Tôhoku Math. Journ.
30 (1978), 59-68.

CLOSED GEODESICS IN A LOCALLY SYMMETRIC SPACE

TOSHIKAZU SUNADA

(Received September 25, 1976)

1. Introduction. The purpose of this paper is to investigate the differential geometric structure of the set of closed geodesics in a compact locally Riemannian symmetric space with non-positive sectional curvature. In our argument we shall use largely Morse theory for closed curves developed by R. S. Palais [7] and the others, which gives a connection between the set of closed geodesics and the topological structure of the space of all closed curves.

Let Geo(M) be the set of all closed geodesics in a Riemannian manifold (M, g), which is the critical point set of the energy function defined on a Hilbert manifold of closed curves (see H. I. Eliasson [2], W. Klingenberg [4]). Though Geo(M) is of finite dimension, it has, in general, complicated aspect. But, in our case, we obtain the following result which will be proved in §4.

THEOREM A. *Let (M, g) be a compact locally Riemannian symmetric space with non-postive sectional curvature. Then,* Geo(M) *is, in the sense of R. Bott* [1], *W. Meyer* [5], *a disjoint union of finite dimensional non-degenerate critical manifolds of index* 0.

In §5, we shall discuss the existence and uniqueness of maximal family of closed geodesics. Then Theorem A and the argument in §3 allow us to describe more precisely the differential geometric aspect of Geo(M).

2. The energy function. Let us begin with a review of Morse theory for closed curves; the basic reference here is H. I. Eliasson [2].

Let (M, g) be a compact connected, n-dimensional Riemannian manifold of class C^∞, without boundary. As is usual, $\pi: TM \to M$ will denote its tangent bundle, and $D: \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ the covariant differentiation associated with the metric g. We shall use W^1 to denote the class of mappings which have square integrable derivatives. Then $W^1(S^1, M)$, the set of mappings $c: S^1 \to M$ of class W^1, is a C^∞-Hilbert manifold and the tangent space at c can be identified with the Hilbert space of W^1-vector fields $X: S^1 \to TM$ along c, which is denoted by $W^1(c^{-1}TM)$. The

Riemannian metric g of M induces a Riemannian metric $\langle\ ,\ \rangle_1$ on $W^1(S^1, M)$ which is given by

$$\langle X, Y\rangle_1 = \int_{S^1} g(X(t), Y(t))dt + \int_{S^1} g\Big(\frac{D}{dt}X(t), \frac{D}{dt}Y(t)\Big)dt ,$$

where D/dt is the covariant derivative along c. With this metric, W^1 is a complete Riemannian manifold of class C^∞.

On $W^1(S^1, M)$ we consider the energy function:

$$E(c) = \frac{1}{2}\int_{S^1} g\Big(\frac{dc}{dt}, \frac{dc}{dt}\Big)dt ,$$

which is a C^∞-function on $W^1(S^1, M)$ and its derivative is

$$d_cE(X) = \int_{S^1} g\Big(\frac{dc}{dt}, \frac{D}{dt}X\Big)dt , \quad X \in W^1(c^{-1}TM) .$$

This means that the critical points of E are exactly the closed geodesics in M. Furthermore, E satisfies the condition (C) of Palais and Smale, cf. R. S. Palais [7]: Given any sequence c_k in $W^1(S^1, M)$ such that $E(c_k)$ is bounded and $\|d_{c_k}E\|$ converges to zero, then (c_k) possesses a convergent subsequence.

We denote by $R: TM\otimes TM\otimes TM \longrightarrow TM$ the curvature tensor on (M, g) which is given by

$$R(X, Y)Z = -D_XD_YZ + D_YD_XZ + D_{[X,Y]}Z .$$

We define an operator $R_c: W^1(c^{-1}TM) \longrightarrow W^1(c^{-1}TM)$ by

$$R_c(X) = R\Big(\frac{dc}{dt}, X\Big)\frac{dc}{dt} \qquad \text{for} \quad c \in C^\infty(S^1, M) .$$

Then the Hessian of E at a critial point c of E is given by

$$H(E)_c(X, Y) = \int_{S^1} g\Big(\frac{D}{dt}X, \frac{D}{dt}Y\Big)dt - \int_{S^1} g(R_c(X), Y)dt .$$

A C^∞-field X along a geodesic c is called a Jacobi field, iff it satisfies the differential equation:

$$\frac{D^2}{dt^2}X + R_c(X) = 0 .$$

The space of Jacobi fields along c will be denoted by J_c. Then we put

$$\text{Nullity}\ (c) = \dim J_c = \dim \operatorname{Ker}\Big(\frac{D^2}{dt^2} + R_c\Big) \qquad (< \infty) .$$

$$\text{Index}\ (c) = \dim \sum_{\lambda<0} \operatorname{Ker}\Big(\frac{D^2}{dt^2} + R_c + \lambda\Big) \qquad (< \infty) .$$

Finally, we denote by $\mathrm{Geo}(M)$ the set of closed geodesics in M.

