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ON PARABOLIC SUBGROUPS OF CHEVALLEY GROUPS OVER LOCAL RINGS

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Introduction. Let G be a Chevalley-Demazure group scheme associated with a connected complex semi-simple Lie group G_C (as for definition, see [1] 1.1), $\varDelta$ be the root system associated with G and a maximal torus T of G, and R be a commutative ring with a unit. We shall fix a fundamental root system $\varPi$ of $\varDelta$ once for all. Denote by $x_\alpha(t)$ the unipotent element of $G(R)$ associated with a root α of $\varDelta$ and $t \in R$. Let $V(R)$ be the subgroup of $G(R)$ generated by $x_\alpha(t)$ for all negative roots α of $\varDelta$ and all $t \in R$. Then a subgroup P of $G(R)$ containing $V(R)T(R)$ is called a parabolic subgroup of $G(R)$ associated with $\varPi$. Following J. Tits, it is well known that if R is a field, then the set of parabolic subgroups of $G(R)$ associated with $\varPi$ is lattice isomorphic to the family of subsets of $\varPi$.

N. S. Romanovskii [4] has given a discription of parabolic subgroups of $GL_n(R)$ for a local ring R. In this note, for a simple Chevalley-Demazure group scheme G and a local ring R, we shall give a generalization of the Tits' theorem in the same situation as Romanovskii's result. The main theorem is stated in Section 1, and we shall prove our main theorem in Sections 2 and 3. The author wishes to express his hearty thanks to professor E. Abe for his many helpful comments and encouragement.

1. The statement of the main theorem.

1.1. Let G be a Chevalley-Demazure group scheme and R be a commutative ring with a unit. A collection of ideals $\{\mathfrak{A}_\alpha\}_{\alpha \in \varDelta}$ which corresponds bijectively to the set $\varDelta$ of roots, is called a carpet of R associated with $\varDelta$. Furthermore, a carpet $\{\mathfrak{A}_\alpha\}_{\alpha \in \varDelta}$ is called a permissible (resp. semi-permissible) carpet associated with $(\varDelta, \varPi)$, if the following conditions (1) and (2) (resp. (1) and (2′)) are satisfied,

(1) for any roots α and β of $\varDelta$ such that $\alpha + \beta \in \varDelta$

$$\mathfrak{A}_\alpha \mathfrak{A}_\beta \subset \mathfrak{A}_{\alpha+\beta}$$

(2) for each negative root α of $\varDelta$, $\mathfrak{A}_\alpha = R$,

(2′) if $\mathfrak{A}_\alpha$ is a proper ideal of R, then $\mathfrak{A}_{-\alpha} = R$.

1.2. Assume G is simple. Let R be a local ring, $\mathfrak{M}$ be the maximal ideal of R, k be the residue class field $R/\mathfrak{M}$ and $ch(k)$ be the characteristic of k. We shall set up the following assumptions,

(a) $ch(k) \neq 2$ for any type of G,

(b) if G are of types A_3, $B_m (m \geqq 2)$, C_n, $D_n (n \geqq 3)$ and F_4, then $k \neq F_3$ where F_3 is a field with three elements and if G is of type G_2, then $ch(k) \neq 3$. Then our main theorem is the following.

1.3. THEOREM. *Let G be a simple Chevalley-Demazure group scheme and R be a local ring. Assume G and R satisfy* (*a*) *and* (*b*) *in* 1.2. *Let P be a parabolic subgroup of $G(R)$ associated with Π and denote $\mathfrak{A}_\alpha = \{t \in R \mid x_\alpha(t) \in P\}$ for each root α of Δ. Then $\{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ is a permissible carpet, and further, the mapping $\Psi: P \rightarrow \{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ is a bijection of the set of parabolic subgroups of $G(R)$ associated with Π onto the set of permissible carpets associated with (Δ, Π).*

REMARK. If G is not simple, examining the proof of lemma in 2.5, we can see that, if we assume $ch(k) \neq 2$ and $ch(k) \neq 3$ instead of (a) and (b) in 1.2, our main theorem also holds.

