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RESOLUTIONS OF GENERALIZED POLYHEDRAL MANIFOLDS

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Introduction. This paper is concerned with the existence and classification of resolutions of polyhedra, satisfying a certain link condition, to PL-manifolds. The starting points of this theory are the existence theory of Cohen [9], Martin [18] and others for homology manifolds and the subsequent classification theorem of Edmonds and Stern [10]. The purpose of this paper is to generalize these results to other classes of manifolds (e.g. rational homology, Euler).

An obstruction theory for blowing up singularities of geometric cycles has been described by Sullivan [27] and our obstruction theory is modeled on this. Resolutions are classified via a space constructed by the theory of Brownian functors.

Generalized polyhedral manifolds are defined as follows. Let $\mathscr{F}$ be a collection of compact polyhedra containing S^0 and closed under link and join. Elements of $\mathscr{F}$ of dimension n are called $\mathscr{F}$ n-spheres; polyhedra of the form $\Sigma^n - \mathring{S}t(x, \Sigma)$, $\Sigma \in \mathscr{F}$, are called $\mathscr{F}$ n-pseudodiscs. The elementary theory of $\mathscr{F}$-manifolds parallels the theory of PL-manifolds, using $\mathscr{F}$-spheres and pseudodiscs instead of PL-spheres and discs. Standard topics, such as regular neighborhoods, handle decompositions and orientability, are developed in §1.

An $\mathscr{F}$-resolution is defined to be a simplicial map whose dual cells are $\mathscr{F}$-pseudodiscs. These are discussed at length in §2. In §§3 and 4, we develop the obstruction theory for finding an $\mathscr{F}$-resolution of an $\mathscr{F}$-manifold from a PL-manifold.

In §§5 and 6, we construct a space which classifies concordance classes of $\mathscr{F}$-resolutions of PL-manifolds (with certain conditions on $\mathscr{F}$). This is done using the product structure theorem of [10] and Adams' representability theorem [1].

1. Generalized manifolds.

Let $\mathscr{F} = \bigcup_{n=0}^{\infty} \mathscr{F}_n$, where each $\mathscr{F}_n$ is a set of PL-isomorphism classes of n-dimensional compact polyhedra satisfying

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(i) $S^0 \in \mathscr{F}_0$

(ii) if $\Sigma \in \mathscr{F}_n$ and $x \in \Sigma$, then $Lk(x, \Sigma) \in \mathscr{F}_{n-1}$

and (iii) if $\Sigma_1 \in \mathscr{F}_n$ and $\Sigma_2 \in \mathscr{F}_m$, then $\Sigma_1 * \Sigma_2 \in \mathscr{F}_{n+m+1}$.

We interpret "$K \in \mathscr{F}$" to mean that K is a polyhedron whose PL-isomorphism class is in $\mathscr{F}$. The set $\mathscr{F}$ is called a *manifold class.*

The elements of $\mathscr{F}_n$ are called the $\mathscr{F}$ *n-spheres.* The cone $c\Sigma$ of an $\mathscr{F}(n-1)$-sphere is called an $\mathscr{F}$ *n-disc,* and a polyhedron Δ of the form $\Sigma - \mathring{S}t(x, \Sigma)$, $\Sigma \in \mathscr{F}_n$, $x \in \Sigma$, is called an $\mathscr{F}$ *n-pseudodisc*; $\partial\Delta = Lk(x, \Sigma)$, $\mathrm{Int}(\Delta) = \Delta - \partial\Delta$. Note that an $\mathscr{F}$ n-disc $c\Sigma$ is an $\mathscr{F}$ n-pseudodisc, since $c\Sigma = S^0 * \Sigma - \mathring{S}t(1, S^0 * \Sigma)$.

A polyhedron M is said to be a PL $\mathscr{F}$ *n-manifold* if M has a subdivision M' so that each $Lk(x, M')$ is either an $\mathscr{F}$ $(n-1)$-sphere or $(n-1)$-pseudodisc, and $\partial M = \{x \in M' : Lk(x, M') \notin \mathscr{F}_{n-1}\}$, the *boundary* of M, is a PL $\mathscr{F}(n-1)$-manifold without boundary.

We list some important examples of manifold classes:

(1) $\mathscr{P}\ell$: Define $\mathscr{P}\ell_n = \{S^n\}$; clearly $\mathscr{P}\ell \subset \mathscr{F}$ for any manifold class and a PL $\mathscr{P}\ell$ n-manifold is simply a PL n-manifold. (The symbol $\mathscr{F}$ is usually omitted when $\mathscr{F} = \mathscr{P}\ell$.)

(2) $\mathscr{H}_K$: Let K be a set of primes and $\Lambda = \mathbf{Z}[1/p : p \in K] \subset \mathbf{Q}$. Define $(\mathscr{H}_K)_n$ inductively by $(\mathscr{H}_K)_0 = \{S^0\}$, $(\mathscr{H}_K)_n = \{\Sigma : Lk(x, \Sigma) \in (\mathscr{H}_K)_{n-1}, H_*(\Sigma; \Lambda) \cong H_*(S^n; \Lambda)\}$. A PL $\mathscr{H}_K$-manifold is usually called a *Λ-homology manifold.*

(3) $\mathscr{h}_K$: As in (2), let $(\mathscr{h}_K)_n = \{S^n\}$ $n = 0, 1, 2$, $(\mathscr{h}_K)_n = \{\Sigma \in (\mathscr{H}_K)_n : Lk(x, \Sigma) \in (\mathscr{h}_K)_{n-1}, \pi_1(\Sigma) = 0\}$, $n \geqq 3$. A PL $\mathscr{h}_K$-manifold is called a *Λ-homotopy manifold.*

(4) $\mathscr{E}$: Define $\mathscr{E}_0 = \{S^0\}$, $\mathscr{E}_n = \{\Sigma : Lk(x, \Sigma) \in \mathscr{E}_{n-1}, \chi(\Sigma) = 1 + (-1)^n\}$. A PL $\mathscr{E}$-manifold is called an *Euler manifold.* (See [11], [26], [28].)

(5) $\mathscr{E}^{(2)}$: Define $\mathscr{E}_0^{(2)} = \{(S^0)^t : t = 1, 2, \cdots\}$, $\mathscr{E}_n^{(2)} = \{\Sigma : Lk(x, \Sigma) \in \mathscr{E}_{n-1}^{(2)}, \chi(M) \equiv 0 \bmod(2)\}$. A PL $\mathscr{E}^{(2)}$-manifold is called a mod(2) *Euler manifold.* By [26], any real algebraic variety is a PL $\mathscr{E}^{(2)}$-manifold.

Clearly, if $\mathscr{F} \subset \mathscr{F}'$ are manifold classes, then a PL $\mathscr{F}$-manifold is a PL $\mathscr{F}'$-manifold. Also, if $\Delta = \Sigma - \mathring{S}t(x, \Sigma)$ is an $\mathscr{F}$ n-pseudodisc, then Δ is a PL $\mathscr{F}$ n-manifold.

