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ON THE COMPACTNESS OF OPERATORS OF HANKEL TYPE

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1. **Introduction.** Let P be the orthogonal projection from $L^2(T)$ onto $H^2(T)$, where $T = \{z \in C \mid |z| = 1\}$ and $H^2(T)$ is the Hardy space on T, that is, $\{f \in L^2(T) \mid \int_T f(e^{i\theta})e^{-ik\theta}d\theta = 0,\ k = -1, -2, \cdots\}$. For a function $\varphi \in H^2(T)$ satisfying $\int \varphi d\theta = 0$ the Hankel operator L_φ is defined by $L_\varphi(\psi) = P(\varphi\bar{\psi})$, $\psi \in H^2(T) \cap L^\infty(T)$, where the bar denotes complex conjugation.

Nehari [5], Hartman [3], and Coifman, Rochberg and Weiss [1] considered some properties of the Hankel operators. In this paper we are concerned with the following theorems.

THEOREM A ([5], [2]). *L_φ is a bounded operator from H^2 to H^2 if and only if $\varphi \in$ BMO. Furthermore the operator norm $\|L_\varphi\|$ is equivalent to $\|\varphi\|_{\mathrm{BMO}}$.*

THEOREM B ([3], [7]). *L_φ is a compact operator if and only if $\varphi \in$ CMO.*

The definitions of BMO, BMO-norm and CMO will be given at the end of Section 1. We note that more general situations are considered in [1].

In the following all the functions considered will be real valued functions defined on R^n. For a measurable function b we define $B(f) = bf$. As pointed out in [1] for the one dimensional case the study of $[H, B] = HB - BH$, where H is the Hilbert transform, is often essentially equivalent to that of L_φ.

Suppose that K is a Caldéron-Zygmund singular integral operator with smooth kernel. That is, there is an $\Omega(x)$ which is homogeneous of degree zero, which satisfies $\int_{|x|=1} \Omega = 0$, $\Omega \not\equiv 0$ and $|\Omega(x) - \Omega(y)| \leqq |x - y|$ when $|x| = |y| = 1$, and that

$$(Kf)(x) = P.V. \int \frac{\Omega(x - y)}{|x - y|^n} f(y)dy .$$

THEOREM A$'$ ([1]). *If b is in* BMO, *then $[K, B]$ is a bounded map of $L^p(R^n)$ to itself, $1 < p < \infty$, with operator norm $\|[K, B]\|_{(p)} \leqq C_{K,p}\|b\|_{\mathrm{BMO}}$. Conversely, if $[B, R_i]$, where $R_1, R_2, \cdots, R_n$ are the Riesz transforms,* *are bounded on* $L^p(R^n)$ *for some* p, $1<p<\infty$ *and* $i=1,\cdots,n$ *then* b *is in* BMO *and* $\|b\|_{\mathrm{BMO}}\leqq A\sum_{i=1}^{n}\|[B,R_i]\|_{(p)}$.

We shall improve Theorem A′ in Section 2 and extend Theorem B on R^n in Section 3. In the latter case we shall find some difficulties in the functions of CMO over R^n which do not occur in the unit circle case. To avoid it we shall use the characterization of CMO over R^n which is announced in Neri [6].

NOTATION. i, j, k and m mean always integers. A dyadic cube is a cube of the form $\{x=(x_1,\cdots,x_n)\in R^n\,|\,k_i2^j\leqq x_i<(k_i+1)2^j$ for $i=1,\cdots,n\}$. For a measurable set E, $|E|$, $m(f,E)$, $\bar{E}$ and χ_E mean the Lebesgue measure of E, $|E|^{-1}\int_E f(y)dy$, the closure of E and the characteristic function of E respectively. For a cube Q in R^n, $M(f,Q)$ means $\inf\left\{|Q|^{-1}\int_Q|f(y)-c|\,dy\;\middle|\,c\in R\right\}$. R_p and $R(x,a,b)$ mean $\{x\in R^n\,|\,|x_i|<2^p$ for $i=1,\cdots,n\}$ and $\{y\in R^n\,|\,a<|x-y|<b\}$ respectively.