Throughout the following section, let G be a Chevalley-Demazure group scheme, and let R be a local ring and $\mathfrak{M}$ be the maximal ideal of R. Denote by Δ the root system associated with G and a maximal torus T of G, by Π a system of fundamental roots of Δ, by Δ^+ (resp. Δ^-) the set of positive (resp. negative) roots of Δ. Let S be a closed subset of Δ^+ and $\mathfrak{K} = \{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ be a carpet of R associated with Δ. Then we denote by $U_S(\mathfrak{K})$ the subgroup of $G(R)$ generated by $x_\alpha(t)$ for all $t \in \mathfrak{A}_\alpha$, $\alpha \in S$. In particular, if $\mathfrak{A}_\alpha = \mathfrak{A}$ for all $\alpha \in S$, we denote $U_S(\mathfrak{K})$ by $U_S(\mathfrak{A})$, and if $S = \Delta^+$, denote $U_S(\mathfrak{A})$ by $U(\mathfrak{A})$. In the above notation, replacing Δ^+ by Δ^-, we can construct $V_S(\mathfrak{K})$, $V_S(\mathfrak{A})$ and $V(\mathfrak{A})$ which are same as $U_S(\mathfrak{K})$, $U_S(\mathfrak{A})$ and $U(\mathfrak{A})$ respectively.

2. Proof of injectivity.

2.1. LEMMA. *Let R be a local ring in which 2 is invertible. Let N be a subgroup of $G(R)$ normalized by the maximal torus $T(R)$. Then, for each root α of Δ, $\mathfrak{A}_\alpha = \{t \in R \mid x_\alpha(t) \in N\}$ is an ideal of R.*

PROOF. Assume $x_\alpha(t) \in N$, then it is sufficient to prove $x_\alpha(bt) \in N$ for any $b \in R$. Every element b of R can be written in the form

$$b = \left(\frac{b+1}{2}\right)^2 - \left(\frac{b-1}{2}\right)^2.$$

Thus it is sufficient to show that $x_\alpha(a^2t) \in N$ for any $a \in R$. If a is invertible, setting $w_\alpha(a) = x_\alpha(a)x_{-\alpha}(-a^{-1})x_\alpha(a)$ and $h_\alpha(a) = w_\alpha(a)w_\alpha(-1)$, we have $h_\alpha(a)x_\alpha(t)h_\alpha(a)^{-1} = x\alpha(a^2t) \in N$. If a is not invertible, then a^2+1, a^2-1 are invertible, and we have

$$x_\alpha(a^2t) = x_\alpha\Big(\Big(\frac{a^2+1}{2}\Big)^2 t\Big)x_\alpha\Big(-\Big(\frac{a^2-1}{2}\Big)^2 t\Big).$$

Thus our assertion can be reduced to the former (cf. [4] Lemma 1).

q.e.d.

For roots α and β of Δ, write $\langle\alpha, \beta\rangle = 2(\alpha, \beta)/(\beta, \beta)$, and these are called Cartan integers where (α, β) is the scalar product of α and β. Then we have the following lemma.

2.2. LEMMA. *Let Δ be of rank > 1. If α and β are any positive roots of Δ and $\alpha \neq \beta$, then there exists a root $\gamma \in \Delta$ such that*

(a) $\langle\alpha, \gamma\rangle = \pm 2$, $\langle\beta, \gamma\rangle = 0$

or

(b) $\langle\alpha, \gamma\rangle \equiv 1$, $\langle\beta, \gamma\rangle \equiv 0 \pmod 2$.

PROOF. If $(\alpha, \beta) = 0$, then taking $\gamma = \alpha$, we have $\langle\alpha, \gamma\rangle = 2$ and $\langle\beta, \gamma\rangle = 0$. Suppose $(\alpha, \beta) \neq 0$, then we have $\alpha + \beta \in \Delta$. Let Δ_2 be a subsystem of roots in Δ of rank 2 consisting of the roots $i\alpha + j\beta$, $i, j \in Z$, then our assurtion follows easily from the following tables of Cartan integers with respect to the roots of Δ_2.

Δ_2: of type A_2 $\Delta_2^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$

	α_1	α_2	$\alpha_1+\alpha_2$
α_1	2	-1	1
α_2	-1	2	1
$\alpha_1+\alpha_2$	1	1	2

Δ_2: of type B_2 $\Delta_2^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2\}$

	α_1	α_2	$\alpha_1+\alpha_2$	$\alpha_1+2\alpha_2$
α_1	2	-2	2	0
α_2	-1	2	0	1
$\alpha_1+\alpha_2$	1	0	2	1
$\alpha_1+2\alpha_2$	0	2	2	2

Δ_2: of type G_2, $\Delta_2^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$

	α_1	α_2	$\alpha_1+\alpha_2$	$2\alpha_1+\alpha_2$	$3\alpha_1+\alpha_2$	$3\alpha_1+2\alpha_2$
α_1	2	-1	-1	1	1	0
α_2	-3	2	3	0	-1	1
$\alpha_1+\alpha_2$	1	1	2	1	0	1
$2\alpha_1+\alpha_2$	1	0	1	2	1	1
$3\alpha_1+\alpha_2$	3	-1	0	3	2	1
$3\alpha_1+2\alpha_2$	0	1	3	0	1	2

q.e.d.

