G. R. Allan [1]. Then the following is easily shown

PROPOSITION. *If X is a barreled space, then $L_s(X)$ is sequencially complete.*

Before going into the discussion, the author wishes to express his hearty thanks to Prof. M. Fukamiya for his many valuable suggestions in the presentation of this paper.

2. Spectra of continuous linear operators on locally convex spaces. We shall define a bounded operator on a locally convex space.

DEFINITION 2.1. An operator $T \in L(X)$ is said to be a bounded operator in $L_s(X)$ (or on X) if there exists a constant $\delta \gneqq 0$ such that $\{(\delta T)^n\}_{n=0}^{\infty}$ is a bounded subset of $L_s(X)$. The family of all bounded operators in $L_s(X)$ is denoted by $\mathscr{B}(X)$.

We notice the following proposition in [5].

PROPOSITION 2.2. *Let X be a barreled space, and $T \in L(X)$. Then the following are equivalent:*

(1) *T is a bounded operator in $L_s(X)$.*

(2) *There exists a constant $\delta \gneqq 0$ such that $\{(\delta T)^n\}_{n=0}^{\infty}$ is a equicontinuous family in $L(X)$.*

(3) *There exist a fundamental semi-norm system (hereafter F.S.N.S. stands for this term), $\boldsymbol{P}$, for X and a constant $c \geqq 0$ such that $p(Tx) \leqq cp(x)$ for all $p \in \boldsymbol{P}$ and all $x \in X$.*

We remark that if we fix a F.S.N.S, $\boldsymbol{P}$, for X, then for any bounded operator $T \in L(X)$ the F.S.N.S. satisfying (3) is given by $\boldsymbol{P}_T = \{p_T \mid p \in \boldsymbol{P}\}$ where $p_T(x) = \sup\{p((\delta T)^n x) \mid n = 0, 1, 2, \cdots\}$.

DEFINITION 2.3. Let $T \in L(X)$. Then the spectrum of T, denoted by $\sigma(T)$, is the complex number λ for which $T - \lambda I$ has no inverse in $\mathscr{B}(X)$. The complement of $\sigma(T)$ in $\boldsymbol{C}$, denoted by $\rho(T)$, is said to be the resolvent set of T.

For a locally convex space X, we have the dual system $(X, X^*, \langle\ \rangle)$, and for $T \in L(X)$, we denoted ${}^tT$ the transpose of T. We provide the bounded convergence topology to X^*, denoted by X_β^*, and consider the spectrum of the transpose, ${}^tT$, in $L_s(X_\beta^*)$, and we state the following theorem.

THEOREM 2.4. *Let X be a Fréchet space whose dual space is a barreled space. Then we have $\sigma(T) = \sigma({}^tT)$ for $T \in L(X)$.*

To prove Theorem, we need following three lemmas, since the strong dual space X_β^* is a barreled space, we can use the criterion of the boundedness of the operators in $L(X_\beta^*)$ as stated in Proposition 2.2.

LEMMA 1. *Let* $\{T_\alpha\}_{\alpha\in A}$ *be an equicontinuous family in* $L(X)$, *then* $\{{}^tT_\alpha\}_{\alpha\in A}$ *is also an equicontinuous family in* $L(X_\beta^*)$.

As the proof is done by the elementary calculas, we omit the proof of this lemma.

LEMMA 2. *Let* $T\in L(X)$ *whose range space is dense in* X. *If both* T *and* ${}^tT$ *are one to one, then we have* $({}^tT)^{-1} = {}^t(T^{-1})$.

PROOF. For $x\in R[T]$, the range of T, and $x^*\in D[{}^tT]$, the domain of ${}^tT$, we have $\langle x, x^*\rangle = \langle TT^{-1}x, x^*\rangle = \langle T^{-1}x, {}^tTx^*\rangle$. Then $\langle T^{-1}(\cdot), {}^tTx^*\rangle$ is same as the restriction of x^* to $R[T]$, so that ${}^tTx^*\in D[{}^t(T^{-1})]$ and ${}^t(T^{-1}){}^tTx^* = x^*$. Therefore ${}^t(T^{-1})$ is the restriction of $({}^tT)^{-1}$.

For $x\in X$ and $x^*\in D[{}^t(T^{-1})]$, we have

$$\langle x, x^*\rangle = \langle T^{-1}Tx, x^*\rangle = \langle Tx, {}^t(T^{-1})x^*\rangle .$$

Then ${}^t(T^{-1})x^*\in D[{}^tT]$ and ${}^tT{}^t(T^{-1})x^* = x^*$, thus ${}^t(T^{-1})$ is the restriction of $({}^tT)^{-1}$. q.e.d.

LEMMA 3. *Let* X *be a Fréchet space whose dual space is a barreled space. Then* T *is a bounded operator in* $L_s(X)$ *if and only if* ${}^tT$ *is a bounded operator in* $L_s(X_\beta^*)$.