3. **Locally Riemannian symmetric space.** In this section we assume the basic facts on symmetric spaces (see S. Helgason [3]). In what follows (M, g) is a compact locally Riemannian symmetric space with non-positive sectional curvature, that is to say, each point $x \in M$ has an open neighborhood on which the geodesic symmetry is an isometry, and

$$g(R(X, Y)X, Y) \leqq 0 \quad \text{for} \quad X, Y \in TM .$$

$\tilde{\omega}: \tilde{M} \to M$ will denote its universal covering and $\mathrm{Exp}: TM \to M$ (resp. $\widetilde{\mathrm{Exp}}: T\tilde{M} \to \tilde{M}$) the exponential mapping of M (resp. of $\tilde{M}$). Then the following diagram is commutative:

$$\begin{array}{ccc} T\tilde{M} & \xrightarrow{\widetilde{\mathrm{Exp}}} & \tilde{M} \\ {\scriptstyle d\tilde{\omega}}\downarrow & & \downarrow{\scriptstyle \tilde{\omega}} \\ TM & \xrightarrow[\mathrm{Exp}]{} & M . \end{array}$$

Since $\tilde{M}$ is a globally Riemannian symmetric space with non-positive sectional curvature, the largest connected group G of isometries of $\tilde{M}$ operates transitively on $\tilde{M}$ and for each $x \in \tilde{M}$ the restriction $\widetilde{\mathrm{Exp}}_x: T_x\tilde{M} \to \tilde{M}$ is a diffeomorphism. Moreover

$$\widetilde{\mathrm{Exp}}\, dg(X) = g\, \widetilde{\mathrm{Exp}}\,(X) \quad \text{for} \quad g \in G,\ X \in T\tilde{M} .$$

Now let $c: S^1 \to M$ be a geodesic, and $\tilde{c}: \boldsymbol{R} \to \tilde{M}$ a lifting of c. We set $\tilde{c}(0) = \mathbf{0}$. Then the Cartan involution $\mathrm{Ad}\,(s)$ of G where s is the symmetry of $\tilde{M}$ at $\mathbf{0}$ yields the decomposition of Lie algebra $\boldsymbol{g}$ of G:

$$\boldsymbol{g} = \boldsymbol{k} + \boldsymbol{p}$$

satisfying $[\boldsymbol{k}, \boldsymbol{p}] \subset \boldsymbol{p}$ and $[\boldsymbol{p}, \boldsymbol{p}] \subset \boldsymbol{k}$, where $\boldsymbol{k}$ is the Lie algebra corresponding to the isotropy subgroup $K = \{g \in G;\ g\mathbf{0} = \mathbf{0}\}$. $\boldsymbol{p}$ can be naturally identified with $T_0\tilde{M}$ and for $X \in \boldsymbol{p}$ we have

$$\widetilde{\mathrm{Exp}}_0\, X = \exp \cdot \mathbf{0}$$

where $\exp: \boldsymbol{g} \to G$ is the exponential mapping of Lie group G. Further, for $X, Y \in \boldsymbol{p}$, $d(\exp Y)X$ is the parallel translate of X along the geodesic $\mathrm{Exp}\, tY (0 \leqq t \leqq 1)$.

LEMMA 1. *Let X be a Jacobi field along c. Then X is parallel* (i.e., $(D/dt)X = 0$) *and* $g(R(\dot{c}, X)\dot{c}, X) = 0$ ($\dot{c} = dc/dt$).