DEFINITION. For $f\in L^1_{\mathrm{loc}}(R^n)$, $\|f\|_{\mathrm{BMO}}$ will denote $\sup\{M(f,Q)\,|\,Q$ is a cube in $R^n\}$. Identifying functions which differ by a constant, the set of functions satisfying $\|f\|_{\mathrm{BMO}}<\infty$ is a Banach space under the norm $\|\cdot\|_{\mathrm{BMO}}$ and we call this space BMO. The BMO-closure of $\mathscr{D}$, where $\mathscr{D}$ is the set of C^∞-functions with compact support, is denoted by CMO. [See [6], p 186.]

2. THEOREM 1. *Let* $1<p<\infty$ *and* $b\in\bigcup_{q>1}L^q_{\mathrm{loc}}(R^n)$. *Then* $\|b\|_{\mathrm{BMO}}\leqq A(p,K)\|[K,B]\|_{(p)}$.

PROOF. In this proof for $i=1,\cdots,10$ A_i is a positive constant depending only on K, p and $A_j(1\leqq j<i)$. We may assume $\|[K,B]\|_{(p)}=1$. We want to prove

$$(*)\qquad \sup_Q M(b,Q)\leqq A(p,K)\,.$$

Since $\|[K,B]\|_{(p)}=\|[K,B_{r,x_0}]\|_{(p)}$ for every $x_0\in R^n$ and $r\in R_+$, where $B_{r,x_0}(f)(x)=b(r^{-1}x+x_0)f(x)$, it suffices to prove the inequality $(*)$ for $Q=Q_1=\{x\in R^n\,|\,|x_j|<(2\sqrt{n})^{-1}$ for $j=1,\cdots,n\}$. Let $M=M(b,Q_1)=|Q_1|^{-1}\int_{Q_1}|b(y)-a_0|\,dy$. Since $[K,B-a_0]=[K,B]$, we may assume $a_0=0$. Let ψ be such that

$$\|\psi\|_{L^\infty}=1\,,$$
$$\operatorname{supp}\psi\subset Q_1\,,$$
$$\int\psi\,dx=0\,,$$

$$\psi(x)b(x) \geqq 0$$

and
$$|Q_1|^{-1}\int \psi(x)b(x)dx = M .$$

Let $\sum_K$, a closed subset of $\sum = \{x \in R^n \mid |x| = 1\}$, and A_1, a positive number, be such that $m(\sum_K) > 0$, where m is the measure on $\sum$ which is induced from the Lebesgue measure on R^n, and $|\Omega(x) - \Omega(y)| < 2^{-1}\Omega(x)$ for every $x \in \sum_K$ and every $y \in \sum$ satisfying $|x - y| < A_1$. Then for $x \in G = \{x \in R^n \mid |x| > A_2 = 2A_1^{-1} + 1 \text{ and } |x|^{-1}x \in \sum_K\}$

$$\begin{aligned} |[K, B]\psi(x)| &\geqq |K(b\psi)(x)| - |b(x)K(\psi)(x)| \\ &\geqq A_3M|x|^{-n} - A_4|b(x)||x|^{-n-1}. \end{aligned}$$

Let
$$F = \{x \in G \mid |b(x)| > (MA_3/2A_4)|x| \text{ and } |x| < M^{p'/n}\} ,$$

where $p^{-1} + p'^{-1} = 1$, then

$$\begin{aligned} 1 &\geqq \int_{R^n} |[K, B]\psi(x)|^p \, dx \\ &\geqq \int_{(G\setminus F)\cap\{|x|<M^{p'/n}\}} (2^{-1}A_3M|x|^{-n})^p dx \\ &\geqq \int_{\{A_5(|F|+A_2^n)^{1/n}<|x|<M^{p'/n}\}\cap G} (2^{-1}A_3M|x|^{-n})^p dx . \end{aligned}$$

Thus

$$|F| \geqq A_6M^{p'} - A_2^n \geqq A_6M^{p'}/2 \quad \text{if } M > (2A_2^nA_6^{-1})^{1/p'} .$$

Let $g(x) = (\operatorname{sgn}(b(x)K(x)))\chi_F(x)$, then for $x \in Q_1$

$$\begin{aligned} |[K^*, B]g(x)| &\geqq A_7\int_F |y|^{-n}(A_3M/2A_4)|y|\,dy - |b(x)||K^*(g)(x)| \\ &\geqq A_8M^{1+p'/n} - A_9|b(x)|\log M , \end{aligned}$$

where $K^*f(x) = P.V.\int \Omega(y - x)|y - x|^{-n}f(y)dy$. Since $[K^*, B]$ is the adjoint operator of $[K, B]$, $\|[K^*, B]\|_{(p')} = 1$. Thus

$$\begin{aligned} A_{10}M &\geqq \|g\|_{p'} \geqq \|[K^*, B]g\|_{p'} \\ &\geqq \int_{Q_1} |[K^*, B]g(x)|\,dx \\ &\geqq \int_{Q_1\cap\{b(x)<2M\}} |[K^*, B]g(x)|\,dx \\ &\geqq 2^{-1}(A_8M^{1+p'/n} - 2A_9M\log M) . \end{aligned}$$

Then, $M \leqq A(K, p)$.