2.3. Corollary. *Let Δ be a simple root system of type A_l $l \geqq 2$ $l \neq 3$, E_6, E_7 or E_8. Then for any positive root α and β of Δ, there exists a root $\gamma \in \Delta$ such that $\langle\alpha, \gamma\rangle \equiv 1$, $\langle\beta, \gamma\rangle \equiv 0 \pmod 2$.*

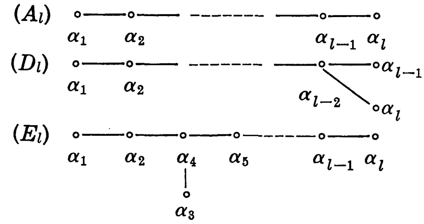
Proof. Assume $(\alpha, \beta) \neq 0$, then the subsystem Δ_2 of Δ generated by α and β is of type A_2. Thus our assertion can be checked by the table of Cartan integers of type A_2. Suppose $(\alpha, \beta) = 0$. Since $\langle\alpha, \beta\rangle = \langle w\alpha, w\beta\rangle$ for any element w of the Weyl group W, we may assume $\alpha = \alpha_l$ where $\Pi = \{\alpha_1, \cdots, \alpha_l\}$, $l \geqq 4$.*) Thus $Z_0(\alpha_l) = \{\gamma \in \Delta \mid (\alpha_l, \gamma) = 0\}$ is a simple subsystem of type A_{l-2}, A_5, D_6 or E_7, if Δ is of type A_l, E_6, E_7 or E_8 respectively (cf. M. R. Stein [5]). Therefore there exists an element w of W such that $w(\alpha) = \alpha_l$, $w(\beta) = \alpha_{l-2}$. Thus, there exists a subsystem Δ' of type A_4 in which we may assume $\alpha = \alpha_4$, $\beta = \alpha_2$, where $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a fundamental system of Δ'. Taking $\gamma = \alpha_1 + \alpha_2 + \alpha_3$, we have

$$\langle\alpha, \gamma\rangle \equiv 1\,, \qquad \langle\beta, \gamma\rangle \equiv 0 \pmod 2\,.$$

q.e.d.

2.4. Lemma. *Let $\mathfrak{R} = \{\mathfrak{A}_\alpha\}_{\alpha\in\Delta}$ be a carpet of R associated with Δ such that $\mathfrak{A}_\alpha\mathfrak{A}_\beta \subset \mathfrak{A}_{\alpha+\beta}$ for α, β and $\alpha+\beta \in \Delta$, and S be a closed subset of Δ^+. Let $\beta_1, \cdots, \beta_M$ be any given ordering of S. Then each element*

*) In the proof of above corollary, we shall set the fundamental root system as follows



$l = 6, 7, 8.$

of $U_S(\mathfrak{R})$ *is expressed in the form*

$$x_{\beta_1}(s_1)\cdots x_{\beta_M}(s_M)$$

where $s_i \in \mathfrak{A}_{\beta_i}$ $i = 1, \cdots, M$.

PROOF. Let U' be the set of elements expressible in the form as stated in the lemma. To prove our assertion, it is sufficient to show that $x_\alpha(t)U' \subset U'$ for any $x_\alpha(t)$, $t \in \mathfrak{A}_\alpha$ and $\alpha \in S$. By the same way as in [1] 2.7, we can show this easily. q.e.d.

2.5. LEMMA. *Assume that* G *is simple, and* G *and* R *satisfy* (*a*) *and* (*b*) *in* 1.2. *Let* N *be a subgroup of* $U(R)$ *normalized by* $T(R)$. *If we express an element* x *of* N *in the form*

$$x = x_{\beta_1}(s_1)\cdots x_{\beta_M}(s_M)$$

where $\beta_1 < \cdots < \beta_M$ *be any regular ordering of* Δ^+, *then* $x_{\beta_i}(s_i) \in N$ *for* $i = 1, 2, \cdots, M$.