PROOF. By the definition of Jacobi field, we have

$$\int_{S^1} g\left(\frac{D}{dt}X, \frac{D}{dt}X\right)dt - \int_{S^1} g(R(\dot{c}, X)\dot{c}, X)dt = 0 .$$

Since M has non-positive sectional curvature, the second integrand is a non-positive function, so

$$\int_{S_1} g\left(\frac{D}{dt}X, \frac{D}{dt}X\right)dt = 0 , \quad \int_{S^1} g(R(\dot{c}, X)\dot{c}, X)dt = 0 .$$

This implies that $(D/dt)X = 0$ and $g(R(\dot{c}, X)\dot{c}, X) = 0$.

LEMMA 2. *Let X be a Jacobi field along c, and $\tilde{X}_0$ be a (unique) tangent vector in $T_0\tilde{M}(=\boldsymbol{p})$ satisfying $d\tilde{\omega}(\tilde{X}_0) = X(0)$. Then*

$$[\dot{\tilde{c}}(0), \tilde{X}_0] = 0 .$$

PROOF. We denote by $\tilde{g}$ (resp. $\tilde{R}$) the Riemannian metric of $\tilde{M}$ (resp. curvature tensor). The standard symmetric space theory says

$$\tilde{R}_0(X, Y)Z = [[X, Y], Z] \quad \text{for} \quad X, Y, Z \in \boldsymbol{p} ,$$

so that

$$0 = g_{c(0)}(R(\dot{c}, X)\dot{c}, X) = \tilde{g}_0(\tilde{R}(\dot{\tilde{c}}(0), \tilde{X}_0)\dot{\tilde{c}}(0), \tilde{X}_0) = \tilde{g}_0([[\dot{\tilde{c}}(0), \tilde{X}_0]\dot{\tilde{c}}(0)], \tilde{X}_0) .$$

Thus it is enough to show that $\tilde{g}_0(\tilde{R}(X, Y)X, Y) = 0$ if and only if $[X, Y] = 0$. Let b denote the endomorphism of $\boldsymbol{p}$ given by

$$\tilde{g}_0(bX, Y) = B(X, Y) , \quad X, Y \in \boldsymbol{p} ,$$

B denoting the Killing form of $\boldsymbol{g}$. Since $\tilde{g}_0(bX, Y) = \tilde{g}_0(X, bY)$, the eigenvalues $\beta_1, \cdots, \beta_k$ of b are non-negative. Let $\boldsymbol{p}_1, \cdots, \boldsymbol{p}_k$ be the corresponding eigenspaces of b. Then it is easy to see that if $i \neq j$, the spaces $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ are orthogonal with respect to B as well as $\tilde{g}_0$, $[\boldsymbol{k}, \boldsymbol{p}_i] \subset \boldsymbol{p}_i$ and $[\boldsymbol{p}_i, \boldsymbol{p}_j] = 0$. Further if $\beta_i = 0$, then $B(\boldsymbol{p}_i, \boldsymbol{p}_i)$ and $[\boldsymbol{p}_i, \boldsymbol{p}_i] = 0$. Let X_i, $Y_i (1 \leqq i \leqq k)$ be the components of X and Y, respectively, in the eigenspaces $\boldsymbol{p}_i$. Then

$$[X, Y] = \sum_{i=1}^{k} [X_i, Y_i] , \quad [[X_i, Y_i], X] = [[X_i, Y_i], X_i]$$

so

$$\begin{aligned} \tilde{g}_0(\tilde{R}(X, Y)X, Y) &= \sum_{i=1}^{k} \tilde{g}_0([[X_i, Y_i], X_i], Y_i) \\ &= \sum_{\beta_i \neq 0} \frac{1}{\beta_i} B([[X_i, Y_i], X_i], Y_i) \\ &= \sum_{\beta_i \neq 0} \frac{1}{\beta_i} B([X_i, Y_i], [X_i, Y_i]) . \end{aligned}$$

Since B is strictly negative definite on $\boldsymbol{k}$, we have

$$\tilde{g}_0(\tilde{R}(X, Y)X, Y) = 0 \quad \text{iff} \quad [X_i, Y_i] = 0 \quad \text{for any} \quad i\,.$$

This proves our assertion.