LEMMA 1.1. *If M and N are* PL *$\mathscr{F}$-manifolds with collared boundaries, then $M \times N$ is a* PL *$\mathscr{F}$-manifold with boundary $M \times \partial N \cup \partial M \times N$.*

PROOF. Let $(x, y) \in M \times N$. Then

$$Lk((x, y), M \times N) \cong Lk(x, M) * Lk(y, N)$$

and is in $\mathscr{F}$ if $x \notin \partial M$, $y \notin \partial N$ by condition (iii) above. If $x \in \partial M$, then

$Lk(x, M) \cong cLk(x, \partial M)$ since ∂M is collared in M. Therefore, if $y \notin \partial N$, $Lk(x, M) * Lk(y, N) \cong c(Lk(x, M) * Lk(y, N))$ is an $\mathscr{F}$-disc, and if $y \in \partial N$, $Lk(x, M) * Lk(y, N) \cong cLk(x, \partial M) * cLk(y, \partial N)$ is easily seen to be PL-homeomorphic to the star of a vertex of S^1 in $S^1 * Lk(x, \partial M) * Lk(y, \partial N)$, and so is also an $\mathscr{F}$-disc.

By a similar proof, we have:

LEMMA 1.2. *If M and N are* PL *$\mathscr{F}$-manifolds with $\partial M = \partial N$ collared in M and N, then $M \cup N$ is a* PL *$\mathscr{F}$-manifold.*

We will need this result for manifolds without collared boundaries, and to get this we must restrict our manifold classes.

A manifold class $\mathscr{F}$ is said to be *connected* if for each pair Δ_1, Δ_2 of *$\mathscr{F}$ n-pseudodiscs* with $\partial\Delta_1 = \partial\Delta_2$, $\Delta_1 \cup \Delta_2 \in \mathscr{F}_n$. It follows that if $\mathscr{F}$ is connected, and M, N are PL $\mathscr{F}$-manifolds with $\partial M = \partial N$, then $M \cup N$ is a PL $\mathscr{F}$-manifold. Notice that all the manifold classes listed above are connected.

Let M be a PL $\mathscr{F}$-manifold and K a compact subpolyhedron of the interior of M. Define a *regular neighborhood* of K in M to be the simplicial neighborhood of K in some derived subdivision of M.

PROPOSITION 1.3. *A regular neighborhood of K in M^n is a compact* PL *$\mathscr{F}$ n-manifold with boundary, unique up to ambient* PL-*isotopy* rel K.

PROOF. Assume M has been subdivided, and let N be the simplicial neighborhood of K in M. Define $\mathrm{Int}(N)$ to be the union of the interiors of the simplexes in M that meet K and ∂N to be the union of the simplexes in N that do not meet K.

For $x \in \mathrm{Int}(N)$, choose a simplex α so that $x \in \mathrm{Int}(\alpha)$ and $\alpha \cap K \neq \emptyset$. Let v be a vertex common to α and K. Then $St(v, N) = St(v, M)$, $x \in \mathrm{Int}(St(v, N))$, and therefore $Lk(x, N) \in \mathscr{F}_{n-1}$. By [8], Theorem 5.3, ∂N is bicollared in M, and it follows that $Lk(x, M) \overset{\mathrm{PL}}{\cong} S(Lk(x, \partial N))$ and $Lk(x, N) \overset{\mathrm{PL}}{\cong} c(Lk(x, \partial N))$ for $x \in \partial N$. Therefore ∂N is a PL $\mathscr{F}(n-1)$-manifold and $Lk(x, N)$ is an $\mathscr{F}$ n-disc. Thus N is a PL $\mathscr{F}$ n-manifold. Uniqueness follows from [8].

A *handle of index q* on a PL $\mathscr{F}$ n-manifold M is an $\mathscr{F}$ n-disc Δ so that $\Delta \cap M \subset \partial M$, together with a PL-homeomorphism $f : \Delta_1^q \times \Delta_2^{n-q} \to \Delta$, where $\Delta_i = c\Sigma_i$ is an $\mathscr{F}$-disc, $i = 1, 2$, so that $f(\Sigma_1 \times \Delta_2) = \Delta \cap M$. We define a handle decomposition of M in the usual way.

LEMMA 1.4. *Let M^n be a compact* PL $\mathscr{F}$*-manifold. Then $Lk(\alpha, M) \in \mathscr{F}_{n-i-1}$ for each i-simplex α of $M - \partial M$.*

PROOF. This follows easily by induction, since $Lk(\alpha^0, M) \in \mathscr{F}_{n-1}$ by definition, and if $\alpha^i = v*\alpha^{i-1}$, then $Lk(\alpha^i, M) = Lk(v, Lk(\alpha^{i-1}, M))$.

PROPOSITION 1.5. *Let M^n be a compact* PL $\mathscr{F}$*-manifold. Then M has a handle decomposition.*

PROOF. Assume $\partial M \neq \varnothing$ by replacing M by $M - \mathring{\Delta}^n$ for some n-simplex Δ^n in M if $\partial M = \varnothing$. Let $\alpha_1, \alpha_2, \cdots, \alpha_k$ be the simplexes of $M - \partial M$, arranged in order of non-decreasing dimension.

Let N be a regular neighborhood of ∂M in the second derived M''. Let $H_i = St(b_{\alpha_i}, M'')$; then $M = N \cup H_1 \cup \cdots \cup H_k$. Let $W_0 = N$, $W_i = W_{i-1} \cup H_j$. We have $H_i = St(b_{\alpha_i}, \alpha_i'')*\partial D(\alpha_i, M'')$, and therefore is PL-homeomorphic to $D^q*Lk(\alpha_i, M'')$, $q = \dim(\alpha_i)$. Let $\Sigma_i = Lk(\alpha_i, M'') \in \mathscr{F}_{n-q-1}$. It follows that there exists a PL-homeomorphism $f: D^q \times c\Sigma_i \to H_i$ so that $f(D^q \times \Sigma_i) = H_i \cap W_{i-1}$. Therefore, M has a handle decomposition.

Let M^n be a compact $\mathscr{F}$-manifold. We recall the classical definition of orientability of [17]. The pair $(M, \partial M)$ is called an *n-circuit* if $H_n(M, \partial M; \boldsymbol{Z}/2) = \boldsymbol{Z}/2$, generated by the sum of all n-simplexes of M; $(M, \partial M)$ is a *simple n-circuit* if, in addition, every $(n-1)$-simplex of $M - \partial M$ is the face of exactly two n-simplexes. By [17], if $(M, \partial M)$ is a simple n-circuit, then either $H_n(M, \partial M) = 0$ (in which case $(M, \partial M)$ is called non-orientable) or $H_n(M, \partial M) = \boldsymbol{Z}$ ($(M, \partial M)$ is orientable) and each n-simplex σ may be oriented with sign $(-1)^\sigma$ so that $\sum (-1)^\sigma \sigma$ generates $H_n(M, \partial M)$.