PROOF. For a unit element u of R, we have

$$[h_\gamma(u), x_\beta(t)] = x_\beta((u^{\langle\beta,\gamma\rangle} - 1)t)$$

where $[a, b] = aba^{-1}b^{-1}$ for $a, b \in G(R)$. If there exists $\gamma \in \Delta$ such that $\langle\beta_1, \gamma\rangle \equiv 1$, $\langle\beta_2, \gamma) \equiv 0 \pmod 2$, then by 2.4, we obtain the following,

$$\begin{aligned}[h_\gamma(-1), x] &= [h_\gamma(-1), x_{\beta_1}(s_1)]^{x_{\beta_1}(s_1)}[h_\gamma(-1), x_{\beta_2}(s_2)] \\ &\quad \cdots {}^{x_{\beta_1}(s_1)\cdots x_{\beta_{M-1}}(s_{M-1})}[h_\gamma(-1), x_{\beta_M}(s_M)] \\ &= x_{\beta_1}(-2s_1)x_{\beta_3}(s'_3)\cdots x_{\beta_M}(s'_M) \in N\end{aligned}$$

where ${}^a y = aya^{-1}$. If u and $u^2 - 1$ are units of R and there exists a root $\gamma \in \Delta$ such that $\langle\beta_1, \gamma\rangle = \pm 2$, $\langle\beta_2, \gamma\rangle = 0$, then we have the following,

$$\begin{aligned}[h_\gamma(u), x] &= [h_\gamma(u), x_{\beta_1}(s_1)]^{x_{\beta_1}(s_1)}[h_\gamma(u), x_{\beta_2}(s_2)] \\ &\quad \cdots {}^{x_{\beta_1}(s_1)\cdots x_{\beta_{M-1}}(s_{M-1})}[h_\gamma(u), x_{\beta_M}(s_M)] \\ &= x_{\beta_1}((u^{\pm 2} - 1)s_1)x_{\beta_3}(s'_3)\cdots x_{\beta_M}(s'_M) \in N\,.\end{aligned}$$

By Lemma 2.2, its Corollary 2.3 and the assumptions (a) and (b) in 1.2, we can see easily that, repeating the above process, we obtain $x_{\beta_1}(vs_1) \in N$ for some unit element v of R. Thus by 2.1, we have $x_{\beta_1}(s_1) \in N$. By induction on the indices i of roots β_i, we have $x_{\beta_i}(s_i) \in N$ for $i = 1, 2, \cdots, M$. q.e.d.

2.6. PROPOSITION. *Let* P *be a parabolic subgroup of* $G(R)$. *Then* P *is generated by the elements of* $B(R)$ *and* $P \cap U(R)$ *where* $B(R) = V(R)T(R)$.

PROOF. Let $\tilde{\phi}$ be a group homomorphism $G(R) \rightarrow G(k)$ induced by the natural ring homomorphism $\phi: R \rightarrow k = R/\mathfrak{M}$, then $\tilde{\phi}(P) = P'$ is a parabolic subgroup of $G(R)$ and by Tits' theorem, we have that P' is generated by $B(k)$ and $x_\alpha(1)$ for all root $\alpha \in I$ where I is a subset of Π. On the other hand, since $\operatorname{Ker} \phi = V(\mathfrak{M})T(\mathfrak{M})U(\mathfrak{M})$ (cf. [1] 3.3), we can choose generators of P among the elements of $B(R)$ and $U(R)$. q.e.d.

2.7. Proof of injectivity. Let P be a parabolic subgroup and set up $\mathfrak{A}_\alpha = \{t \in R \mid x_\alpha(t) \in P\}$ for each root α of Δ. By 2.1 and the definition of P, it is clear that $\mathfrak{A}_\alpha$ is an ideal of R for each $\alpha \in \Delta$, and $\mathfrak{A}_\alpha = R$ for each negative roots α of Δ. On the other hand, let α and β be roots of Δ such that $\alpha + \beta \in \Delta$, and Δ_2 be the subsystem of Δ of rank $= 2$ generated by α and β. If $\alpha > 0$ and $\beta < 0$, then there is an element w of the Weyl group W_2 of Δ_2 such that $w(\alpha) > 0$ and $w(\beta) > 0$. Thus, by the commutator relations for $x_\alpha(t)$, $t \in \mathfrak{A}_\alpha$, $\alpha \in \Delta^+$ (cf. [1] 2.2) and by 2.5, we see easily $\mathfrak{A}_\alpha\mathfrak{A}_\beta \subset \mathfrak{A}_{\alpha+\beta}$ for any roots α and β of Δ. That is, $\{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ is a permissible carpet. From 2.4, 2.5 and 2.6, it is clear that P is generated by $x_\alpha(t)$, $t \in \mathfrak{A}_\alpha$, $\alpha \in \Delta$ and $T(R)$. Namely, the mapping $\Psi: \{P\} \rightarrow \{\{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}\}$ is injective. q.e.d.

