Tôhoku Math. Journ.
31 (1979), 549-552.

CHANGES OF LAW, MARTINGALES AND THE CONDITIONED SQUARE FUNCTION

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(Received December 26, 1978)

Let (Ω, F, P) be a complete probability space, given an increasing sequence (F_n) of sub σ-fields of F such that $F = \bigvee_{n \geqq 0} F_n$. If $f = (f_n, F_n)$ is a martingale with difference sequence $d = (d_n)_{n \geqq 1}$, we shall set $f^* = \sup_{n \geqq 0} |f_n|$, $S(f) = (\sum_{n=1}^{\infty} d_n^2)^{1/2}$ and $s(f) = (\sum_{n=1}^{\infty} E[d_n^2 | F_{n-1}])^{1/2}$. Let us assume that $f_0 = 0$. The operator $s(f)$, which is not of matrix type, is called the conditioned square function. It was studied by Burkholder and Gundy [3]. Let $s_n(f) = (\sum_{k=1}^{n} E[d_k^2 | F_{k-1}])^{1/2}$. Clearly, $s_n(f)$ is F_{n-1}-measurable. Throughout the paper, we fix a BMO-martingale $M_n = \sum_{k=1}^{n} m_k$, $M_0 = 0$ such that $-1 + \delta < m_k$, $(k \geqq 1)$ for some constant δ with $0 < \delta \leqq 1$, and consider the process Z given by the formula $Z_n = \prod_{k=1}^{n} (1 + m_k)$, $Z_0 = 1$. Z is a positive uniformly integrable martingale which satisfies the condition

$$(A_p) \qquad Z_n E[Z_\infty^{-1/(p-1)} | F_n]^{p-1} \leqq C_p\,, \quad n \geqq 0$$

for some $p > 1$; see [6]. As $Z_\infty > 0$ a.s., the weighted probability measure $d\hat{P} = Z_\infty dP$ is equivalent to dP. Note that for every $\hat{P}$-integrable random variable Y

$$\hat{E}[Y | F_n] = E[Z_\infty Y | F_n]/Z_n \qquad \text{a.s., under } dP \text{ and } d\hat{P}\,,$$

where $\hat{E}$ denotes the expectation over Ω with respect to $d\hat{P}$.

Our aim is to prove the following:

THEOREM. *Let $0 < p \leqq 2$. Then the inequality*

$$(1) \qquad \hat{E}[(f^*)^p] \leqq c_p \hat{E}[s(f)^p]$$

is valid for all martingales $f = (f_n)$.

Furthermore, if $2 \leqq p < \infty$ and Z satisfies the (A_p) condition, then we have

$$(2) \qquad \hat{E}[s(f)^p] \leqq C_p \sup_{n \geqq 0} \hat{E}[|f_n|^p]\,.$$

Here, the choice of c_p and C_p depends only on p.

This result is well-known for the case where $Z \equiv 1$; see Theorem 5.3 of [3]. To prove the theorem, we need several lemmas, which will be stated without proof in the following. The letter C_p denotes a positive constant, not neccessarily the same number from line to line.

LEMMA 1. *If* $\{a_n\}$ *is a sequence of non-negative random variables, then for* $p \geqq 1$

$$E\left[\left(\sum_{n=1}^{\infty} E[a_n | F_n]\right)^p\right] \leqq p^p E\left[\left(\sum_{n=1}^{\infty} a_n\right)^p\right].$$

See Theorem 3.2 of [2].

LEMMA 2. *Let* $1 < p < \infty$. *If* Z *satisfies* (A_p), *then the inequality*

$$\hat{E}[(f^*)^p] \leqq C_p \sup_{n \geqq 0} \hat{E}[|f_n|^p]$$

is valid for all martingales $f = (f_n)$.

In our case, (A_p) implies $(A_{p-\varepsilon})$ for some $\varepsilon > 0$, and so this inequality follows from Theorem 2 of [5]. It is proved in [5] that the converse to Lemma 2 is true.

LEMMA 3. *Let* $1 \leqq p < \infty$. *If* $f = (f_n)$ *is a martingale, then*

$$c_p \hat{E}[S(f)^p] \leqq \hat{E}[(f^*)^p] \leqq C_p \hat{E}[S(f)^p].$$

For the proof, see [4]. The inequality corresponding to the continuous parameter case was obtained by Bonami and Lépingle [1] and Sekiguchi [8] independently.

LEMMA 4. *Let* $0 < \varepsilon \leqq 1$. *Then we have*

$$Z_n^{\varepsilon} \leqq C_{\varepsilon} E[Z_{\infty}^{\varepsilon} | F_n], \quad n \geqq 1.$$

This is Lemma 1 of [7]. It is easy to see that for a martingale $f = (f_n)$ with difference sequence $d = (d_n)$, the process $\hat{f} = (\hat{f}_n)$ defined by $\hat{f}_n = \sum_{k=1}^{n} d_k/(1 + m_k)$ is a martingale with respect to $d\hat{P}$. The following lemma is proved in [7].