PROOF. Let T be a bounded operator in $L_s(X)$. Since $\{(\delta T)^n\}_{n=0}^\infty$ is an equicontinuous family in $L(X)$ for some $\delta \gneqq 0$, each $(\delta T)^n$ maps bounded subsets into bounded subsets. For p_B, defined by

$$p_B(x^*) = \sup_{x\in B} |\langle x, x^*\rangle| ,$$

where B is a bounded subset of X,

$$p_B((\delta^tT)^nx^*) = \sup_{x\in B} |\langle x, (\delta^tT)^nx^*\rangle| = \sup_{x\in B} |\langle(\delta T)^nx, x^*\rangle| < M \quad \text{(const.)} ,$$

where M is independent to n. Thus $\{(\delta^tT)^n\}_{n=0}^\infty$ is bounded in $L_s(X_\beta^*)$. It follows that ${}^tT\in\mathscr{B}(X_\beta^*)$.

Next if we exchange X by X_β^* and X_β^* by $(X_\beta^*)_\beta^*$ in the discussion above, we obtain that ${}^tT\in\mathscr{B}(X_\beta^*)$ implies ${}^t({}^tT)\in\mathscr{B}((X_\beta^*)_\beta^*)$. Since the topology of X is same as the topology induced by $(X_\beta^*)_\beta^*$, T, the restriction of ${}^t({}^tT)$, is a bounded operator in $L_s(X)$. Therefore if ${}^tT\in\mathscr{B}(X_\beta^*)$ then $T\in\mathscr{B}(X)$. q.e.d.

PROOF OF THEOREM 2.4. By the fact ${}^t(T-\lambda I) = {}^tT-\lambda I$, it is sufficient to show the case $\lambda = 0$ especially.

Let $0\in\rho(T)$. Then $R[T] = T$ and there exists a constant $\delta \gneqq 0$ such that $\{(\delta T^{-1})^n\}_{n=0}^{\infty}$ are equicontinuous in $L(X)$ by (2) of Proposition 2.2. By virtue of Lemma 1, $\{{}^t((\delta T^{-1})^n)\}_{n=0}^{\infty}$ are equicontinuous in $L(X_\beta^*)$. The fact $R[T] = X$ shows that ${}^tT$ is one to one, so that $\{(\delta({}^tT)^{-1})^n\}_{n=0}^{\infty}$ are also so by Lemma 2. Hence $0 \in \rho({}^tT)$.

Conversely let $0 \in \rho({}^tT)$. Then there exists a constant $\delta \gneqq 0$ such that $\{(\delta({}^tT)^{-1})^n\}_{n=0}^{\infty}$ are equicontinuous in $L(X_\beta^*)$. The fact that $R[{}^tT] = X^*$ and ${}^tT$ is one to one, shows that T is one to one and $R[T]$ is dense in X, then ${}^t(T^{-1}) = ({}^tT)^{-1}$ by Lemma 2. Of course $D[{}^tT] = X^*$ by the continuity of ${}^t(T^{-1})$. As above, $\{(\delta({}^{tt}(T^{-1}))^n\}_{n=0}^{\infty}$ are equicontinuous in $L((X_\beta^*)_\beta^*)$. From the discussion in Lemma 3, $\{(\delta(T^{-1}))^n\}_{n=0}^{\infty}$ are equicontinuous in $L(X)$, therefore $0 \in \rho(T)$. q.e.d.

Now we assume that X is a barreled space. Let S be the complex numbers λ for which there exist a F.S.N.S., $\boldsymbol{P}$, for X and a constant $c \gneqq 0$ such that $p((T - \lambda I)x) \geqq cp(x)$ for all $p \in \boldsymbol{P}$ and all $x \in X$. The complement of S, denoted by $\pi(T)$, is a subset of $\sigma(T)$ and closed in $\boldsymbol{C}$ since S is open in $\boldsymbol{C}$.

Let $\gamma(T)$ be the complex numbers λ in S for which $R[T - \lambda I]$, the range of $T - \lambda I$, is not dense in X. If $\lambda \in S$, $R[T - \lambda I]$ is closed from the completeness of X, then $\gamma(T) = \{\lambda \in S \mid R[T - \lambda I] \neq X\}$. Thus $\sigma(T)$ is the disjoint union of $\pi(T)$ and $\gamma(T)$. Moreover $\sigma(T)$ is closed (Cor. 3.9. [1]) and, if $T \in \mathscr{B}(X)$, bounded in $\boldsymbol{C}$.

REMARK. $\pi(T)$ is the set of all $\lambda \in \boldsymbol{C}$ satisfying the condition; for an arbitrary F.S.N.S., $\boldsymbol{P}$, there exist $\{x_n\}_{n=1}^{\infty} \subset X$ such that $p_n(x_n) = 1$ and $p_n((T - \lambda I)x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ for some sequence $\{p_n\}_{n=1}^{\infty}$ of $\boldsymbol{P}$.

THEOREM 2.5. *Let X be a barreled space and $T \in L(X)$. Then we have $\partial\sigma(T) \subset \partial\pi(T)$. (Let ∂ stand for "boundary of")*

PROOF. Firstly we shall show that $\rho(T)$ is closed in S. Let $\lambda = \lim_{n\to\infty} \lambda_n$ where $\lambda \in S$ and $\lambda_n \in \rho(T)$. For $\lambda \in S$, there exist a F.S.N.S., $\boldsymbol{P}$, for X and a constant $c \gneqq 0$ such that

$$p((T - \lambda I)x) \geqq cp(x) \qquad \text{for all } p \in \boldsymbol{P} \text{ and all } x \in X .$$

If $|\lambda - \lambda_n| \leqq c/2$, then

$$\begin{aligned} p((T - \lambda_n I)x) &\geqq p((T - \lambda I)x) - |\lambda_n - \lambda| p(x) \\ &\geqq cp(x) - (c/2)p(x) \\ &= (c/2)p(x) . \end{aligned}$$