3. Proof of surjectivity. Now in order to prove that the mapping Ψ in 2.7 is surjective, we shall first prove the following lemmas.

3.1. Let $\mathfrak{R} = \{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ be a permissible carpet of R with respect to (Δ, Π). Setting $\Delta'_{\mathfrak{R}} = \{\alpha \in \Delta \mid \mathfrak{A}_\alpha = R\}$ and $\Delta''_{\mathfrak{R}} = \{\alpha \in \Delta \mid \mathfrak{A}_\alpha \neq R\}$, we have that i) $\Delta'_{\mathfrak{R}} \supset \Delta^-$ and $\Delta''_{\mathfrak{R}} \subset \Delta^+$, ii) $\Delta'_{\mathfrak{R}}$ and $\Delta''_{\mathfrak{R}}$ are closed, iii) $\Delta''_{\mathfrak{R}}$ is an ideal of Δ^+, that is, if $\alpha \in \Delta''_{\mathfrak{R}}$, $\beta \in \Delta^+$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta''_{\mathfrak{R}}$. Thus we can see easily the following lemma.

3.2. LEMMA. *Using the same notation as in* 3.1, *we have the following.*

i) *Let α be a positive root, then $\alpha \in \Delta''_{\mathfrak{R}}$ if and only if there exists a root $\alpha_i \in \Delta''_{\mathfrak{R}} \cap \Pi$ such that $n_i \neq 0$ for $\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l$ where $\Pi = \{\alpha_1, \quad , \alpha_l\}$,*

ii) *we set $Z^+\Delta'_{\mathfrak{R}} = \{n_1\beta_1 + \cdots + n_s\beta_s \mid n_i \in Z^+, \beta_i \in \Delta'_{\mathfrak{R}}\}$ and $Z^+\Delta''_{\mathfrak{R}} = \{m_1\gamma_1 + \cdots + m_t\gamma_t \mid m_j \in Z^+, \gamma_j \in \Delta''_{\mathfrak{R}}\}$ where Z^+ is the set of positive rational integers. Then we have $Z^+\Delta'_{\mathfrak{R}} \cap Z^+\Delta''_{\mathfrak{R}} = \varnothing$.*

3.3. LEMMA. *Let $\mathfrak{R} = \{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ be a permissible carpet of R associated with (Δ, Π) and $\Delta'_{\mathfrak{R}}$, $\Delta''_{\mathfrak{R}}$ be the same as in* 3.1. *Let Q be a subgroup of $G(R)$ generated by $x_\alpha(t)$ for all $t \in R$, $\alpha \in \Delta'_{\mathfrak{R}}$ and elements of $T(R)$. Then we have $Q \cap U_{\Delta''_{\mathfrak{R}}}(R) = \{1\}$.*

PROOF. Let $\mathfrak{G}_C$ be a simple Lie algebra over the complex field C,

$(d\rho, V)$ be a faithful representation of $\mathfrak{G}_C$ with $\mathfrak{G}_C$-module V over C, $\{H_{\alpha_1}, \cdots, H_{\alpha_l}, X_\alpha, \alpha \in \Delta\}$ be a Chevalley basis of $\mathfrak{G}_C$ and V_Z be an admissible lattice of V, then for any root α and $t \in R$, we can construct a unipotent automorphism $x_\alpha(t) = \exp t d\rho(X_\alpha)$ of $V_z \otimes R$. Let $\{\lambda\}$ be the set of weights of $d\rho$ and denote by V^λ a weight space associated with a weight λ, then we have $V_z = \sum_\lambda \oplus V_z^\lambda$ where $V_z^\lambda = V_z \cap V^\lambda$. For any element $v \in V_z^\lambda$, $d\rho(X_\alpha)v \in V_z^{\lambda+\alpha}$ (cf. Steinberg [6]). Therefore, for each $x_\alpha(t)$,

$$\begin{aligned} x_\alpha(t)v &= v + td\rho(X_\alpha)v + t^2\frac{1}{2}d\rho(X)v + \cdots \\ &= v + u \end{aligned}$$

where $u \in \sum_{i=1} \oplus V_z^{\lambda+i\alpha} \otimes R$. On the other hand, for any $h(\chi) \in T(R)$ and $v \in V_z^\lambda$, $h(\chi)v = c_{\lambda,\chi}v$ for some $c_{\lambda,\chi} \in R$. Thus for any $x \in Q \cap U_{\Delta''_{\mathfrak{R}}}(R)$ and $v \in V_z^\lambda$, we have $xv \in cv + \sum_{\mu \in Z+\Delta'_{\mathfrak{R}}} \oplus V_z^{\lambda+\mu} \otimes R$ and $xv \in v + \sum_{\nu \in Z+\Delta''_{\mathfrak{R}}} \oplus V^{\lambda+\nu} \otimes R$, thus by 3.2 ii), $xv = v$. Since λ can be chosen arbitrary, we have $x = 1$. q.e.d.