THEOREM 1.6. *Suppose $\tilde{H}_0(Lk(\alpha^i, M)) \cong \tilde{H}_0(S^{n-i-1})$ for each i-simplex α of $M - \partial M$ and $\tilde{H}_0(Lk(\alpha^i, M)) = 0$, $\tilde{H}_0(Lk(\alpha^i, \partial M)) \cong \tilde{H}_0(S^{n-i-2})$ for each i-simplex α of ∂M. Then $(M, \partial M)$ is a simple n-circuit.*

PROOF. First assume $\partial M = \varnothing$ and proceed by induction on n. If $n = 0, 1$ or 2, then M is a PL-manifold by the condition in the hypothesis. Assume the result for $n - 1$. Then for each $x \in M$, $Lk(x, M)$ is a simple $(n-1)$-circuit, so that $H_{n-1}(Lk(x, M); \boldsymbol{Z}/2) = \boldsymbol{Z}/2$. The result now follows by the usual proof in the $\mathscr{F} = \mathscr{H}$ case (e.g., [19], Theorem 5.3.3). Compare Kato [13], Lemma 3.1.

If $\partial M \neq \varnothing$, then let DM denote the double of M. Note that DM need not be an $\mathscr{F}$-manifold, but the proof above shows that DM is a simple n-circuit since for a simplex α of ∂M, $Lk(\alpha, DM) = DLk(\alpha, M)$. Again by the first case, ∂M is a simple n-circuit and it follows easily that $(M, \partial M)$ is a simple n-circuit.

Let us generalize the condition of Theorem 1.6 and say that $(M, \partial M)$ is a PL $\mathscr{F}^{(k)}$-manifold, $k=0, 1, \cdots, n$, if for each i-simplex α of $M-\partial M$, $\tilde{H}_j(Lk(\alpha, M)) \cong \tilde{H}_j(S^{n-i-1})$ for $j \leqq k$ and for each i-simplex α of ∂M, $\tilde{H}_j(Lk(\alpha, M)) = 0$, $\tilde{H}_j(Lk(\alpha, \partial M)) \cong \tilde{H}_j(S^{n-i-2})$ for $j \leqq k$. Thus a PL $\mathscr{F}^{(0)}$-manifold is a relative simple n-circuit and a PL $\mathscr{F}^{(n)}n$-manifold is an $\mathscr{H}$-manifold. A PL $\mathscr{F}^{(0)}$-manifold M is *orientable* if M and ∂M are orientable as simple circuits. (Compare [24] and [27].)

A PL $\mathscr{F}^{(0)}$-manifold M is said to be *locally orientable* if for each x, $Lk(x, M)$ is orientable. Note that $Lk(x, M)$ is also a PL $\mathscr{F}^{(0)}$-manifold, and it follows easily that M is locally orientable if and only if each $Lk(\alpha, M)$ is orientable.

Let M^n be a closed, locally orientable PL $\mathscr{F}^{(0)}$-manifold and $\pi(M)$ its fundamental groupoid. We can define a local coefficient system $\Gamma_M : \pi(M) \to \mathscr{A}\!\ell$, by the usual correspondence $x \to H_{n-1}(Lk(x, M))$. Let $H_*^t(M)$ denote the homology of M with respect to this local system. We have the following *partial Poincaré duality theorem* (compare Kato [14]).

THEOREM 1.7. *If M^n is a closed, locally orientable* PL $\mathscr{F}^{(k)}$*-manifold, then $H^i(M) \cong H_{n-i}^t(M)$ for $i \geqq n-k$.*

PROOF. Let M_j be the union of the dual cells of M of dimension $\leqq j$, $D(\sigma^{n-l}, M)$, $l < j$. Define $\bar{C}_j = H_j^t(M_j, M_{j-1})$ and $d_j : \bar{C}_j \to \bar{C}_{j-1}$ by the composition $H_j^t(M_j, M_{j-1}) \to H_{j-1}^t(M_{j-1}) \to H_{j-1}^t(M_{j-1}, M_{j-2})$. By the proof of Poincaré duality for $\mathscr{H}$-manifolds, there is a chain isomorphism $C^*(M) \to \bar{C}_{n-*}$. Since M is $\mathscr{F}^{(k)}$, it follows that $H_l^t(M_j, M_{j-1}) = 0$ if $l > j$ or $l < \min\{j, k\}$. By the proof of [19], Theorem 4.4.14, $H_j(\bar{C}_*) \cong H_j^t(M)$ for $j \leqq k$. Therefore, for $i \geqq n-k$, $H^i(M) \cong H_{n-i}(\bar{C}_*) \cong H_{n-i}^t(M)$.

There are corresponding statements concerning Lefshetz duality and duality with coefficients in a ring which we leave to the reader to state. We close this section with a characterization of 3-manifolds.

PROPOSITION 1.8. *Let M^3 be a closed, connected $\mathscr{F}^{(0)}$-manifold. Then M is a* PL *3-manifold if and only if $\chi(M) = 0$.*

The proof follows from Wall [28]; see also Seifert and Threlfall [23], §60.

2. **$\mathscr{F}$-Resolutions.** Let $\mathscr{F}$ be a manifold class and M^n, N^n PL $\mathscr{F}$-manifolds. A proper PL-surjection $f : N \to M$ is said to be an *$\mathscr{F}$-resolution* if for every $x \in M$, there exist subdivisions N', M' so that $f : N' \to M'$ is simplicial, x is a vertex of M' and the regular neighborhood $N(f^{-1}(x), N')$ of $f^{-1}(x)$ in N' is an $\mathscr{F}$-pseudodisc. If $\partial M, \partial N \neq \emptyset$, we assume $f \mid \partial N : \partial N \to \partial M$ is also an $\mathscr{F}$-resolution.

EXAMPLES. (1) $\mathscr{F} = \mathscr{K}$: f is an $\mathscr{K}$-resolution if and only if each $f^{-1}(x)$ is contractible, and is called a contractible resolution ([9]).

(2) $\mathscr{F} = \mathscr{H}_K : f$ is an $\mathscr{H}_K$-resolution if and only if each $\tilde{H}_*(f^{-1}(x); \Lambda) = 0$, and is called a Λ-acyclic resolution ([16], [18], [27]).

(3) $\mathscr{F} = \mathscr{E} : f$ is an $\mathscr{E}$-resolution if and only if each $\chi(f^{-1}(x)) = 1$, and is called an Euler resolution ([16]); the case $\mathscr{F} = \mathscr{E}^{(2)}$ is similar.

We write $N \overset{\mathscr{F}}{\searrow} M$ if there is an $\mathscr{F}$-resolution $f: N \to M$ and $N \underset{\mathscr{F}}{\wedge} M$ if there is a sequence $N = N_0, N_1, \cdots, N_k = M$ so that for each i, $N_i \overset{\mathscr{F}}{\searrow} N_{i+1}$ or $N_{i+1} \overset{\mathscr{F}}{\searrow} N_i$.

LEMMA 2.1. *Let $\Sigma \in \mathscr{F}_n$. Then $\Sigma \overset{\mathscr{F}}{\searrow} S^n$.*

PROOF. Choose x in the interior of an n-simplex σ of Σ, and let A be the complement of an open star neighborhood of x. Then A is an $\mathscr{F}$-pseudodisc and $f : \Sigma \to \Sigma/A \cong S^n$ is an $\mathscr{F}$-resolution. (We may choose f to be PL by assuming $\partial\sigma$ has a PL collar in Σ.)