LEMMA 3. *Let X be a Jacobi field along c. Then the mapping c_X: $S^1 \to M$ given by $c_X(t) = \mathrm{Exp}_{c(t)} X(t)$ is a geodesic in M.*

PROOF. Let $\tilde{X}: \boldsymbol{R} \to T\tilde{M}$ be a lifting of X which is a filed along $\tilde{c}$. Then the mapping $\tilde{c}_X: \boldsymbol{R} \to \tilde{M}$ defined by $\tilde{c}_X(t) = \widetilde{\mathrm{Exp}}_{\tilde{c}(t)}\tilde{X}(t)$ is a lifting of c_X. We shall show that $\tilde{c}_X$ is a geodesic in $\tilde{M}$. For this porpose, we set

$$\tilde{c}(t) = \mathrm{Exp}\, tY = \exp tY\cdot\boldsymbol{0}\,, \quad Y \in \boldsymbol{p}\,.$$

Since $\tilde{X}$ is parallel along $\tilde{c}$,

$$\tilde{X}(t) = d(\exp tY)\tilde{X}(0)\,.$$

Thus

$$\begin{aligned}\tilde{c}_X(t) &= \widetilde{\mathrm{Exp}}_{\tilde{c}(t)}\tilde{X}(t) = \widetilde{\mathrm{Exp}}_{\tilde{c}(t)} d(\exp tY)\tilde{X}(0)\\ &= \exp tY\, \widetilde{\mathrm{Exp}}_{\tilde{c}(0)}(\tilde{X}(0))\\ &= \exp tY \exp \tilde{X}(0)\cdot\boldsymbol{0}\\ &= \exp\tilde{X}(0)\exp tY\cdot\boldsymbol{0} = \exp\tilde{X}(0)\tilde{c}(t)\,,\end{aligned}$$

where we have used the fact $[\tilde{X}(0), Y] = 0$. This implies that $\tilde{c}_X$ is a geodesic.

4. **The space of closed geodesics.** Again, let (M, g) be a compact locally Riemannian symmetric space with non-positive sectional curvature. The notations of the preceding section will be preserved.

We consider the mapping

$$\mu_c\colon J_c \to W^1(S^1, M)$$

defined by $\mu_c(X) = c_X$, which is of class C^∞ as is easily checked. Moreover, by Lemma 3, $\mu_c(J_c) \subset \mathrm{Geo}\,(M)$.

LEMMA 4. *The differential of J_c at 0 (zero section of $\Gamma(c^{-1}TM)$) coincides with the injection: $J_c \subset W^1(c^{-1}TM)$. Here J_c is regarded as a manifold in the usual way and whose tangent space at 0 is identified with J_c itself.*

PROOF. For $X \in T_0J_c(=J_c)$, the differential is given by

$$d_0\mu_c(X)(s) = \frac{d}{dt}\Big|_{t=0}\mu_c(tX)(s) = \frac{\partial}{\partial t}\Big|_{t=0}\mathrm{Exp}_{c(s)}\, tX(s) = X(s)\,,$$

where we have used that the differential of Exp_p: $T_pM \to M$ at 0 is the identity mapping. Thus $d_0\mu_c$ is the injection: $J_c \subset W^1(c^{-1}TM)$.

PROPOSITION. *$\mathrm{Geo}(M)$ is a finite dimensional, closed C^∞-submanifold in $W^1(S^1, M)$ (with many components). Moreover, μ_c: $J_c \to \mathrm{Geo}(M)$ yields a local diffeomorphism in a neighborhood of* 0.

This follows from the implicit function theorem, in view of Lemma 4. For the sake of completeness we shall prove this in appendix.

PROOF OF THEOREM A. By assumption of sectional curvature

$$H(E)_c(X, X) = \int_{S^1} g\Big(\Big(\frac{D^2}{dt^2} + R_c\Big)X,\ X\Big)dt \leqq 0\ ,$$

so $\mathrm{Index}(c) = 0$ for each geodesic c. Further

$$\dim T_c \mathrm{Geo}(M) = \dim J_c = \mathrm{Nullity}(c)$$

which implies that each component of $\mathrm{Geo}(M)$ is a non-degenerate. q.e.d.

5. Family of closed geodesic. In this section, we shall investigate the differential geometric structure of $\mathrm{Geo}(M)$, by using the result of §4.

Generally, by a C^∞-family of closed geodesics in a Riemannian manifold (M, g) we mean a connected C^∞-manifold F with a C^∞-mapping φ: $F \times S^1 \to M$ such that for each point p of F the mapping φ_p: $S^1 \to M$ given by $\varphi_p(t) = \varphi(p, t)$ is a geodesic in M. For instance, a geodesic, c: $S^1 \to M$ always yields a one-parameter family φ: $S^1 \times S^1 \to M$ given by $\varphi(s, t) = c(s + t)$.