3.4. PROPOSITION. *Let $\mathfrak{R} = \{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ be a permissible carpet, and use the same notation as in* 3.3. *Then $U_{\Delta''_{\mathfrak{R}}}(\mathfrak{R})Q$ is a subgroup of $G(R)$.*

PROOF. To prove the proposition, it is sufficient to show the following,

$$\text{(a)} \qquad x_\alpha(t)U_{\Delta''_{\mathfrak{R}}}(\mathfrak{R}) \subset U_{\Delta''_{\mathfrak{R}}}(\mathfrak{R})Q$$

for all $t \in \mathfrak{A}_\alpha$, $\alpha \in \Delta$. Assume $\alpha \in \Delta^+$. Since $\Delta''_{\mathfrak{R}}$ is an ideal of Δ^+, $U_{\Delta''_{\mathfrak{R}}}(\mathfrak{R})$ is a normal subgroup of $U(R)$, thus (a) holds. For a negative root α of Δ, (a) follows from the following two lemmas.

3.5. LEMMA. *Let $\mathfrak{R} = \{\mathfrak{A}_\alpha\}_{\alpha \in \Delta}$ be a semi-permissible carpet. Set $\Delta'^+_{\mathfrak{R}} = \Delta^+ \cap \Delta'_{\mathfrak{R}}$ and $\Delta''^+_{\mathfrak{R}} = \Delta^+ \cap \Delta''_{\mathfrak{R}}$. Then we have*

$$\text{(b)} \qquad x_{-\alpha}(t)U_{\Delta''^+_{\mathfrak{R}}}(\mathfrak{R}) \subset U_{\Delta''^+_{\mathfrak{R}}}(\mathfrak{R})U_{\Delta'^+_{\mathfrak{R}}}(\mathfrak{M})x_{-\alpha}(R)T(R)$$

for any $\alpha \in \Pi$ and $t \in \mathfrak{A}_{-\alpha}$.

PROOF. By 2.4, any element x of $U_{\Delta''^+_{\mathfrak{R}}}(\mathfrak{R})$ is expressed by the form

$$x = x_{\beta_1}(s_1) \cdots x_{\beta_M}(s_M)$$

where $\{\beta_1, \cdots, \beta_M\} = \Delta''^+_{\mathfrak{R}}$ and $s_i \in \mathfrak{A}_{\beta_i}$, $i = 1, \cdots, M$. Set up $x_i = x_{\beta_i}(s_i) \cdots x_{\beta_M}(s_M)$. Then we shall prove (b) by induction on i. If $\beta_{i-1} \neq \alpha$, we have

$$\begin{aligned} x_{-\alpha}(t)x_{i-1} &= x_{-\alpha}(t)x_{\beta_{i-1}}(s_{i-1})x_i \\ &= x_{\beta_{i-1}}(s_{i-1})\prod_{j,k>0} x_{-j\alpha+k\beta_{i-1}}(c_{j,k}t^j s_{i-1}^k)x_{-\alpha}(t)x_i \end{aligned}$$

where $-j\alpha + k\beta_{i-1} > 0$ and $c_{j,k}t^j s_{i-1}^k \in \mathfrak{A}_{-j\alpha+k\beta_{i-1}} \cap \mathfrak{M}$. If $\beta_{i-1} = \alpha$ then

$$x_{-\alpha}(t)x_{i-1} = x_{-\alpha}(t)x_\alpha(s_{i-1})x_i = x_\alpha(v)x_{-\alpha}(w)x_i z$$

where $v \in \mathfrak{A}_\alpha$ and $z \in T(R)$. Therefore by 2.4 and the assumption of the induction, we have

$$x_{-\alpha}(t)x_{i-1} \in U_{\Delta''_{\mathfrak{R}}+}(\mathfrak{R})U_{\Delta'_{\mathfrak{R}}+}(\mathfrak{M})x_{-\alpha}(R)T(R) \,.$$

q.e.d.