LEMMA 2.2. *If Δ^n is an $\mathscr{F}$-pseudodisc, then $\Delta^n \overset{\mathscr{F}}{\searrow} D^n$.*

PROOF. Choose x in the interior of an $(n-1)$-simplex of $\partial\Delta$ and define A as in Lemma 2.1. Then $A \cong \Delta$ and $\Delta/A \cong D^n$, $\partial\Delta/A \cap \partial\Delta \cong S^{n-1}$.

Let M, N be PL $\mathscr{F}$-manifolds and $K \subset M$ a subcomplex. A map $f: N \to M$ is an *$\mathscr{F}$-resolution* $\mathrm{rel}(K)$ if f is an $\mathscr{F}$-resolution and $f \mid f^{-1}(K) : f^{-1}(K) \to K$ is a PL-homeomorphism. We define the symbols $N \overset{\mathscr{F}}{\searrow} M \,\mathrm{rel}(K)$, $N \underset{\mathscr{F}}{\wedge} M \,\mathrm{rel}(K)$ as before.

LEEMA 2.3. *Let Δ^n be an $\mathscr{F}$-pseudodisc. Then $\Delta^n \overset{\mathscr{F}}{\searrow} c(\partial\Delta)\,\mathrm{rel}\,(\partial\Delta)$.*

PROOF. The proof is immediate by [20], Lemma 3.1.

LEMMA 2.4. *Let M be a* PL *$\mathscr{F}$-manifold. Then $M \cup \partial M \times I \overset{\mathscr{F}}{\searrow} M$.*

The proof is obvious.

We now consider two additional axioms for a manifold class: $\mathscr{F}$ is said to be a *homotopy class* if every compact, contractible PL $\mathscr{F}$-manifold is an $\mathscr{F}$-pseudodisc; $\mathscr{F}$ is *connected* if for every $\Sigma \in \mathscr{F}_n$ and $\mathscr{F}$ n-pseudodisc Δ embedded as a full subcomplex of Σ, $\Sigma - \mathrm{Int}(\Delta)$ is an $\mathscr{F}$-pseudodisc. For example, $\mathscr{K}_K$, $\mathscr{H}_K$, $\mathscr{E}$, and $\mathscr{E}^{(2)}$ are connected homotopy classes.

An $\mathscr{F}$-resolution $f: N^n \to M^n$ is said to be a *strong $\mathscr{F}$-resolution* if either $n = 0$ or, as in the definition of $\mathscr{F}$-resolutions, each $f \mid \partial N(f^{-1}(x), N')$: $\partial N(f^{-1}(x), N') \to \partial N(x, M')$ is a strong $\mathscr{F}$-resolution. Any $\mathscr{H}_K$-resolution is a strong $\mathscr{H}_K$-resolution.

PROPOSITION 2.5. *Let M^n, N^n be* PL *$\mathscr{F}$-manifolds and $f: N \to M$ a proper, simplicial surjection, where $\mathscr{F}$ is a connected, coconnected homotopy class. Then f is a strong $\mathscr{F}$-resolution if and only if the simplicial mapping cylinder C_f is a* PL *$\mathscr{F}$-manifold.*

PROOF. Let $A_n(f)$ denote the statement "f is a strong $\mathscr{F}$-resolution" and $B_n(f)$ the statement "C_f is a PL $\mathscr{F}$-manifold". We show that $C_n = \forall f(A_n(f) \Leftrightarrow B_n(f))$ is true by induction on n.

C_0: This is obvious since for $x \in M$, $f^{-1}(x)$ is its own regular neighborhood and $Lk(x, C_f) = f^{-1}(x)$.

$(C_{n-1}, A_n(f)) \Rightarrow B_n(f)$: Assume $f: N^n \to M^n$ is a strong $\mathscr{F}$-resolution. To show that C_f is a PL $\mathscr{F}$-manifold we need only show that $Lk(x, C_f)$ is an $\mathscr{F}$-pseudodisc for each vertex x of M.

We have $N(f^{-1}(x), N) = D(x, f) = N \cap Lk(x, C_f)$ by [7], and $Lk(x, C_f) = D(x, f) \cup C_{f_0}$, where $f_0 = f \mid \partial D(x, f) : \partial D(x, f) \to \partial D(x, M)$. Since f is a strong $\mathscr{F}$-resolution, f_0 is also, so that C_{f_0} is a PL $\mathscr{F}(n-1)$-manifold by C_{n-1}.

Since $\mathscr{F}$ is connected, $C_{f_0} \cup D(x, M)$ is a PL $\mathscr{F}$-manifold, and in fact an $\mathscr{F}$-pseudodisc since it is contractible and $\mathscr{F}$ is a homotopy class. Again by connectedness, $Lk(x, C_f) \cup D(x, M) \in \mathscr{F}_{n-1}$. Therefore $Lk(x, C_f)$ is an $\mathscr{F}$-pseudodisc. $(C_{n-1}, B_n(f)) \Rightarrow A_n(f)$: Assume C_f is a PL $\mathscr{F}$-manifold, so that for $x \in M$, $Lk(x, C_f)$ is an $\mathscr{F}$-pseudodisc. As before, $Lk(x, C_f) = D(x, f) \cup C_{f_0}$, and since $D(x, f)$ is a PL $\mathscr{F}$-manifold, C_{f_0} is also. By C_{n-1}, f_0 is a strong $\mathscr{F}$-resolution.

Again, $\mathscr{F}$ is a homotopy class, so that $C_{f_0} \cup D(x, M)$ is an $\mathscr{F}$-pseudodisc, and so $D(x, f)$ is an $\mathscr{F}$-pseudodisc, since $\mathscr{F}$ is coconnected and $Lk(x, C_f) \cup D(x, M) \in \mathscr{F}_{n-1}$. Therefore, f is an $\mathscr{F}$-resolution. Since each f_0 above is a strong $\mathscr{F}$-resolution, so is f.

COROLLARY 2.6. *If $\mathscr{F}$ is a connected, coconnected homotopy class, then the composition of strong $\mathscr{F}$-resolutions is a strong $\mathscr{F}$-resolution.*

Let $f: N \to M$ be a strong $\mathscr{F}$-resolution. We say that f is a PL *$\mathscr{F}$-resolution* if N is a PL-manifold. Two PL $\mathscr{F}$-resolutions $f_i: N_i \to M$ $i = 1, 2$ are *concordant* if there exists a PL $\mathscr{F}$-resolution $F: W \to M \times I$ defining a cobordism between f_1 and f_2. It is easily checked that concordance is an equivalence relation.

Let M be a PL $\mathscr{F}$-manifold, and define $\mathscr{R}_{\mathscr{F}}(M)$ to be the set of concordance classes of PL $\mathscr{F}$-resolutions $f: N \to M$. Much of the remainder of this paper is devoted to the following questions: Is $\mathscr{R}_{\mathscr{F}}(M) \neq \emptyset$? (Existence of PL $\mathscr{F}$-resolutions.) If so, compute $\mathscr{R}_{\mathscr{F}}(M)$. (Classification of PL $\mathscr{F}$-resolutions.)