For $x \in X$, let $x_n = (T - \lambda_n I)^{-1}x$. Since

$$p(x_n) \leqq (2/c)p((T - \lambda_n I)x_n) = (2/c)p(x)$$

for large n, we have the following

$$\begin{aligned} p(x - (T-\lambda I)(T-\lambda_n I)^{-1}x) &= p(x-(T-\lambda I)x_n) \\ &\leqq p(x-(T-\lambda_n I)x_n) + p((\lambda-\lambda_n)x_n) \\ &= p(x-x) + |\lambda-\lambda_n|\, p(x_n) \\ &= |\lambda-\lambda_n|\,(2/c)p(x) \longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty\,. \end{aligned}$$

Thus $x \in \overline{R[T-\lambda I]} = R[T-\lambda I]$. Therefore λ belongs to $\rho(T)$. Now for $A \subset \boldsymbol{C}$, $\operatorname{Int} A$ denotes the set of interior points in $\boldsymbol{C}$. Since $\gamma(T) = S\backslash\rho(T)$ is open in S, it is also open in $\boldsymbol{C}$. Then $\gamma(T) = \operatorname{Int}\gamma(T) \subset \operatorname{Int}\sigma(T)$. Thus $\gamma(T) \cup \operatorname{Int}\pi(T) \subset \operatorname{Int}\sigma(T)$. Consequently

$$\begin{aligned} \partial\sigma(T) &= \overline{\sigma(T)}\backslash\operatorname{Int}\sigma(T) \subset \sigma(T)\backslash(\gamma(T)\cup\operatorname{Int}\pi(T)) \\ &= \pi(T)\backslash\operatorname{Int}\pi(T) \\ &= \partial\pi(T)\,. \end{aligned}$$

3. The spectrum of an element and the joint spectrum of elements in a locally convex algebra. In this section, let A be a quasi-complete commutative locally convex algebra over $\boldsymbol{C}$. (If A is a sequencially complete locally convex algebra, then it is quasi-complete.) (cf. [1]) Moreover let A_0 be the set of all bounded elements of A, i.e., $A_0 = \{a \in A \mid \{(\delta a)^n\}_{n=0}^{\infty}$ is a bounded subset of A for some $\delta \gneqq 0\}$ and $\boldsymbol{B}$ be a fundamental system of multiplicative closed absolutely convex and bounded subset in A.

It is easily shown that $A_0 = \bigcup_{B_\alpha \in \boldsymbol{B}} A(B_\alpha)$ where $A(B_\alpha) = \bigcup_{n=1}^{\infty} nB_\alpha$ is a Banach algebra with the norm $\|\cdot\|_{B_\alpha}$ defined by

$$\|a\|_{B_\alpha} = \inf\{\lambda > 0 \mid \lambda^{-1}a \in B_\alpha\}\,.$$

Let M_0 (resp. M_α) be the set of all non identically zero multiplicative linear functional on A_0 (resp. $A(B_\alpha)$). It has been shown in [1] that M_0 (resp. M_α) is compact with respect to $\sigma(M_0, A_0)$ (resp. $\sigma(M_\alpha, A(B_\alpha))$ topology and M_0 is isomorphic to the projective limit of M_α, i.e., $M \cong \varprojlim (M_\alpha, \pi_{\alpha\beta})$ where for $h_\beta \in M_\beta$, $\pi_{\alpha\beta}(h_\beta)$ is the restriction of h_β to $A(B_\alpha)$ ($\beta > \alpha$ i.e., $B_\beta \supset B_\alpha$).

The spectrum of $a \in A$ is defined as follows;

$$\sigma(a) = \{\lambda \in \boldsymbol{C} \mid a - \lambda e \text{ has no inverse in } A_0\}\,,$$

where e is the identity of A, and

$$\rho(a) = \{\lambda \in \boldsymbol{C} \mid \lambda \notin \sigma(a)\}\,.$$

Then we have $\sigma(a) = \{h(a) \mid h \in M_0\}$ for $a \in A_0$. [1]

Next we define the joint spectrum of element a_i $(1 \leqq i \leqq n)$ of A_0.

DEFINITION 3.1. Let a_i $(1 \leqq i \leqq n)$ be element of A. We say that $(\lambda_1, \cdots, \lambda_n)(\lambda_i \in C, 1 \leqq i \leqq n)$ belongs to the joint resolvent set, $\rho(a_1, \cdots, a_n)$ of $\{a_i\}_{1 \leqq i \leqq n}$, if there exist $b_i \in A_0$ $(1 \leqq i \leqq n)$ such that $\sum_{i=1}^n b_i(a_i - \lambda_i e) = e$. Otherwise it is said to be in the joint spectrum, $\sigma(a_1, \cdots, a_n)$.

THEOREM 3.2. *Let* $a_i \in A_0$ $(1 \leqq i \leqq n)$. *Then* $(\lambda_1, \cdots, \lambda_n)$ *belongs to* $\sigma(a_1, \cdots, a_n)$ *if and only if there exists* $h \in M_0$ *such that* $h(a_i) = \lambda_i$ *for each* i, $1 \leqq i \leqq n$.