COROLLARY. *For f in $H^1(R^n)$*

$$A(K)\|f\|_{H^1} \leqq \inf\{\sum_{i=1}^{\infty}\|g_i\|_{L^2}\|h_i\|_{L^2} \mid f=\sum_{i=1}^{\infty}(g_iK(h_i)-K^*(g_i)h_i)\} \leqq A(K)'\|f\|_{H^1}\,.$$

For the definition of $H^1(R^n)$ we refer to [2]. The corollary will be proved in the same way as in Theorem II of [1] using Theorem A′ and Theorem 1.

3. LEMMA. *Let $f \in \mathrm{BMO}$. Then $f \in \mathrm{CMO}$ if and only if f satisfies the following three conditions.*

(i) $\lim_{a \downarrow 0} \sup_{|Q|=a} M(f, Q) = 0$.

(ii) $\lim_{a \uparrow \infty} \sup_{|Q|=a} M(f, Q) = 0$.

(iii) $\lim_{x \to \infty} M(f, Q+x) = 0$ for each Q .

This lemma, which seems to be due to Herz, Strichartz and Sarason, is announced in Neri [6] without proof.

PROOF. In this proof A is a positive constant depending only on n. From the definition of CMO, it is trivial that CMO satisfies (i) (ii) and (iii). In the following we prove that if f satisfies (i) (ii) and (iii), then for any $\varepsilon > 0$ there exists $g_\varepsilon \in \mathrm{BMO}$ such that

$$\inf_{h \in \mathscr{D}} \|g_\varepsilon - h\|_{\mathrm{BMO}} < A\varepsilon\,. \tag{1}$$

and

$$\|g_\varepsilon - f\|_{\mathrm{BMO}} < A\varepsilon\,. \tag{2}$$

From (i) and (ii) there exist i_ε and k_ε such that

$$\sup\{M(f, Q) \mid |Q| \leqq 2^{ni_\varepsilon}\} < \varepsilon$$

and

$$\sup\{M(f, Q) \mid |Q| \geqq 2^{nk_\varepsilon}\} < \varepsilon\,.$$

From (i), (ii) and (iii) there exists j_ε such that $j_\varepsilon > i_\varepsilon, k_\varepsilon$ and

$$\sup\{M(f, Q) \mid Q \cap R_{j_\varepsilon} = \varnothing\} < \varepsilon\,.$$

We define Q_x as follows. If $x \in R_{j_\varepsilon}$, Q_x means the dyadic cube of side length 2^{i_ε} that contains x. If $x \in R_m \backslash R_{m-1}$ where $j_\varepsilon < m$, Q_x means the dyadic cube of side length $2^{i_\varepsilon+m-j_\varepsilon}$. We set $g'_\varepsilon(x) = m(f, Q_x)$. From (ii) there exists $m_\varepsilon > j_\varepsilon$ such that

$$\sup\{|g'_\varepsilon(x) - g'_\varepsilon(y)| \mid x, y \in R_{m_\varepsilon} \backslash R_{m_\varepsilon - 1}\} < \varepsilon\,.$$

If $x \in R_{m_\varepsilon}$, we define $g_\varepsilon(x) = g'_\varepsilon(x)$ and if $x \in R_{m_\varepsilon}^c$, we define $g_\varepsilon(x) =$

$m(f, R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})$. Note the fact that

$$\text{(3)} \qquad \text{if} \quad \bar{Q}_x \cap \bar{Q}_y \neq \varnothing \ , \quad \operatorname{diam} Q_x \leqq 2 \operatorname{diam} Q_y \ .$$