3.6. Lemma. *We use the same notation as in 3.5. For a given negative root $-\alpha$, we assume that*

$$x_{-\alpha}(t)U_{\Delta''_{\mathfrak{R}}+}(\mathfrak{R}) \subset U_{\Delta''_{\mathfrak{R}}+}(\mathfrak{R})U_{\Delta'_{\mathfrak{R}}+}(\mathfrak{M})V(\mathfrak{M})T(R)x_{-\alpha}(R)$$

for all semi-permissible carpet $\mathfrak{R} = \{\mathfrak{A}_\alpha\}_{\alpha\in\Delta}$. Then, for any element w of the Weyl group associated with Δ such that $w(\alpha) > 0$, we have

$$x_{w(-\alpha)}(t)U_{\Delta''_{\mathfrak{R}'}+}(\mathfrak{R}') \subset U_{\Delta''_{\mathfrak{R}'}+}(\mathfrak{R}')U_{\Delta'_{\mathfrak{R}'}+}(\mathfrak{M})V(\mathfrak{M})x_{w(-\alpha)}(R)T(R)\,,$$

where $\mathfrak{R}' = \{\mathfrak{A}'_\alpha\}_{\alpha\in\Delta}$ is any semi-permissible carpet.

Proof. Denote by w_σ the reflection with respect to hyperplane orthogonal to a root σ, then for any element w of the Weyl group W such that $w(\alpha) > 0$, we can choose an element w' of W as follows i) $w(\alpha) = w'(\alpha)$ ii) $w' = w_{\alpha_1}w_{\alpha_2}\cdots w_{\alpha_L}$ where $\alpha_i \in \Pi$, $i = 1, 2, \cdots, L$, and $w_{\alpha_j}w_{\alpha_{j+1}}\cdots w_{\alpha_L}(\alpha) > 0$ for $1 \leqq j \leqq L$. Therefore, without-loss of generality, we may assume $w = w_\sigma$ for some $\sigma \in \Pi$. Let $\mathfrak{R}' = \{\mathfrak{A}'_\alpha\}_{\alpha\in\Delta}$ be any semi-permissible carpet. For each element x of $U_{\Delta''_{\mathfrak{R}'}+}(\mathfrak{R}')$ we write $x = x_{\beta_1}(s_1)\cdots x_{\beta_N}(s_N)$ where $\{\beta_1, \cdots, \beta_N\} = \Delta''^{+}_{\mathfrak{R}'}$, $s_i \in \mathfrak{A}'_{\beta_i}$ $(i = 1, \cdots, N)$ and $\beta_i \neq \sigma$, $i = 1, \cdots, N-1$. Now, taking the conjugation of $x_{w(\alpha)}(t)x$ with $w(1)$, we have

$$(\,*\,)\quad x_{-\alpha}(\pm t)w(1)xw(1)^{-1} = x_{-\alpha}(\pm t)x_{w(\beta_1)}(\pm s_1)\cdots x_{w(\beta_N)}(\pm s_N)$$

where $w(\beta_j) > 0$ for $i = 1, \cdots, N-1$ and $w(\beta_N) < 0$ (resp. > 0) if $\beta_N = \sigma$ (resp. $\beta_N \neq \sigma$). Setting $\mathfrak{A}'_\beta = \mathfrak{A}_{w(\beta)}$, we have semi-permissible carpet $\{\mathfrak{A}_\gamma\}_{\gamma\in\Delta}$. First assume $\beta_N = \sigma$. Then, using the assumption of this lemma, $(*)$ is equal to

$$\begin{aligned} (**)\quad & x_{w(\beta_1)}(v_1)\cdots x_{w(\beta_{N-1})}(v_{N-1})x_{\gamma_1}(u_1)\cdots x_{\gamma_M}(u_M)x_{\gamma_{M+1}}(u_{M+1}) \\ & \cdots x_{\gamma_T}(u_T)x_{-\alpha}(t')x_{-\sigma}(s'_N)z \end{aligned}$$

where $\gamma_1, \cdots, \gamma_m$ are positive, $\gamma_{M+1}, \cdots, \gamma_T$ are negative, $v_i \in \mathfrak{A}_{w(\beta_i)}$ $i = 1, \cdots, N-1$, $u_j \in \mathfrak{A}_{\gamma_j} \cap \mathfrak{M}$, $j = 1, \cdots, T$, $s'_N \in \mathfrak{M} \cap \mathfrak{A}_{w(\beta_N)}$ and $z \in T(R)$. By 2.4, we may assume $\gamma_M = \sigma$ and $\gamma_{M+1} = -\sigma$, and $(**)$ is equal to