PROOF. Let $(\lambda_1, \cdots, \lambda_n) \in \rho(a_1, \cdots, a_n)$, then there exist n elements $b_i (1 \leqq i \leqq n)$ of A_0 such that $\sum_{i=1}^n b_i(a_i - \lambda_i e) = e$. For every $h \in M_0$, $h(\sum_{i=1}^n b_i(a_i - \lambda_i e)) = \sum_{i=1}^n h(b_i)(h(a_i) - \lambda_i)$. On the other hand $h(e) = 1$, thus $h(a_i) \neq \lambda_i$ for some λ_i.

Conversely let $(\lambda_1, \cdots, \lambda_n) \in \sigma(a_1, \cdots, a_n)$. Then there exist a Banach algebra $A(B_\delta)$, $(B_\delta \in \boldsymbol{B})$, which contains $\{a_i\}_{1 \leqq i \leqq n}$ and $h_\delta \in M_\delta$ such that $h_\delta(a_i) = \lambda_i$ for each i, [9]. For every $\alpha \geqq \delta$, we put $N_\alpha = \{h \in M_\alpha \mid h(a_i) = \lambda_i \ 1 \leqq i \leqq n\}$.

Then N_α is closed, and therefore compact, subset of M_α, and by the supposition N_α is non-empty. Since $\pi_{\alpha\beta}(N_\beta) \subset N_\alpha$, $(\delta \leqq \alpha \leqq \beta)$, we can form the projective limit, Q_δ, of $\{N_\alpha \mid \alpha \geqq \delta\}$. Choose some element $\{h'_\alpha\} \in Q_\delta$.

Then we may extend it to an element $\{h_\alpha\}$ of M_0 by putting

(i) $h_\alpha = h'_\alpha$ if $\alpha \geqq \delta$;

(ii) $h_\alpha = \pi_{\alpha\beta}(h'_\beta)$ otherwise, where $\beta \geqq \alpha$, $\beta \geqq \delta$.

Then $h(a_i) = h_\delta(a_i) = h'_\delta(a_i) = \lambda_i$ for each i $(1 \leqq i \leqq n)$. q.e.d.

From Theorem 3.2, we obtain the following.

THEOREM 3.3. *Let* a_i $(1 \leqq i \leqq n)$ *be elements of* A_0. *If* P *is a complex polynomial in* n *variables, then*

$$P(\sigma(a_1, \cdots, a_n)) = \sigma(P(a_1, \cdots, a_n)) .$$

PROOF. Following statements are equivalent:

(i) $\lambda \in P(\sigma(a_1, \cdots, a_n))$.

(ii) $\lambda = P(\lambda_1, \cdots, \lambda_n)$ for some $(\lambda_1, \cdots, \lambda_n) \in \sigma(a_1, \cdots, a_n)$.

(iii) $\lambda = P(\lambda_1, \cdots, \lambda_n)$ where $(\lambda_1, \cdots, \lambda_n)$ is a vector such that for some $h \in M_0$, $h(a_i) = \lambda_i$ for each i, $(1 \leqq i \leqq n)$.

(iv) $\lambda = P(h(a_1), \cdots, h(a_n))$ for some $h \in M_0$.

(v) $\lambda = h(P(a_1, \cdots, a_n))$ for some $h \in M_0$.

(vi) $\lambda \in \sigma(P(a_1, \cdots, a_n))$.

Theorem 3.2 says the equivalence of (ii) and (iii). q.e.d.

4. On the spectra of tensor product of linear operators on Fréchet spaces. For Fréchet spaces X and Y, let W be the completion of the tensor product of X and Y endowed with the π topology, say $X \otimes_\pi Y$ (cf. [7]). If $\boldsymbol{P}$ (resp. $\boldsymbol{Q}$) is a F.S.N.S. for X (resp. Y), then $\boldsymbol{P} \otimes \boldsymbol{Q} = \{p \otimes q \mid p \in \boldsymbol{P},\ q \in \boldsymbol{Q}\}$ is a F.S.N.S. for W, where

$$p \otimes q(u) = \inf \left\{ \sum_{i=1}^{n} p(x_i) q(y_i) \,\middle|\, u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

for all $u \in X \otimes_\pi Y$. Conversely we have the following fact.

PROPOSITION 4.1. *Let $\boldsymbol{R}$ be a F.S.N.S. for W. Then for $r \in \boldsymbol{R}$, there exist continuous semi-norms $p = p(r)$ and $q = q(r)$ on X and Y respectively such that r is a cross semi-norm of p and q, moreover $\{p = p(r) \mid r \in \boldsymbol{R}\}$ and $\{q = q(r) | r \in \boldsymbol{R}\}$ are F.S.N.S. for X and Y respectively.*

PROOF. Let $r \in \boldsymbol{R}$ be given, for which we put

$$\boldsymbol{R}_r = \{p \otimes q \mid p \text{ and } q \text{ are continuous semi-norms on } X \text{ and } Y \text{ respectively such that } r \leqq p \otimes q\} ,$$

where $r \leqq p \otimes q$ means that $r(u) \leqq p \otimes q(u)$ for all $u \in W$. Clearly $\boldsymbol{R}_r$ is non-empty set by the continuity of r. For any totally ordered subset $\{p_\alpha \otimes q_\alpha\}_{\alpha \in A}$ of $\boldsymbol{R}_r$, let

$$p_0(x) = \inf_{\alpha \in A} p_\alpha(x) \quad \text{and} \quad q_0(y) = \inf_{\alpha \in A} q_\alpha(y) .$$

Then $p_0 \otimes q_0$ belongs to $\boldsymbol{R}_r$ and it is smaller than all elements of $\{p_\alpha \otimes q_\alpha\}_{\alpha \in A}$. By Zorn's Lemma, there exists at least a minimal element $p_1 \otimes q_1$ in $\boldsymbol{R}_r$.