In the following we fix a geodesic c: $S^1 \to M$. We are interested in how we can so to speak deform c. To be more precise, by "*deforming* c" we mean a connected C^∞-manifold F together with a reference point p_0 and a C^∞-family of closed geodesic in M: $F \times S^1 \xrightarrow{\varphi} M$ with $\varphi_{p_0} = c$. Furthermore, a C^∞-family (F, p_0) of deformations of c will be called maximal if for any C^∞-family (F', p_0') of deformations of c there exists a unique C^∞-mapping μ: $(F', p_0') \to (F, p)$ such that $\varphi\cdot(\mu \times id) = \varphi'$. By the definition of maximal family, if it exists, it is uniquely determined up to diffeomorphism. In general, there does not always exist a maximal family of deformations of a given one.

In our case, we have the following results.

THEOREM B. *Let (M, g) be a compact locally Riemannian symmetric space with non-positive sectional curvature. Then, for any closed geodesic c_0 in M there exists a C^∞-maximal family φ: $(F, p_0) \times S^1 \to M$ of deformations of c with the following properties*:

(i) *For any* $t \in S^1$, *the mapping* $\varphi^t: F \to M$ *defined by* $\varphi^t(p) = \varphi(p, t)$ *is an immersion.*

(ii) *The induced metric* $(\varphi^t)^*g$ *of* F *does not depend on* $t \in S^1$, *and with respect to this metric,* F *is a compact locally Riemannian symmetric space with non-positive sectional curvature. Further,* φ^t *is a totally geodesic mapping.*

(iii) *If a closed geodesic* c' *in* M *is homotopic to* c_0, *then there exists a unique point* p *of* F *with* $\varphi_p = c'$.

(iv) *Let* $[c_0] \in \pi_1(M, c_0(0))$ *be the homotopy class determined by* c_0. *Then the fundamental group of* F *is isomorphic to the centralizer of* $[c_0]$. *More precisely, the induced homomorphism* $\varphi^0_\#: \pi_1(F, p_0) \to \pi_1(M, c_0(0))$ *is injective and the image of* $\varphi^0_\#$ *is equal to* $\{\mu \in \pi_1(M, c_0(0)); \mu[c_0] = [c_0]\mu\}$.

PROOF. Let F be the connected component of $\mathrm{Geo}(M)$ containing c which is compact in view of the condition (C). Let $\varphi: F \times S^1 \to M$ be the mapping defined by the evaluation: $(c, t) \mapsto c(t)$. Since

$$\varphi(t, \mu_c(X)) = \mathrm{Exp}_{c(t)} X(t) \quad \text{for} \quad X \in J_c ,$$

F gives a C^∞-family of deformations of c_0. Maximality of F follows at once from the definition.

We identify J_c with the tangent space T_cF via $d_0\mu_c$. Then the differential of $\varphi^t: F \to M$ at c is given by

$$d_c\varphi^t(X) = X(t)$$

which is injective, so that φ^t is an immersion ((i) of Theorem). Further,

$$(\varphi^t)^*g_c(X, Y) = g_{c(t)}(d\varphi^t(X), d\varphi^t(Y)) = g_{c(t)}(X(t), Y(t)) \quad \text{for} \quad X, Y \in J_c .$$

Since X, Y are parallel, the function $g_{c(t)}(X(t), Y(t))$ on S^1 is constant, so the metric $(\varphi^t)^*g$ does not depend on $t \in S^1$.

To prove that φ^t is totally geodesic, let N_c be a normal neighborhood of c in F such that the restriction $\varphi^t: N_c \to M$ is an embedding. Let v be a vector of $T_{c(t)}M$ which is tangent to $\varphi^t(N_c)$. Then there exists a vector $X \in T_cF$ such that $v = (\varphi^t)_*X$ and

$$\mathrm{Exp}\, sv = \mathrm{Exp}_{c(t)}\, s(\varphi^t)_*X = \mathrm{Exp}_{c(t)}\, sX(t) = \varphi(t, \mu_c(sX)) ,$$

which is contained in $\varphi^t(N_c)$ if $|s|$ is taken small enough. This implies that the curve given by $s \to \mu_c(sX)$ is a geodesic in F and φ^t is totally geodesic. From these arguments, it follows immediately that F is locally symmetric and has non-positive sectional curvature.