Then by the definition of i_ε, j_ε and m_ε, if $\bar{Q}_x \cap \bar{Q}_y \neq \varnothing$ or $x, y \in R^c_{m_\varepsilon-1}$, then

$$\text{(4)} \qquad |g_\varepsilon(x) - g_\varepsilon(y)| < A\varepsilon \ .$$

Thus (1) is obvious. From the definition of i_ε and j_ε

$$\text{(5)} \qquad \int_{Q_x} |f(y) - g_\varepsilon(y)|\, dy \leqq A\varepsilon |Q_x|$$

for every $x \in R_{m_\varepsilon}$. Let Q be an arbitrary cube in R^n. First we consider the case such that $Q \subset R_{m_\varepsilon}$ and $\max \{\operatorname{diam} Q_x | Q_x \cap Q \neq \varnothing\} > 4 \operatorname{diam} Q$. Then by (3) the number of Q_x such that $Q_x \cap Q \neq \varnothing$ is bounded by A, and if $Q \cap R_{j_\varepsilon} \neq \varnothing$, $|Q|$ is less than 2^{ni_ε}. Thus from (4) and the definition of i_ε and j_ε, $M(f - g_\varepsilon, Q) < A\varepsilon$. Second if $Q \subset R_{m_\varepsilon}$ and $\max \{\operatorname{diam} Q_x | Q_x \cap Q \neq \varnothing\} \leqq 4 \operatorname{diam} Q$,

$$M(f - g_\varepsilon, Q) \leqq |Q|^{-1} \sum_{Q_x \cap Q \neq \phi} \int_{Q_x} |f(y) - g_\varepsilon(y)|\, dy \leqq A\varepsilon$$

by (5). Third if $Q \subset R^c_{m_\varepsilon-1}$, by the definition of m_ε

$$M(f - g_\varepsilon, Q) \leqq M(f, Q) + A\varepsilon \leqq (1 + A)\varepsilon \ .$$

Lastly we consider the case $Q \cap R^c_{m_\varepsilon} \neq \varnothing$ and $Q \cap R_{m_\varepsilon-1} \neq \varnothing$. Let p_Q be the smallest integer satisfying $Q \subset R_{p_Q}$, then

$$M(f - g_\varepsilon, Q) \leqq AM(f - g_\varepsilon, R_{p_Q}) \ .$$

Since $m_\varepsilon > k_\varepsilon$, $|m(f, R_q) - m(f, R_{q-1})| < A\varepsilon$ for every integer q such that $m_\varepsilon \leqq q$. Then

$$\begin{aligned} M(f - g_\varepsilon, R_{p_Q})|R_{p_Q}| &\leqq \int_{R_{p_Q}\backslash R_{m_\varepsilon}} |f(y) - m(f, R_{p_Q})|\, dy \\ &+ |m(f, R_{p_Q}) - m(f, R_{m_\varepsilon}\backslash R_{m_\varepsilon-1})|\,|R_{m_\varepsilon}| + \sum_{Q_x \subset R_{m_\varepsilon}} \int_{Q_x} |f(y) - g_\varepsilon(y)|\, dy \\ &\leqq \varepsilon |R_{p_Q}| + A\varepsilon(p_Q - m_\varepsilon)|R_{m_\varepsilon}| + A\varepsilon |R_{m_\varepsilon}| \\ &\leqq (1 + 2A)\varepsilon |R_{p_Q}| \ . \end{aligned}$$

Thus (2) is proved.

THEOREM 2. *Let $b \in \bigcup_{q>1} L^q_{\mathrm{loc}}(R^n)$. Then $[K, B]$ is a compact operator from L^p to itself, $1 < p < \infty$, if and only if $b \in \mathrm{CMO}$.*

PROOF. If $[K, B]$ is a compact operator, then from Theorem 1 $b \in \mathrm{BMO}$. Thus we may assume $\|b\|_{\mathrm{BMO}} = 1$. First suppose that b does not satisfy (i) of the previous lemma. Then there exist $\delta > 0$ and a sequence of cubes $\{Q_j\}_{j=1}^{\infty}$ such that

$$M(b, Q_j) > \delta \tag{11}$$

for every j and $\lim_{j\to\infty} q_j = 0$ where q_j is the diameter of Q_j. In the following for $i = 20, \cdots, 36$ A_i is a positive constant depending only on K, p, δ and $A_j (20 \leqq j < i)$. Let b_j be a real number such that $M(b, Q_j) = |Q_j|^{-1} \int_{Q_j} |b(y) - b_j|\, dy$ and x_j the center of Q_j. We define f_j as follows