We shall show that r is a cross semi-norm of p_1 and q_1. If this were not true, then there exists an element $x_0 \otimes y_0 \in X \otimes_\pi Y$ such that

$$p_1(x_0) q_1(y_0) = p_1 \otimes q_1(x_0 \otimes y_0) \gneqq r(x_0 \otimes y_0)$$

and we may assume that

$$p_1(x_0) q_1(y_0) \gneqq 1 \geqq r(x_0 \otimes y_0) ,$$

moreover we assume that $p_1(x_0) \gneqq 1$ and $q_1(y_0) = 1$. This implies that $x_0 \otimes y_0$ belongs to the 0-neighbourhood, $U = \{u \in W \mid r(u) \leqq 1\}$, in W, but x_0 does not belong to the 0-neighbourhood, $U(p_1) = \{x \in X \mid p_1(x) \leqq 1\}$ in X. Now we define a new semi-norm, p_2, on X such that

$$p_2(x) = \inf \{t \gneqq 0 \mid t^{-1} x \in \Gamma(x_0,\ U(p_1))\} ,$$

where $\Gamma(x_0,\ U(p_1))$ is a convex and balanced hull of $\{x_0\}$ and $U(P_1)$. Then

$p_2 \otimes q_1 \gneqq p_1 \otimes q_1$, and by the fact;

$$\begin{aligned}\Gamma(x_0, U(p_1)) &\otimes \{y \in Y \mid q_1(y) \leqq 1\} \\ &= \{x \otimes y \mid x \in \Gamma(x_0, U(p_1)), q_1(y) \leqq 1\} \subset U ,\end{aligned}$$

and r is the gauge function of U, we have $p_2 \otimes q_1 \geqq r$ (cf. 6.3 Chap. III of [7]). Then $p_2 \otimes q_1 \in \boldsymbol{R}_r$. This contradicts the minimality of $p_1 \otimes q_1$, thus r is a cross semi-norm of p_1 and q_1. Therefore for each $r \in \boldsymbol{R}$ there exist continuous semi-norms, $p = p(r)$ and $q = q(r)$, on X and Y respectively such that r is a cross semi-norm of p and q, and the assertion that families of these semi-norms are F.S.N.S.s for X and Y respectively is easily shown by the following formulas;

$$p(x) = r(x \otimes y_1)/q(y_1) , \qquad q(y) = r(x_1 \otimes y)/p(x_1) ,$$

where y_1 and x_1 are elements of X and Y respectively such that $q(y_1) \neq 0$ and $p(x_1) \neq 0$. q.e.d.

For $T_1 \in L(X)$ and $T_2 \in L(Y)$, we define the tensor product, $T_1 \otimes T_2$, of T_1 and T_2 on the algebraic tensor product, $X \otimes Y$, by

$$T_1 \otimes T_2(u) = \sum_{i=1}^{n} T_1(x_i) \otimes T_2(y_i) \qquad \text{for } u = \sum_{i=1}^{n} x_i \otimes y_i ,$$

then $T_1 \otimes T_2$ is a densely defined continuous linear operator on W and its continuous extension to W is again denoted by $T_1 \otimes T_2$. Further the following is easily shown.

PROPOSITION 4.2. *Let X and Y be Fréchet spaces. If T_1 and T_2 are bounded operators on X and Y respectively, then $T_1 \otimes T_2$ is also a bounded operator on W.*

For $T_1 \in L(X)$ and $I \in L(Y)$ (resp. $I \in L(X)$ and $T_2 \in L(Y)$), let A_1 (resp. A_2) be the operator on W defined by $T_1 \otimes I$ (resp. $I \otimes T_2$).

Now if either X or Y is a nuclear space, we have Grothendieck's Theorem, that is, the dual space of W with the β topology, W_β^*, is the completion of the tensor product of X_β^* and Y_β^* with respect to the π topology, (cf. [3]). Then we have ${}^t(T_1 \otimes T_2) = {}^tT_1 \otimes {}^tT_2$ by elementary calculas.

At last, in order to prove the main theorem, we note the lemma about the property of complex valued polynomial in two variables.

LEMMA [8]. *Let $P(\cdot, \cdot)$ be a polynomial in two variables such that $P(0, 0) = 0$. Then there exist two complex-valued functions $g(t)$ and $h(t)$ continuous in $0 \leqq t < \infty$ such that $g(0) = h(0) = 0$, $|g(t)| + |h(t)| \to \infty$ as $t \to \infty$, and $P(g(t), h(t)) = 0$ in $0 \leqq t < \infty$.*

THEOREM 4.3. *Let X and Y be Fréchet spaces and let either X or Y be a nuclear space. If T_1 and T_2 are bounded operators on X and Y and $A_1 = T_1 \otimes I$, $A_2 = I \otimes T_2$ respectively and P is a polynomial in two variables, then the spectrum of $P(A_1, A_2)$ consists of those λ such that $\lambda = P(\lambda_1, \lambda_2)$, where $\lambda_i \in \sigma(T_i)$ $i = 1, 2$. i.e., $\sigma(P(A_1, A_2)) = P(\sigma(T_1), \sigma(T_2))$.*