3. Groups of PL $\mathscr{F}$-spheres. Let $\mathscr{F}$ be a manifold class. Define a PL $\mathscr{F}$ *n-sphere* to be a closed, oriented PL manifold Σ that belongs to $\mathscr{F}_n$. If Σ_1, Σ_2 are PL $\mathscr{F}$ n-spheres, we say that Σ_1, Σ_2 are PL $\mathscr{F}$-*cobordant* if there is a compact, oriented PL-manifold W with $\partial W = \Sigma_1 \cup (-\Sigma_2)$ so that $c\Sigma_1 \cup W \cup c\Sigma_2 \in \mathscr{F}_{n+1}$.

LEMMA 3.1. *If $\mathscr{F}$ is connected, then the relation of* PL *$\mathscr{F}$-cobordism is an equivalence relation.*

PROOF. Suppose W_1 and W_2 are PL $\mathscr{F}$-cobordisms between Σ_1, Σ_2 and Σ_2, Σ_3. Then $c\Sigma_1 \cup W_1$ and $W_2 \cup c\Sigma_3$ are $\mathscr{F}$-pseudodiscs, and so since $\mathscr{F}$ is connected, $c\Sigma_1 \cup W_1 \cup W_2 \cup c\Sigma_3 \in \mathscr{F}_{n+1}$. Therefore $W_1 \cup W_2$ is a PL $\mathscr{F}$-cobordism between Σ_1 and Σ_3.

If $\mathscr{F}$ is connected, we let $\theta_n^{\mathscr{F}}$ denote the set of $\mathscr{F}$-cobordism classes of connected PL $\mathscr{F}$ n-spheres, $n > 0$. (See [3], [5] for the cases $\mathscr{F} = \mathscr{H}_K, \mathscr{h}_K, \mathscr{E}, \mathscr{E}^{(2)}$.)

PROPOSITION 3.2. *$\theta_n^{\mathscr{F}}$ is an abelian group under the operation of connected sum.*

PROOF. Let Σ_1, Σ_2 be PL $\mathscr{F}$ n-spheres, and $\sigma_1 \subset \Sigma_1, \sigma_2 \subset \Sigma_2$ top dimensional simplexes. Then $\Sigma_1 - \mathring{\sigma}_1, \Sigma_2 - \mathring{\sigma}_2$ are $\mathscr{F}$-pseudodiscs, and so $\Sigma_1 \# \Sigma_2 = (\Sigma_1 - \mathring{\sigma}_1) \cup (\Sigma_2 - \mathring{\sigma}_2)$, identified along $\partial\sigma_1 \cong \partial\sigma_2$, is a $\mathscr{F}$ n-sphere since $\mathscr{F}$ is connected.

The PL $\mathscr{F}$-cobordism class of $\Sigma_1 \# \Sigma_2$ depends in fact, only on the PL $\mathscr{F}$-cobordism classes of Σ_1, Σ_2. To see this, let W_1, W_2 be PL $\mathscr{F}$-cobordisms from Σ_1, Σ_2 to Σ_1', Σ_2'. Choose simplicially embedded paths α_1, α_2 in W_1, W_2 joining a vertex in Σ_1, Σ_2 to a vertex in Σ_1', Σ_2', so that $\alpha_1 \cap \Sigma_1, \alpha_1 \cap \Sigma_1', \alpha_2 \cap \Sigma_2, \alpha_2 \cap \Sigma_2'$ are singletons. Let R_1, R_2 be closed regular neighborhoods of α_1, α_2 in W_1, W_2 and define $V_1 = W_1 - (\mathrm{Int}(R_1) \cup \mathrm{Int}(R_1 \cap \partial W_1))$, $V_2 = W_2 - (\mathrm{Int}(R_2) \cup \mathrm{Int}(R_2 \cap \partial W_2))$. Then $U = V_1 \cup V_2$, with the obvious identifications, is a PL cobordism between $\Sigma_1 \# \Sigma_2$ and $\Sigma_1' \# \Sigma_2'$. Finally, $c(\Sigma_1 \# \Sigma_2) \cup U \cup c(\Sigma_1' \# \Sigma_2')$ is PL-homeomorphic to $(c\Sigma_1 \cup W_1 \cup c\Sigma_1' - \mathring{\tau}_1) \cup (c\Sigma_2 \cup W_2 \cup c\Sigma_2' - \mathring{\tau}_2)$, where $\tau_1 = c(\partial(R_1 \cap \Sigma_1)) \cup R_1 \cup c(\partial(R_1 \cap \Sigma_1'))$, $\tau_2 = c(\partial(R_2 \cap \Sigma_2)) \cup R_2 \cup c(\partial(R_2 \cap \Sigma_2'))$, and is therefore in $\mathscr{F}_{n+1}$, since it is a union of 2 $\mathscr{F}$-pseudodiscs along their common boundary.

Thus $\#$ defines an abelian semigroup operation on $\theta_n^{\mathscr{F}}$, and clearly S^n

represents an identity element. We show that $\Sigma \#(-\Sigma)$ is PL $\mathscr{F}$-cobordant to S^n, which shows that $\theta_n^{\mathscr{F}}$ is a group. Choose $x \in \Sigma$ and $y \in \Sigma - St(x, \Sigma)$ so that $Lk(y, \Sigma) \subset \Sigma - St(x, \Sigma)$. Then (subdividing if necessary) $\Sigma \times I - (\mathring{S}t(x, \Sigma) \times I \cup Lk((y, 1/2), \Sigma \times I))$ is the desired PL $\mathscr{F}$-cobordism.

COROLLARY 3.3. *If Σ is a* PL *$\mathscr{F}$ n-sphere, $n > 0$, and $\Sigma_1, \cdots, \Sigma_k$ the connected components of Σ, then Σ and $\Sigma_1 \# \cdots \# \Sigma_k$ are* PL *$\mathscr{F}$-cobordant.*

Define a PL *$\mathscr{F}$-pseudodisc* to be a PL-manifold that is also an $\mathscr{F}$-pseudodisc. Clearly, if Σ^n is an orientable PL $\mathscr{F}$-sphere, then $[\Sigma] = 0$ in $\theta_n^{\mathscr{F}}$ if and only if Σ bounds an orientable PL $\mathscr{F}$-pseudodisc.

4. **Existence theory for PL $\mathscr{F}$-resolutions.** Let $\mathscr{F}$ be a connected coconnected homotopy class and M^n a compact connected locally orientable PL $\mathscr{F}^{(0)}$-manifold. In this section, we develop an obstruction theory to determine when $\mathscr{R}_{\mathscr{F}}(M) \neq \varnothing$. The methods are due to Sullivan [27] and Cohen [9].

Define the *singularity set* of M by $S(M) = \{x \in M : bLk(x, M) \neq \varnothing\}$; it is easily seen that $S(M)$ is a subcomplex of M. We seek to built a PL $\mathscr{F}$-resolution of M by constructing a strong $\mathscr{F}$-resolution $f: N \to M$ with $\dim(S(N)) < \dim(S(M))$.