$$(***) \quad x_{w(\beta_1)}(v_1) \cdots x_{w(\beta_{N-1})}(v_{N-1}) x_{\gamma_1}(u_1) \cdots x_{-\sigma}(v) x_\sigma(y) x_{\gamma_{M+2}}(u'_{M+2}) \cdots x_{\gamma_T}(u'_T) x_{-\alpha}(t'') z'$$

where $v \in \mathfrak{A}_{-\sigma} \cap \mathfrak{M}$, $y \in \mathfrak{A}_\sigma \cap \mathfrak{M}$, $u'_j \in \mathfrak{A}_{\gamma_j}$ $j = M+2, \cdots, T$, and $z' \in T(R)$. Taking the conjugation of the above form with $w(1)$ again, and using 2.4, we have

$$x_{w(-\alpha)}(t)x \in U_{\Delta''^+_{\mathfrak{R}}}(\mathfrak{R}) U_{\Delta'^+_{\mathfrak{R}}}(\mathfrak{M}) V(\mathfrak{M}) x_{w(-\alpha)}(R) T(R) .$$

If $\beta_N \neq \sigma$, we can prove our assertion by the same way as above without calculation $(***)$. q.e.d.

3.7. LEMMA. *Let $\{\mathfrak{A}_\alpha\}_{\alpha\in\Delta}$ be a permissible carpet of R with respect to (Δ, Π), and let $\Delta'_{\mathfrak{R}}$, $\Delta''_{\mathfrak{R}}$ and Q be same as in* 3.1 *and* 3.3 *respectively. Then we have $Q \cap U(R) = U_{\Delta'^+_{\mathfrak{R}}}(R)$.*

PROOF. To prove our lemma, it is sufficient to show $Q \cap U(R) \subset U_{\Delta'^+_{\mathfrak{R}}}(R)$. If $x \in Q \cap U(R)$, then by 2.4, we have $x = yz$ where $y \in U_{\Delta''_{\mathfrak{R}}}(R)$, $z \in U_{\Delta'^+_{\mathfrak{R}}}(R)$. Since $z \in Q$, we have $y \in Q \cap U_{\Delta''_{\mathfrak{R}}}(R)$, and by 3.3, $y = 1$. Therefore we have $x \in U_{\Delta'^+_{\mathfrak{R}}}(R)$. q.e.d.

3.8. PROOF OF SURJECTIVITY. Let $\mathfrak{R} = \{\mathfrak{A}_\alpha\}_{\alpha\in\Delta}$ be any permissible carpet of R associated with (Δ, Π), and P be a parabolic subgroup of $G(R)$ generated by $x_\alpha(t)$ for all $t \in \mathfrak{A}_\alpha$, $\alpha \in \Delta$ and elements of $T(R)$. Set $\mathfrak{A}'_\alpha = \{t \in R \mid x_\alpha(t) \in P\}$. Then, to show our assertion, it is sufficient to prove that $\mathfrak{A}_\alpha = \mathfrak{A}'_\alpha$ for all roots $\alpha \in \Delta$. It is clear $\mathfrak{A}_\alpha \subset \mathfrak{A}'_\alpha$. If $\alpha < 0$, then $\mathfrak{A}_\alpha = \mathfrak{A}'_\alpha = R$. In order to prove $\mathfrak{A}_\alpha \supset \mathfrak{A}'_\alpha$ for $\alpha > 0$, from 2.5, it is sufficient to show $P \cap U(R) = U_{\Delta^+}(\mathfrak{R})$. By 3.4, we have $P = U_{\Delta''_{\mathfrak{R}}}(\mathfrak{R})Q$, thus, for any $x \in P \cap U(R)$, $x = yz$ where $y \in U_{\Delta''_{\mathfrak{R}}}(\mathfrak{R})$, $z \in Q$, and from 3.7, $z \in U(R) \cap Q \subset U_{\Delta^+}(\mathfrak{R})$, therefore $x \in U_{\Delta^+}(\mathfrak{R})$, that is $P \cap U(R) \subset U_{\Delta^+}(\mathfrak{R})$. On the other hand, it is clear that $P \cap U(R) \supset U_{\Delta^+}(\mathfrak{R})$. Thus we have $P \cap U(R) = U_{\Delta^+}(\mathfrak{R})$. q.e.d.

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