PROOF. The proof of $\sigma(P(A_1, A_2)) \subset P(\sigma(T_1), \sigma(T_2))$. It is easily shown that if X and Y are Fréchet space, then W is also Fréchet space, thus $L_s(W)$ is sequencially complete by Proposition in § 1, moreover it is quasi-complete. Let $\mathfrak{A}$ be the double commutant of $\{A_1, A_2\}$ in $L_s(W)$. Since A_1 and A_2 are commute each other, $\mathfrak{A}$ is a commutative quasi-complete locally convex algebra, and we put $\sigma_{\mathfrak{A}}(T)$, the algebraic spectrum of T, be considered as an element of $\mathfrak{A}$, then $\sigma_{\mathfrak{A}}(T) = \sigma(T)$.

Since $\sigma_{\mathfrak{A}}(P(A_1, A_2)) = P(\sigma_{\mathfrak{A}}(A_1, A_2))$ by virtue of Theorem 3.3, and by Theorem 3.2, $\lambda \in \sigma_{\mathfrak{A}}(A_1, A_2)$ implies that $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_1 \in \sigma_{\mathfrak{A}}(A_1)$, $\lambda_2 \in \sigma_{\mathfrak{A}}(A_2)$. Then

$$\begin{aligned}\sigma(P(A_1, A_2)) &= P(\sigma_{\mathfrak{A}}(A_1, A_2)) \subset P(\sigma_{\mathfrak{A}}(A_1), \sigma_{\mathfrak{A}}(A_2)) \\ &= P(\sigma(A_1), \sigma(A_2)) = P(\sigma(T_1), \sigma(T_2)) .\end{aligned}$$

The proof of $P(\sigma(T_1), \sigma(T_2)) \subset \sigma(P(A_1, A_2))$. We shall divide the proof into three cases. Let $\lambda = (\lambda_1, \lambda_2) \in \sigma(T_1) \times \sigma(T_2)$.

Case 1. $\lambda = (\lambda_1, \lambda_2) \in \pi(T_1) \times \pi(T_2)$.

Since $P(A_1, A_2) = \sum_{i,j=0}^{N} a_{ij} T_1^i \otimes T_2^j$ $(a_{ij} \in \boldsymbol{C})$, the proof is reduced to show the following.

(i) If $\mu \in \pi(T_1)$, then $\mu^n \in \pi(T_1^n)$ for every n.

(ii) If $(\mu_1, \mu_2) \in \pi(T_1) \times \pi(T_2)$, then $\mu_1 \cdot \mu_2 \in \pi(T_1 \otimes T_2)$.

Let $\mu \in \pi(T_1)$ and $\boldsymbol{P}$ be a F.S.N.S. for X. For the polynomial, $\boldsymbol{Q}(\cdot)$, such that $T_1^n - \mu^n I = Q(T_1)(T_1 - \mu I)$, we define the F.S.N.S., $\boldsymbol{P}_Q = \{p_Q \mid p \in \boldsymbol{P}\}$, for X such that

$$p_Q(x) = \sup \{p((\delta Q(T_1))^m x) \mid m = 1, 2, \cdots\}$$

where δ is some positive constant. For any $\varepsilon \gneqq 0$, there exist $p_Q \in \boldsymbol{P}_Q$ and $x \in X$ with $p_Q(x) = 1$ such that $p_Q((T_1 - \mu I)x) \leqq \varepsilon$. Then for some m, $p((\delta Q(T_1))^m x) \geqq 1/2$, and we have

$$\begin{aligned}&p((T_1^n - \mu^n I)(\delta Q(T_1))^m x) \\ &\quad = p(Q(T_1)(T_1 - \mu I)(\delta Q(T_1))^m x) \\ &\quad = (1/\delta) p((\delta Q(T_1))^{m+1}(T_1 - \mu I)x) \\ &\quad \leqq (1/\delta) p_Q((T_1 - \mu I)x) \\ &\quad \leqq \varepsilon/\delta .\end{aligned}$$

Therefore $\mu^n \in \pi(T_1^n)$. This completes the proof of (i).

To show (ii), let $\boldsymbol{R}$ be a F.S.N.S. for W. We may assume that $\boldsymbol{R}$ is the countable set, then we put $\boldsymbol{R} = \{r_m\}_{m=1}^{\infty}$. From Proposition 4.1, there exist F.S.N.S.s $P = \{p_m\}_{m=1}^{\infty}$ and $Q = \{q_m\}_{m=1}^{\infty}$ for X and Y respectively such that r_m is a cross semi-norm of p_m and q_m. Since $\mu_1 \in \pi(T_1)$, we can take a subsequence $\{p_{\alpha_i}\}_{i=1}^{\infty}$ from $\{p_m\}_{m=1}^{\infty}$ and $\{x_{\alpha_i}\}_{i=1}^{\infty}$ in X satisfying following two conditions;

(a) $p_{\alpha_i}(x_{\alpha_i}) = 1$ and $p_{\alpha_i}((T_1 - \mu_1)x_{\alpha_i}) \to 0$ as $i \to \infty$.

(b) $\{q_{\alpha_i}\}_{i=1}^{\infty} \subset \{q_m\}_{m=1}^{\infty}$ is also a F.S.N.S. for Y.