Assume $\partial M = \varnothing$ and $\dim(S(M)) = k$. Choose a k-simplex σ_0 and an orientation of $St(\sigma_0, M)$. Let T be a fixed maximal tree in M. For every k-simplex σ of M, the local coefficient system of §1, the orientation of $St(\sigma_0, M)$ and T determine an orientation of $St(\sigma, M)$, and so an orientation of $Lk(\sigma, M)$. Since $S(M)$ is k-dimensional, $Lk(\sigma, M)$ is a PL $\mathscr{F}(n-k-1)$-sphere. Define a chain $\bar{\mu}_k(M) \in C_k^t(M; \theta_{n-k-1}^{\mathscr{F}})$ by $\bar{\mu}_k(M) = \Sigma[Lk(\sigma, M)]\sigma$, taken over all k-simplexes σ.

LEMMA 4.1. *$\bar{\mu}_k(M)$ is a cycle.*

PROOF. Let τ be a $(k-1)$-simplex of M. For each k-simplex σ with $\sigma > \tau$, let α_σ be the 1-simplex $b_\sigma b_\tau$ of M'. Then $Lk(\sigma, M) * b_{\alpha_\sigma} - b_{\alpha_\sigma}$ is a PL-manifold with boundary $Lk(\sigma, M)$. Let R_σ denote the boundary of a PL-collared neighborhood of $Lk(\sigma, M)$ in $Lk(\sigma, M) * b_{\alpha_\sigma} - b_{\alpha_\sigma}$.

Define $W_\tau = Lk(\tau, M) - \bigcup_{\sigma > \tau} \mathrm{Int}(R_\sigma * b_{\alpha_\sigma})$. Then W is an oriented PL-manifold with boundary equal to the disjoint union of the PL $\mathscr{F}$-spheres $Lk(\sigma, M)$, $\sigma > \tau$. Let $\sigma_1, \cdots, \sigma_r$ be a list of the k-simplexes with τ as a face, and V_i a regular neighborhood, in $W_\tau - (\partial W_\tau \cup \bigcup_{j<i} V_j)$, of a PL-embedded path from a (single) point in $Lk(\sigma_i, M)$ to a (single) point in $Lk(\sigma_{i+1}, M)$, $i < r$. (Here we need M to be a PL $\mathscr{F}^0$-manifold, so that $k \leqq n-2$). Then $W_\tau - \bigcup_{i=1}^{r-1} \mathrm{Int}(V_i)$ is a PL-manifold bounding $\#_{i=1}^r Lk(\sigma_i, M)$.

Since $\mathscr{F}$ is connected and coconnected, and $W_\tau \in \mathscr{F}_{n-k}$, $W_\tau - \bigcup_{i=1}^{r-1} \mathrm{Int}(V_i)$ is an $\mathscr{F}$-pseudodisc. This implies that the coefficient of τ in $\partial\bar{\mu}_k(M)$ is 0.

Define $\mu_k(M) \in H_k^t(M; \theta_{n-k-1}^{\mathscr{F}})$ to be the homology class of $\bar{\mu}_k(M)$. Our main result is:

THEOREM 4.2. *Let $\mathscr{F}$ be a connected, coconnected homotopy class and M^n a closed, locally orientable* PL *$\mathscr{F}^{(0)}$-manifold with* $\dim(S(M)) = k$. *Then there exists a strong $\mathscr{F}$-resolution $f\colon N \to M$, with* $\dim(S(N)) < k$, *provided $\mu_k(M) = 0$.*

PROOF. First suppose $\bar{\mu}_k(M) = 0$. Then for every k-simplex σ of M, $Lk(\sigma, M)$ bounds an oriented PL $\mathscr{F}$-pseudodisc V_σ. Let $D(\sigma, M')$ denote the dual cell of σ; we have $\partial D(\sigma, M') \cong Lk(\sigma, M)$. Define N by replacing each $\dot{\sigma} * D(\sigma, M')$ in M' with $\dot{\sigma} * V_\sigma$, with the obvious identifications. Clearly $\dim(S(N)) < k$, and we define $f\colon N \to M$ by collapsing the exterior of an open PL collar of $\partial(\dot{\sigma} * V_\sigma)$ in $\dot{\sigma} * V_\sigma$ to b_σ.

First notice that f is bijective away from the barycenters b_σ and $f^{-1}(b_\sigma) \cong \dot{\sigma} * V_\sigma$. We have $N(f^{-1}(b_\sigma), N) = \dot{\sigma} * V_\sigma$, which is an $\mathscr{F}$-pseudodisc, since it is equal to $\dot{\sigma} * (V_\sigma \cup c\partial V_\sigma) - \mathrm{Int}(c(\dot{\sigma} * \partial V_\sigma))$. Since f is a PL embedding when restricted to $\partial N(f^{-1}(b_\sigma), N)$, f is a strong $\mathscr{F}$-resolution.

Now assume $\bar{\mu}_k(M) = \partial d$, where $d = \Sigma[\Sigma_\tau]\tau$, taken over all $(k+1)$-simplexes τ of M. Since $\dim(S(M)) = k$, $\partial D(\tau, M) \cong S^{n-k-2}$. Let $\Delta_\tau = \Sigma_\tau - \dot{\alpha}$, where α is some $(n-k-1)$-simplex of Σ_τ, oriented so that $\partial\Delta_\tau$ is compatible with $\partial D(\tau, M)$. Replace, as above, each $\dot{\tau} * D(\tau, M)$ with $\dot{\tau} * \Delta_\tau$, to get a strong $\mathscr{F}$-resolution $f_0\colon N_0 \to M$. By construction, $\mu_k(N_0) = 0$, and so there is a strong $\mathscr{F}$-resolution $f'\colon N \to N_0$ with $\dim(S(N)) < k$. By Corollary 2.6, $f_0 \circ f'\colon N \to M$ is the desired strong $\mathscr{F}$-resolution.

We now turn to the relative version of Theorem 4.2. Again assume that $\mathscr{F}$ is a connected coconnected homotopy class, and let M^n be a compact, locally orientable PL $\mathscr{F}^{(0)}$-manifold with $\dim(S(M)) = k$. Assume further that ∂M is collared in M (which we can always do by Lemma 2.4) and that Q^{n-1} is a PL-manifold embedded as a full subcomplex of M with ∂Q collared in ∂M. Homology will have coefficients twisted by the local system on M of §1.

Define $\bar{\mu}_k(M) \in C_k^t(M; \theta_{n-k-1}^{\mathscr{F}})$ by $\bar{\mu}_k(M) = \sum[Lk(\sigma, M)]\sigma$ over all k-simplexes σ not contained in ∂M.

LEMMA 4.3. $\dim(S(\partial M)) \leqq k-1$, $\partial\bar{\mu}_k(M) = \bar{\mu}_{k-1}(\partial M)$, *and* $\bar{\mu}_{k-1}(\partial M)$ *lies in* $C_{k-1}^t(\partial M - Q; \theta_{n-k-1}^{\mathscr{F}})$.