$$f_j(b - b_j) \geqq 0\,, \tag{12}$$

$$\operatorname{supp} f_j \subset Q_j \tag{13}$$

$$\int f_j\, dy = 0 \tag{14}$$

and

$$|f_j(y)| = |Q_j|^{-1/p} \tag{15}$$

for every $y \in Q_j$. Note that $[K, B]f = K((b - b_j)f) - (b - b_j)K(f)$. From (13) and (15)

$$|K((b - b_j)f_j)(y)| \leqq A_{20} |Q_j|^{1-1/p} |x_j - y|^{-n} \tag{16}$$

for $y \notin A_{21}Q_j$. By (11), (12) and the continuity of the kernel

$$|K((b - b_j)f_j)(y)| \geqq A_{22}\delta |Q_j|^{1-1/p} |x_j - y|^{-n} \tag{17}$$

for $y \in (A_{21}Q_j)^c \cap \{y \mid |x_j - y|^{-1}(x_j - y) \in \sum_K\}$, where $\sum_K$ is as in the proof of Theorem 1. On the other hand, by (14) and the smoothness of the kernel

$$|(b(y) - b_j)K(f_j)(y)| \leqq A_{23} |b(y) - b_j|\, |x_j - y|^{-n-1} q_j |Q_j|^{1-1/p} \tag{18}$$

for $y \notin A_{21}Q_j$. Since $\|b\|_{\mathrm{BMO}} = 1$,

$$\int_{R(x_j, 2^k q_j, 2^{k+1} q_j)} |b(y) - b_j|^p\, dy \leqq A_{24} 2^{kn} |Q_j|\, k^p\,.$$

[See for example [2][4].] Thus if $\alpha > A_{21}$

$$\begin{aligned}
&\int_{|x_j - y| > \alpha q_j} |(b(y) - b_j)K(f_j)(y)|^p\, dy \\
&\leqq A_{23}^p A_{24} q_j^p |Q_j|^{p-1} \sum_{k=\log\alpha}^{\infty} (2^k q_j)^{-p(n+1)} 2^{kn} |Q_j|\, k^p \\
&\leqq A_{25} \sum_{k=\log\alpha}^{\infty} k^p 2^{-k(pn+p-n)}
\end{aligned}$$

$$\leqq A_{26} \sum_{k=\log \alpha}^{\infty} 2^{-k(pn+p-n-p/2)}$$
$$\leqq A_{27}\alpha^{-((p-1)n+p/2)} .$$

Then from (17), for $\beta > \alpha > A_{21}$

$$\left(\int_{R(x_j, \alpha q_j, \beta q_j)} |[K, B]f_j|^p \, dy\right)^{1/p}$$
$$\geqq A_{28}\delta(\alpha^{-pn+n} - \beta^{-pn+n})^{1/p} - A_{27}^{1/p}\alpha^{-(1/2+n(p-1)/p)} .$$

So from (16) there exist A_{29}, A_{30} and A_{31} satisfying

(19) $$2 < A_{29} < A_{30},$$
$$\int_{R(x_j, A_{29}q_j, A_{30}q_j)} |[K, B]f_j|^p \, dy \geqq A_{31}$$

and

(20) $$\int_{|x_j-y|>A_{30}q_j} |[K, B]f_j|^p \, dy \leqq A_{31}/4 .$$

By the result of [2] and [4],

(21) $$|\{y \mid |b(y) - b_j| > u + A_{32}\} \cap R(x_j, A_{29}q_j, A_{30}q_j)| \leqq A_{33}|Q_j| e^{-A_{34}u} .$$

Let $E \subset R(x_j, A_{29}q_j, A_{30}q_j)$ be an arbitrary measurable set. Then by (16), (18), (21) and $\|b\|_{\mathrm{BMO}} = 1$

$$\int_E |[K, B]f_j|^p \, dy \leqq A_{35}\frac{|E|}{|Q_j|}\left(1 + \log^+\frac{|Q_j|}{|E|}\right)^p .$$

Thus there exists A_{36} such that

$$\int_E |[K, B]f_j|^p \, dy < A_{31}/4$$

for every measurable set E satisfying

$$E \subset R(x_j, A_{29}q_j, A_{30}q_j) \text{ and } |E| < A_{36}^n q_j^n .$$

If we select a subsequence $\{Q_{j(k)}\}$ satisfying

(22) $$q_{j(k+1)}/q_{j(k)} < A_{36}/A_{30} ,$$

then for $m > 0$ using (19), (20) and (22) we get

$$\|[K, B]f_{j(k)} - [K, B]f_{j(k+m)}\|_p^p$$
$$\geqq \int_{R(x_{j(k)}, A_{29}q_{j(k)}, A_{30}q_{j(k)}) \setminus R(x_{j(k+m)}, 0, A_{30}q_{j(k+m)})} |[K, B]f_{j(k)} - [K, B]f_{j(k+m)}|^p \, dy$$
$$\geqq ((A_{31}/2)^{1/p} - (A_{31}/4)^{1/p})^p$$
$$\geqq ((1/2)^{1/p} - (1/4)^{1/p})^p A_{31} .$$