The proof of (iii), (iv) is broken up into a few lemmas.

LEMMA 5. *The injection*: $\mathrm{Geo}(M) \subset W^1(S^1, M)$ *is a homotopy eqivalence.*

In view of Theorem A, we can apply the results of W. Meyer [5] to the energy function. Our assertion follows from that each component of $\mathrm{Geo}(M)$ is of index 0.

In particular, each component of $W^1(S^1, M)$ contains a unique component of $\mathrm{Geo}(M)$. On the other hand, we know, from a general theorem of Palais [8], that the inclusion $W^1(S^1, M) \subset C^0(S^1, M)$ (the space of all continuous mappings of S^1 into M) is a homotopy equivalence. This proves (iii).

LEMMA 6. *$\varphi^0_\sharp: \pi_1(F, p_0) \to \pi_1(M, c_0(0))$ is injective.*

It is not hard to see that a lifting of φ^0 to the universal covering: $\tilde{F} \to \tilde{M}$ is injective, from which follows lemma.

Now, let $C(F)$ be the component of $C^0(S^1, M)$ containing F, and let $\Phi: C(F) \to M$ be the mapping given by $\Phi(c) = c(0)$. Then the diagram

$$\begin{tikzcd} F \arrow[rr, "\subset"] \arrow[dr, "\varphi^0"'] & & C(F) \arrow[dl, "\Phi"] \\ & M & \end{tikzcd}$$

is commutative. For (iv), it is enough to prove the following lemma, because the inclusion $F \subset C(F)$ is a homotopy equivalence as we remarked above.

LEMMA 7. *The image of the induced homomorphism $\Phi_\sharp: \pi_1(C(F), c_0) \to \pi_1(M, c_0(0))$ is the centralizer of $[c_0]$.*

PROOF. It is known (J. C. Moore [6]) that the evaluation mapping $\Phi: C^0(S^1, M) \to M$ is a Serre fibration (namely, it has a structure to insure the covering homotopy theorem for cells). Thus we have an exact sequence associated with the fibering:

$$\cdots \to \pi_1(C^0(S^1, M), c_0) \xrightarrow{\Phi_\sharp} \pi_1(M, c_0(0)) \xrightarrow{\Delta} \pi_0(\Phi^{-1}(c_0(0)), c_0) \to \cdots .$$

Note $\pi_1(C^0(S^1, M), c_0) = \pi_1(C(F), c_0)$ and we can identify $\pi_0(\Phi^{-1}(c_0(0)), c_0)$ with $\pi_1(M, c(0))$ using the fact that the universal covering $\tilde{M}$ is contractible. Then Δ is given by $\Delta(\mu) = \mu[c_0]\mu^{-1}$, hence we obtain

$$\mathrm{Im}\, \Phi_\sharp = \Delta^{-1}([c_0]) = \{\mu \in \pi_1(M, c_0(0));\ \mu[c_0]\mu^{-1} = [c_0]\} . \qquad \text{q.e.d.}$$

6. Appendix. The appendix will give a proof of Proposition in §4. First we prove the following lemma.

LEMMA. *Let X ane Y be C^∞-Hilbert manifold, X_1 (resp. Y_1) a C^∞-closed submanifold of X (resp. Y), and $f: X \to Y$ a C^∞-mapping with*

$f(X_1) \subset Y_1$. *Suppose that for* $x_0 \in X_1$, $T_{f(x_0)} Y_1 = d_0 f(T_{x_0} X_1)$, $(d_{x_0} f)^{-1}(T_{f(x_0)} Y_1) = T_{x_0} X_1$ and $\operatorname{Im} d_{x_0} f$ *is a closed subspace of* $T_{f(x_0)} Y$. *Then,* X_1 *is a neighborhood of* x_0 *in* $f^{-1}(Y_1)$, *or equivalently* $(X_1, x_0) = (f^{-1}(Y_1), x_0)$ *as germs of subset at* x_0.