PROOF. Let σ be a k-simplex of ∂M. Then $Lk(\sigma, M) = c(Lk(\sigma, \partial M))$ (since ∂M is collared in M) is a PL-manifold and so $Lk(\sigma, \partial M) \cong S^{n-k-2}$.

Write DM as the union of two copies M_1, M_2 of M. It follows that $\bar{\mu}_k(DM) = \bar{\mu}_k(M_1) + \bar{\mu}_k(M_2)$ since $[Lk(\sigma^k, DM)] = 0$ for $\sigma^k \subset \partial M$. But $\bar{\mu}_k(DM)$ is a cycle, and so $\partial\bar{\mu}_k(M_1) = -\partial\bar{\mu}_k(M_2)$ lies in $C^t_{k-1}(\partial M; \theta^{\mathscr{F}}_{n-k-1})$.

Let τ be a $(k-1)$-simplex of ∂M, and $\sigma_1, \cdots, \sigma_r$ the k-simplexes of $M - \partial M$ with τ as a face. Construct $W_\tau - \bigcup_{i=1}^{r-1} \mathrm{Int}(V_i)$ as in Lemma 4.1. Then $W_\tau - \bigcup_{i=1}^{r-1} \mathrm{Int}(V_i)$ is a PL $\mathscr{F}$-pseudodisc bounding $Lk(\tau, \partial M) \cup \#_{i=1}^{r} Lk(\sigma_i, M)$, and so the coefficient of τ in $\partial\bar{\mu}_k(M)$ is $[Lk(\tau, \partial M)]$. Finally, for $\tau \subset Q$, $Lk(\tau, \partial M) = S^{n-k-1}$ since Q is a PL-manifold with collared boundary, so that the coefficient of τ in $\bar{\mu}_{k-1}(\partial M)$ is 0.

Thus $\bar{\mu}_k(M)$ defines a homology class $\mu_k(M, Q) \in H^t_k(M, \partial M - Q; \theta^{\mathscr{F}}_{n-k-1})$.

THEOREM 4.4. *If $\mu_k(M, Q) = 0$, then there exists a strong $\mathscr{F}$-resolution $f: N \to M$, $\mathrm{rel}(Q)$, so that* $\dim(S(N)) < k$.

PROOF. Write $DM = M_1 \cup M_2$, $M_1 \cong M_2 \cong M$, $M_1 \cap M_2 \cong \partial M$. If $\mu_k(M, Q) = 0$, then since $H^t_*(M - Q, \partial M - Q) \cong H^t_*(M, \partial M - Q)$, there exist $d_i \in C^t_{k+1}(M_i - Q; \theta^{\mathscr{F}}_{n-k-1})$, $i = 1, 2$, $c \in C^t_k(\partial M - Q; \theta^{\mathscr{F}}_{n-k-1})$ so that $\bar{\mu}_k(M_i) = \partial d_i + (-1)^i c$. Therefore, $\bar{\mu}_k(DM) = \partial(d_1 + d_2)$ and by Theorem 4.2, we may construct a strong $\mathscr{F}$-resolution $F: W \to DM$ with $\dim(S(W)) < k$. But since $d_1 - d_2 \in C^t_{k+1}(DM - Q; \theta^{\mathscr{F}}_{n-k-1})$, the preliminary modification of DM does not touch Q, and so F is a resolution $\mathrm{rel}(Q)$. The strong $\mathscr{F}$-resolution f is now obtained by letting $N = F^{-1}(M_1)$, $D = F|N$.

Let $\partial_* : H^t_k(M, \partial M - Q; \theta^{\mathscr{F}}_{n-k-1}) \to H^t_{k-1}(\partial M - Q; \theta^{\mathscr{F}}_{n-k-1})$ be the boundary homomorphism, and $j : \partial M - Q \to \partial M$ the inclusion. By Lemma 4.3, we have

PROPOSITION 4.5. $j_*\partial_*\mu_k(M, Q) = \mu_{k-1}(\partial M)$.

We now show that the obstructions μ_k are natural. Let M^n be closed and suppose M_1^n, M_2^n a PL $\mathscr{F}$-manifolds embedded as full subcomplexes of M so that $M = M_1 \cup M_2$ and $M_1 \cap M_2 = \partial M_1 = \partial M_2$ is a PL-manifold. Let $j_i : M_i \to M$ be the inclusions.

PROPOSITION 4.6. $\mu_k(M) = (j_1)_*(\mu_k(M_1, \partial M_1)) + (j_2)_*(\mu_k(M_2, \partial M_2))$.

The proof follows immediately from the fact that $\bar{\mu}_k(M) = \bar{\mu}_k(M_1) + \bar{\mu}_k(M_2)$. There is a similar result if ∂M_1, ∂M_2 are not PL-manifolds.

Let $f: N \to M$ be a map between closed PL $\mathscr{F}^{(0)}$-manifolds. We say that f is *orientation-preserving* if $f^*\Gamma_M = \Gamma_N$. Such a map induces a homomorphism on homology with twisted coefficients, $f_* : H^t_*(N) \to H^t_*(M)$.

Let $f: N \to M$ be an orientation-preserving strong $\mathscr{F}$-resolution so that $f: N \to M$ and $f': N \to M'$ are simplicial and $\dim(S(C_f)) = k + 1$.

THEOREM 4.7. $f_* \mu_k(N) = \mu_k(M)$.

PROOF. Let σ_0 be a k-simplex of M and $\{\sigma_1, \cdots, \sigma_r\}$ the set of k-simplexes of N so that $f(\sigma_i) \subset \sigma_0$. Choose an orientation for $St(\sigma_0, M)$ and give $St(\sigma_1, N)$ the induced orientation.

Since f is a strong $\mathscr{F}$-resolution and $\dim(S(N)) \leqq k$, $D(f, \sigma_0)$ is an $\mathscr{F}$-pseudodisc bounding a PL $\mathscr{F}(n-k-1)$-sphere Σ. By the proof of Lemma 4.1, $D(f, \sigma_0)$ may be modified to obtain a PL $\mathscr{F}$-cobordism W between Σ and $\#_{i=1}^r \partial D(\sigma_i, N') \cong \#_{i=1}^r Lk(\sigma_i, N.)$ Furthermore, since $\dim(S(C_f)) = k+1$, $Lk(\sigma_0, C_f) - \mathrm{Int}(D(\sigma_0, f))$ is a PL-manifold. By Proposition 2.5, C_f is a PL $\mathscr{F}$-manifold and $Lk(\sigma_0, M) \cong \partial D(\sigma_0, M')$ and Σ are PL $\mathscr{F}$-cobordant.

Let $\sigma_1, \cdots, \sigma_s$ be the k-simplexes of N so that $f(\sigma_s) \subsetneqq \sigma_0$, $0 \leqq s \leqq r$. If $s > 0$, then it follows that $Lk(\sigma_0, M) \cong S^{n-k-1}$: Suppose v_0, v_1 are vertexes of σ_0 so that $v_0 \notin f(\sigma_1)$, $v_1 \in f(\sigma_1)$. Then $Lk(\sigma_1 v_1, C_f)$ is a PL-manifold containing $v_0 * \partial D(\sigma_0, M)$ so that $\partial D(\sigma_0, M)$ is a PL $(n-k-1)$-sphere. This also shows that $[Lk(\sigma_i, N)] = 0$ for $i = 1, \cdots, s$.