Thus $\{[K, B]f_j\}_{j=1}^{\infty}$ is not relatively compact in L^p, i.e., $[K, B]$ is not compact. Quite similarly we can prove that if b does not satisfy (ii) or (iii) of the previous lemma, $[K, B]$ is not a compact operator.

Conversely, suppose that $b \in \mathrm{CMO}$. Then for any $\varepsilon > 0$ there exists $b_\varepsilon \in \mathscr{D}$ such that $\|b - b_\varepsilon\|_{\mathrm{BMO}} < \varepsilon$. By Theorem A′

$$\|[K, B] - [K, B_\varepsilon]\|_{(p)} < \varepsilon .$$

Thus for the proof of the converse part it suffices to prove that $[K, B]$ is a compact operator for $b \in \mathscr{D}$. In the following for $i = 40, \cdots, 48$ A_i is a positive constant depending only on b, p, K and A_j $(40 \leqq j < i)$. It is clear that

$$|[K, B]f(x)| \leqq A_{40}\|f\|_p |x|^{-n} \tag{31}$$

for $|x| > A_{41}$ and from Theorem A′

$$\|[K, B]f\|_p \leqq A_{42}\|f\|_p . \tag{32}$$

Take an arbitrary $2^{-1} > \varepsilon > 0$ and $z \in R^n$. Then,

$$\begin{aligned}
&[K, B]f(x) - [K, B]f(x + z) \\
&\quad = P.V. \int K(x - y)(b(y) - b(x))f(y)dy \\
&\quad - P.V. \int K(x + z - y)(b(y) - b(x + z))f(y)dy \\
&\quad = \int_{|x-y|>\varepsilon^{-1}|z|} K(x - y)(b(x + z) - b(x))f(y)dy \\
&\quad + \int_{|x-y|>\varepsilon^{-1}|z|} (K(x - y) - K(x + z - y))(b(y) - b(x + z))f(y)dy \\
&\quad + P.V. \int_{|x-y|<\varepsilon^{-1}|z|} K(x - y)(b(y) - b(x))f(y)dy \\
&\quad - P.V. \int_{|x-y|<\varepsilon^{-1}|z|} K(x + z - y)(b(y) - b(x + z))f(y)dy .
\end{aligned} \tag{33}$$

The first term of (33) is dominated by

$$|b(x + z) - b(x)|\, K_*(f)(x)$$

where $K_*(f)(x) = \sup_{\eta>0}\left|\int_{|x-y|>\eta} K(x - y)f(y)dy\right|$. The second term is dominated by

$$A_{43}\int_{|x-y|>\varepsilon^{-1}|z|} |z|\,|x - y|^{-n-1}|f(y)|\,dy .$$

The last two terms are dominated by

$$A_{44}\Big(\int_{|x-y|<\varepsilon^{-1}|z|}|x-y|^{-n+1}|f(y)|\,dy + \int_{|x-y|<\varepsilon^{-1}|z|}|x+z-y|^{-n+1}|f(y)|\,dy\,.\Big)$$

Note that $\int_{|y|>\varepsilon^{-1}|z|}|z|\,|y|^{-n-1}dy = A_{45}\varepsilon,$

$$\int_{|y|<\varepsilon^{-1}|z|}|y|^{-n+1}dy = A_{46}\varepsilon^{-1}|z|\,,$$

$$\|K_*(f)\|_p \leqq A_{47}\|f\|_p$$

[see [8], p42] and that b is uniformly continuous. Then by taking $|z|$ sufficiently small depending on ε, we can get

$$\Big(\int|[K, B]f(x) - [K, B]f(x+z)|^p\,dx\Big)^{1/p} \leqq \varepsilon A_{48}\|f\|_p\,. \tag{34}$$

Thus from (31), (32), (34) and the theorem of Frechet-Kolmogorov ([9], p275), $[K, B]$ is a compact operator.

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