Then we can take again a subsequence $\{q_{\beta_i}\}_{i=1}^{\infty}$ from $\{q_{\alpha_i}\}_{i=1}^{\infty}$ and $\{y_{\beta_i}\}_{i=1}^{\infty}$ in Y such that $p_{\beta_i}(y_{\beta_i}) = 1$ and $q_{\beta_i}((T_2 - \mu_2)y_{\beta_i}) \to 0$ as $i \to \infty$. Therefore

$$\begin{aligned} & r_{\beta_i}((T_1 \otimes T_2 - \mu_1 \cdot \mu_2 I \otimes I)(x_{\beta_i} \otimes y_{\beta_i})) \\ & \quad = p_{\beta_i} \otimes q_{\beta_i}(((T_1 - \mu_1 I)x_{\beta_i} \otimes (T_2 - \mu_2 I)y_{\beta_i}) \\ & \qquad + (T_1 - \mu_1 I)x_{\beta_i} \otimes \mu_2 y_{\beta_i} + \mu_1 x_{\beta_i} \otimes (T_2 - \mu_2)y_{\beta_i}) \\ & \quad \leqq p_{\beta_i}((T_1 - \mu_1 I)x_{\beta_i})q_{\beta_i}((T_2 - \mu_2 I)y_{\beta_i}) \\ & \qquad + |\mu_2|\, p_{\beta_i}((T_1 - \mu_1 I)x_{\beta_i})q_{\beta_i}(y_{\beta_i}) \\ & \qquad + |\mu_1|\, p_{\beta_i}(x_{\beta_i})q_{\beta_i}((T_2 - \mu_2)y_{\beta_i}) \to 0 \qquad \text{as } i \to \infty . \end{aligned}$$

The other hand $r_{\beta_i}(x_{\beta_i} \otimes y_{\beta_i}) = p_{\beta_i}(x_{\beta_i})q(y_{\beta_i}) = 1$. Therefore $\mu_1 \cdot \mu_2 \in \pi(T_1 \otimes T_2)$.

Case 2. $\lambda = (\lambda_1, \lambda_2) \in \pi(T_1) \times \gamma(T_2)$.

Since $\sigma(T_1) = \sigma({}^tT_1) = \pi({}^tT_1) \cup \gamma({}^tT_1)$, we shall consider two cases;

$$(2.1) \quad \lambda_1 \in \pi({}^tT_1) , \qquad (2.2) \quad \lambda_1 \in \gamma({}^tT_1) .$$

Firstly suppose $\lambda = (\lambda_1, \lambda_2) \in \pi({}^tT_1) \times \gamma(T_2)$. Since λ_2 is an eigenvalue of ${}^tT_2$, $\lambda = (\lambda_1, \lambda_2) \in \pi({}^tT_1) \times \pi({}^tT_2)$. In the proof of (b) of the preceding case, we used the countability of $\boldsymbol{R}$, then this case is not reduced to Case 1, but λ_2^n is also an eigenvalue of $({}^tT_2)^n$ and the following holds.

(iii) If $T_1^*, T_2^* \in L(X_\beta^*)$, $\mu_1 \in \pi(T_1^*)$ and μ_2 is an eigenvalue of T_2^*, then $\mu_1 \cdot \mu_2 \in \pi(T_1^* \otimes T_2^*)$.

To prove (iii), let $\boldsymbol{R}^* = \{r_\alpha^*\}_{\alpha \in A}$ be a F.S.N.S. for W^* and we assume that $r_\alpha^*(x_0^* \otimes y_0^*) \neq 0$ for every $\alpha \in A$, where x_0^* is some vector in X_β^* and y_0^* is an eigenvalue of μ_2. As Case 1, there exist F.S.N.S.s, $P^* = \{p_\alpha^*\}$ and $Q^* = \{q_\alpha^*\}$, for X and Y respectively such that r_α^* is a cross semi-norm of p_α and q_α, and $q_\alpha^*(y_0^*) \neq 0$. Since $\mu_1 \in \pi(T_1^*)$, there exist $\{p_{\alpha_i}^*\}_{i=1}^{\infty} \subset \boldsymbol{P}^*$ and $\{x_{\alpha_i}^*\}_{i=1}^{\infty} \subset X_\beta^*$ with $p_{\alpha_i}^*(x_{\alpha_i}^*) = 1$ such that $p_{\alpha_i}^*((T_1^* - \mu_1)x_{\alpha_i}^*) \to 0$ as $i \to \infty$. If we put $y_{\alpha_i}^* = y_0^*/q_{\alpha_i}^*(y_0^*)$, $r_{\alpha_i}^*((T_1^* \otimes T_2^* - \mu_1\mu_2 I \otimes I)x_{\alpha_i}^* \otimes y_{\alpha_i}^*) \to 0$ as $i \to \infty$, and $r_{\alpha_i}^*(x_{\alpha_i}^* \otimes y_{\alpha_i}^*) = 1$. Then $\mu_1\mu_2 \in \pi(T_1^* \otimes T_2^*)$. Therefore

$$P(\lambda_1, \lambda_2) \in \pi({}^tP(A_1, A_2)) \subset \sigma({}^t(P(A_1, A_2)) = \sigma(P(A_1, A_2)) .$$