LEMMA 5. *Let* $1 \leqq p < \infty$, *and set* $\hat{d}_k = d_k/(1 + m_k)$. *Then we have*

$$\hat{E}\left[\left(\sum_{k=1}^{\infty} \hat{d}_k^2\right)^{p/2}\right] \leqq C_p E\left[\left(\sum_{k=1}^{\infty} Z_{k-1}^{2/p} d_k^2\right)^{p/2}\right].$$

PROOF OF THEOREM. Let $s(\hat{f})$ denote the conditioned square function $(\sum_{k=1}^{\infty} \hat{E}[\hat{d}_k^2 | F_{k-1}])^{1/2}$ relative to $d\hat{P}$. Since $\delta < Z_k/Z_{k-1} = 1 + m_k \leqq \|M\|_{\mathrm{BMO}}$ and $\hat{E}[\hat{d}_k^2 | F_{k-1}] = E[d_k^2/(1 + m_k) | F_{k-1}]$, we have $(1 + \|M\|_{\mathrm{BMO}})^{-1/2} s(f) \leqq s(\hat{f}) \leqq \delta^{-1/2} s(f)$ and $\hat{E}[d_k^2 | F_{k-1}] \leqq (1 + \|M\|_{\mathrm{BMO}}) E[d_k^2 | F_{k-1}]$.

We strat with the case $0 < p \leqq 2$. From Lemma 3 it follows that

$$\hat{E}[(f^*)^2] \leqq C\hat{E}[S(f)^2] = C\hat{E}\Big[\sum_{n=1}^{\infty} \hat{E}[d_n^2 \mid F_{n-1}]\Big]$$
$$\leqq C\hat{E}\Big[\sum_{n=1}^{\infty} E[d_n^2 \mid F_{n-1}]\Big] = C\hat{E}[s(f)^2]\,.$$

Thus (1) is proved for $p = 2$. Now, let us consider the case $0 < p < 2$. Following the idea of Garsia, we define a martingale transform g by the formula $g_n = \sum_{k=1}^{n} s_k(f)^{(p-2)/2} d_k$. Then by the definition of $s(g)$

$$s(g)^2 = \sum_{n=1}^{\infty} s_n(f)^{p-2} E[d_n^2 \mid F_{n-1}] = \sum_{n=1}^{\infty} s_n(f)^{p-2}\{s_n(f)^2 - s_{n-1}(f)^2\}\,.$$

But, if $0 < a \leqq b$, then $b^{p-2}(b^2 - a^2) \leqq 2(b^p - a^p)/p$. This gives $s(g)^2 \leqq 2s(f)^p/p$. Therefore, $\hat{E}\,[(g^*)^2] \leqq C\hat{E}[s(g)^2] \leqq C_p\hat{E}[s(f)^p]$. On the other hand,

$$f_n = \sum_{k=1}^{n} s_k(f)^{1-p/2}(g_k - g_{k-1}) = g_n s_n(f)^{1-p/2} - \sum_{k=1}^{n} g_k\{s_k(f)^{1-p/2} - s_{k-1}(f)^{1-p/2}\}$$

and so $f^* \leqq 2g^* s(f)^{1-p/2}$. Then we apply Hölder's inequality with exponents $2/p$ and $2/(2-p)$:

$$\hat{E}[(f^*)^p] \leqq 2^p\hat{E}[(g^*)^p s(f)^{p(1-p/2)}] \leqq 2^p\hat{E}[(g^*)^2]^{p/2}\hat{E}[s(f)^p]^{1-p/2}$$
$$\leqq C_p\hat{E}[s(f)^p]^{p/2}\hat{E}[s(f)^p]^{1-p/2} \leqq C_p\hat{E}[s(f)^p]\,.$$

Thus the desired inequality (1) is obtained.

Next we deal with the case $2 \leqq p < \infty$. Let us assume that Z satisfies (A_p); namely, the weighted norm inequality stated in Lemma 2 holds. As $s(f) \leqq cs(\hat{f})$, we have $\hat{E}[s(f)^p] \leqq C_p\hat{E}[s(\hat{f})^p] \leqq C_p\hat{E}[(\hat{f}^*)^p]$; the right hand side inequality is well-known. See Theorem 5.3 (i) of [3]. By Lemmas 4 and 5 we have

$$\hat{E}[(\hat{f}^*)^p] \leqq C_pE\Big[\Big(\sum_{n=1}^{\infty} Z_{n-1}^{2/p} d_n^2\Big)^{p/2}\Big] \leqq C_pE\Big[\Big(\sum_{n=1}^{\infty} Z_n^{2/p} d_n^2\Big)^{p/2}\Big]$$
$$\leqq C_pE\Big[\Big(\sum_{n=1}^{\infty} E[Z_\infty^{2/p} d_n^2 \mid F_n]\Big)^{p/2}\Big]$$

and by Lemma 1 the expectation on the right hand side is smaller than $E[(\sum_{n=1}^{\infty} Z_\infty^{2/p} d_n^2)^{p/2}] = \hat{E}[(\sum_{n=1}^{\infty} d_n^2)^{p/2}]$. Then this combined with Lemmas 2 and 3 yields (2) as desired. Thus the theorem is established.

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