PROOF. We put $Z = \operatorname{Im} d_{x_0} f / T_{f(x_0)} Y_1$. Without loss of generality we can assume that there exists a C^∞-mapping $p: Y \to Z$ such that $p(Y_1) = 0$ and the differential $d_{f(x_0)} p: \operatorname{Im} d_{x_0} f \to Z$ coincides with the natural projection, because the statement of lemma is local in its nature. Then, the composition: $X \xrightarrow{f} Y \xrightarrow{p} Z$ is of maximal rank, namely $d_{x_0}(p \circ f): T_{x_0} X \to Z$ is surjective. Hence the inverse image $(p \circ f)^{-1}(0)$ gives a germ of closed submanifold at x_0 with the tangent space $T_{x_0}(p \circ f)^{-1}(0) = \operatorname{Ker} d_{x_0}(p \circ f)$. On the other hand

$$\begin{aligned} (d_{x_0} f)^{-1}(\operatorname{Ker} d_{f(x_0)} p) &= (d_{x_0} f)^{-1}(\operatorname{Im} d_{x_0} f \cap \operatorname{Ker} d_{f(x_0)} p) \\ &= (d_{x_0} f)^{-1}(T_{f(x_0)} Y_1) \\ &= T_{x_0} X_1 . \end{aligned}$$

According to the inverse function theorem, X_1 is a neighborhood of x_0 in $(p \circ f)^{-1}(0)$. Since $X_1 \subset f^{-1}(Y_1) \subset (p \circ f)^{-1}(0)$, it follows that $(f^{-1}(Y_1), x_0) = (X_1, x_0)$. q.e.d.

We now return to the proof of proposition. Let U_c be an open neighborhood of 0 in J_c such that the restriction $\mu_c: U_c \to W^1(S^1, M)$ is injective. We shall show that $\mu(U_c)$ contains an open neighborhood of c in $\operatorname{Geo}(M)$. For this, we apply the above lemma as

$X =$ an open subset of $W^1(S^1, M)$ in which $\mu(U_c)$ is closed ,
$X_1 = \mu(U_c)$, $Y = T^* X$, $Y_1 = X \subset T^* X$ (as zero section)
$f = dE: X \to T^* X$, $x_0 = c$.

Notice that for $x \in X (\subset T^* X)$ there exists a canonical identification:

$$T_x T^* X = T_x^* X \oplus T_x X .$$

In order to check that the assumption of Lemma A is satisfied in our situation, it suffices to prove:

LEMMA B. *The differential of* $f = dE$ *at* x_0 *is given by*

$$d_{x_0} f(v) = (H(v), v) \in T_{x_0}^* X \oplus T_{x_0} X \qquad (v \in T_{x_0} X) ,$$

where $H: T_{x_0} X \to T_{x_0}^* X$ *is a homomorphism corresponding to the Hessian* $H(E)_{x_0}$ *via the identification* $\operatorname{Hom}(T_{x_0} X, T_{x_0}^* X) = T_{x_0}^* X \otimes T_{x_0}^* X$. *Furthermore,* $\operatorname{Im} H$ *is a closed subspace in* $T_{x_0}^* X$.

The first statement follows by direct computation. For the second, we identify $T^*_{x_0}X$ with $T_{x_0}X$ using the Riemann metric $\langle\ ,\ \rangle_1$. Then $H\colon T_{x_0}X \longrightarrow T_{x_0}X$ is given by

$$H = 1 - \left(1 - \frac{D^2}{dt^2}\right)^{-1}(R_{x_0} + 1)\,,$$

which is the self-adjoint Fredholm operator. In particular, $\operatorname{Im} H$ is closed. q.e.d.

REFERENCES

[1] R. BOTT, Non-degenerate critical manifolds, Ann. of Math., (2) 60 (1954), 248-261.
[2] H. I. ELIASSON, Morse theory for closed curves, Symp. on infinite dimensional topology, Ann. of Math. Studies, 69 (1972), 63-77.
[3] S. HELGASON, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.
[4] W. KLINGENBERG, Closed geodesics, Ann. of Math., 89 (1969), 68-91.
[5] W. MEYER, Kritische Mannigfaltigkeiten in Hilbertmannigfaltigkeiten, Math. Ann., 170 (1967), 45-66.
[6] J. C. MOORE, On a theorem of Borsuk, Fund. Math., 43 (1956), 195-201.
[7] R. S. PALAIS, Morse theory on Hilbert manifolds, Topology, 2 (1963), 299-340.
[8] R. S. PALAIS, Foundations of Global Non-Linear Analysis, Benjamin, 1968.

DEPARTMENT OF MATHEMATICS
NAGOYA UNIVERSITY
NAGOYA, JAPAN