To show that Case (2.2) is reduced to the case above, we put

$$P(\xi, \eta) = P(\lambda_1 + \xi, \lambda_2 + \eta) - P(\lambda_1, \lambda_2) ,$$

then $P(0, 0) = 0$. Hence by the Lemma [8] there exist continuous functions $g(t)$, $h(t)$ on $[0, \infty)$ such that $g(0) = 0$, $h(0) = 0$,

$$P(\lambda_1 + g(t), \lambda_2 + h(t)) = P(\lambda_1, \lambda_2) ,$$

and $|g(t)| + |h(t)| \longrightarrow \infty$ as $t \longrightarrow \infty$. Since λ_1 (resp. λ_2) is belonging to the bounded open subset, $\gamma({}^tT_1)$ (resp. $\gamma(T_2)$) of C, there exists $t_0 \geqq 0$ such that either

(2.2.1) $$\lambda_1 + g(t_0) \in \partial\gamma({}^tT_1) \subset \pi({}^tT_1) \quad \text{and} \quad \lambda_2 + h(t) \in \gamma(T_2) \quad \text{in } 0 \leqq t \lneqq t_0 ,$$

or

(2.2.2) $$\lambda_2 + h(t_0) \in \partial\gamma(T_2) \subset \pi(T_2) \quad \text{and} \quad \lambda_1 + g(t) \in \gamma({}^tT_1) \subset \pi(T_1) \quad \text{in } 0 \leqq t \lneqq t_0 .$$

Case (2.2.1) and (2.2.2) are reduced to Case (2.1) and Case 1 respectively. Thus

$$P(\lambda_1, \lambda_2) = P(\lambda_1 + g(t_0), \lambda_2 + h(t_0)) \in \sigma(P(A_1, A_2)) .$$

Case 3. $\lambda = (\lambda_1, \lambda_2) \in \gamma(T_1) \times \gamma(T_2)$.

Since $\gamma(T_1) \subset \pi({}^tT_1)$, this case is a particular case of (2.1). q.e.d.

Now we shall apply Theorem 4.3 to the following example.

EXAMPLE. The space $C_p^\infty(R^k)$ of complex valued periodic infinitely partial differentiable functions in R^k with I^k, hypercube $[0, 1]^k$, as period, with the topology of uniform convergence in all derivatives, is a nuclear Fréchet space and we have known $C_p^\infty(R^2) = C_p^\infty(R^1) \otimes_\pi C_p^\infty(R^1)$. On the space $C_p^\infty(R^1)$, we consider translations T_α and T_β $(0 \lneqq \alpha, \beta \lneqq 1)$ defined by

$$(T_\alpha f)(x) = f(x + \alpha) , \qquad (T_\beta f)(x) = f(x + \beta) ,$$

for $f(x) \in C_p^\infty(R^1)$. Then

$$((T_\alpha \otimes T_\beta)h)(x, y) = h(x + \alpha, y + \beta) ,$$
$$((T_\alpha \otimes I + I \otimes T_\beta)h)(x, y) = h(x + \alpha, y) + h(x, y + \beta) ,$$

for $h(x, y) \in C_p^\infty(R^2)$.

If $\alpha \in Q$, the set of rational number, there exists an integer n such that $n\alpha \equiv 0$ (mod. 1) and we put $n_\alpha = \min\{n \gneqq 0 \mid n\alpha \equiv 0 \text{ (mod. 1)}\}$. Otherwise $n\alpha \neq 0$ (mod. 1) for any integer n. Then we have the following;

$$\sigma(T_\alpha) = \begin{cases} \{\lambda \mid \lambda^{n_\alpha} = 1\} & \text{if } \alpha \in Q , \\ \{\lambda \mid |\lambda| = 1\} & \text{if } \alpha \notin Q . \end{cases}$$

Since T_α and T_β are bounded operators on $C_p^\infty(R^1)$, we have the following;

$$\sigma(T_\alpha \otimes T_\beta) = \begin{cases} \{\lambda\mu \mid \lambda^{n_\alpha} = 1,\ \mu^{n_\beta} = 1\} & \text{if } \alpha \in Q \text{ and } \beta \in Q\ , \\ \{\lambda \mid |\lambda| = 1\} & \text{otherwise}\ . \end{cases}$$

$$\sigma(T_\alpha \otimes I + I \otimes T_\beta) = \begin{cases} \{\lambda \mid |\lambda| \leqq 2\} & \text{if } \alpha \notin Q \text{ and } \beta \notin Q\ , \\ \{\lambda + \mu \mid \lambda^{n_\alpha} = 1,\ |\mu| = 1\} & \text{if } \alpha \in Q \text{ and } \beta \notin Q\ , \\ \{\lambda + \mu \mid \lambda^{n_\alpha} = 1,\ \mu^{n_\beta} = 1\} & \text{if } \alpha \in Q \text{ and } \beta \in Q\ . \end{cases}$$

Furthermore for a polynomial, $P(\cdot, \cdot)$, in two complex variables

$$\sigma(P(T_\alpha \otimes I, I \otimes T_\beta)) = \{P(\lambda, \mu) \mid \lambda \in \sigma(T_\alpha),\ \mu \in \sigma(T_\beta)\}\ .$$

Moreover we can know the spectrum of ${}^tP(T_\alpha \otimes I, I \otimes T_\beta)$ acting on the dual space of $(C_p^\infty(R^2))$.

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