It is easily checked that the orientations involved are correct, and we have

$$f_*\Big(\sum_{i=1}^{n} [Lk(\sigma_i, N)]\sigma_i\Big) = \sum_{i=s+1}^{r} [Lk(\sigma_i, N)](\pm\sigma_0) = \Big[\mathop{\#}_{i=s+1}^{r} Lk(\sigma_i, N)\Big]\sigma_0$$
$$= \Big[\mathop{\#}_{i=1}^{r} Lk(\sigma_i, N)\Big]\sigma_0 = [\Sigma]\cdot\sigma_0 = [Lk(\sigma_0, M)]\sigma_0 \,.$$

Doing this for every k-simplex of M (retaining the orientations on $St(\sigma_0, M)$, $St(\sigma_1, M)$), we get that $f_* \bar{\mu}_k(N) = \bar{\mu}_k(M)$.

There is a corresponding result for resolutions relative to a codimension 0 submanifold of ∂M. Theorem 4.7 implies the converse to Theorem 4.2:

COROLLARY 4.8. *If $\mathscr{R}_{\mathscr{F}}(M) \neq \emptyset$, then $\mu_k(M) = 0$.*

5. A product structure theorem. Let M be a compact PL $\mathscr{F}^{(0)}$-manifold with ∂M a PL-manifold. We have the following generalization of the product structure theorem of Edmonds and Stern [10]. Let $J = [-1, 1]$.

THEOREM 5.1. *Let $f: N \to M \times J^k$ be a PL $\mathscr{F}$-resolution $\mathrm{rel}(\partial M \times J^k)$. Then there is a PL $\mathscr{F}$-resolution $\mathrm{rel}(\partial M)$ $f_0 : N_0 \to M$, where N_0 is a PL submanifold of N with trivial normal block bundle, and a proper PL embedding $h: M \to M \times J^k$, isotopic to the standard embedding $M \to M \times 0$ so that*

$$\begin{tikzcd} N \arrow[r, "f"] & M \times J^k \\ N_0 \arrow[u, hook] \arrow[r, "f_0"] & M \arrow[u, "h"'] \end{tikzcd}$$

commutes.

PROOF. The proof is identical to the proof of Theorem 3.1 of [10] (treating the case $\mathscr{F} = \mathscr{H}$), which we give for completeness sake.

Assume f is simplicial and $M \times J^{k-1} \times 0$ is a full subcomplex of $M \times J^k$. Let $V = N(M \times J^{k-1} \times 0, (M \times J^k)')$ and $W = f^{-1}(V)$. By [7], W is a PL-manifold, and since f is a strong $\mathscr{F}$-resolution, so is $f | W : W \to V$. The boundary of V has two components; let M_+ be one of them, and $N_0 = f^{-1}(M_+)$. Since N_0 is a boundary component of a codimension 0 submanifold of W, the normal bundle of N_0 is trivial.

By the theory of derived neighborhoods, there is a proper PL-homeomorphism $\phi : (M \times J^{k-1}, M \times J^{k-1}) \to (M_+, M_+ \cap (\partial M \times J^{k-1}))$ so that h, defined to be the composition $M \times J^{k-1} \xrightarrow{\phi} M_+ \subset M \times J^k$ is isotopic to $M \times J^{k-1} \to M \times J^{k-1} \times 0 \subset M \times J^k$. The result follows by letting $f_0 = \phi^{-1} \circ (f | N_0)$ and proceeding by induction.

We will need a variant of this result for our classification theorem. Let M be a PL $\mathscr{F}$-manifold and $x_0 \in M$. A *based $\mathscr{F}$-resolution* is a PL $\mathscr{F}$-resolution $f : N \to M$ so that for some subdivision M' of M, $f : f^{-1}(St(x_0, M')) \to St(x_0, M')$ is a PL-homeomorphism. Two based $\mathscr{F}$-resolutions f_0, f_1 of M are (*based*) *concordant* if there exists a concordance $F : W \to M \times I$ between f_0 and f_1 so that for some subdivision $(M \times I)'$, $F : F^{-1}(N(x_0 \times I, (M \times I)') \to N(x_0 \times I, (M \times I)')$ is PL-homeomorphism. Let $\mathscr{R}_{\mathscr{F}}(M, x_0)$ denote the set of concordance classes of based $\mathscr{F}$-resolutions of M.

THEOREM 5.1 (based version). *Let $f : N \to M \times J^k$ be a* PL *$\mathscr{F}$-resolution* rel$(\partial M \times J^k)$ *based at $(x_0, 0)$. Then there is a* PL *$\mathscr{F}$-resolution $f_0 : N_0 \to M$* rel(∂M), *based at x_0, satisfying the conclusion of Theorem* 5.1.

The proof is the same, assuming $M \times J^k$ is subdivided so that $f | f^{-1}(St((x_0, 0), M \times J^k)$ is a PL-homeomorphism.

LEMMA 5.2. *Suppose $\mathscr{F}$ is a connected, coconnected homotopy class and there exists an n so that $\theta_i^{\mathscr{F}} = 0$, $i \neq n$. Then for any* PL *$\mathscr{F}$-manifold M and $x_0 \in M - S(M)$, the natural map $\mathscr{R}_{\mathscr{F}}(M, x_0) \to \mathscr{R}_{\mathscr{F}}(M)$ is bijective.*

PROOF. We show surjectivity; injectivity follows from the appropriate relative argument. Let $f: N \to M$ be a simplicial PL $\mathscr{F}$-resolution and $U = St(x_0, M)$. Since U is a contractible PL-manifold, and $\theta_i^{\mathscr{F}} = 0$ for $i \neq n$, any PL $\mathscr{F}$-resolution of U is concordant to 1_U. (If $f: V \to U$ is a PL $\mathscr{F}$-resolution, the only obstruction to such a concordance lies in $H_{m-n}(U \times I \cup C_g, \partial U \times I \cup C_{g|\partial V}; \theta_n^{\mathscr{F}}) \cong H_{m-n}(U \times I, \partial U \times I; \theta_n^{\mathscr{F}}) = 0$ by Theorem 4.2, where $m = \dim(M)$.)

Let $F: W \to U \times I$ be a concordance from $f | f^{-1}(U)$ to 1_U. Define $P = N \times I \cup W$, identifying $f^{-1}(U) \times 1$ and $F^{-1}(U \times 0)$, and $G: P \to M \times I \cup U \times I \overset{\cong}{\to} M \times I$ by $G | N \times I = f \times 1_I$, $G | W = F$. Then G is a concordance from f to the based $\mathscr{F}$-resolution $G | G^{-1}(M \times 1)$. By [15], the condition of the lemma holds if $\mathscr{F} = \mathscr{h}$ or $\mathscr{H}$.

Let M be a compact PL $\mathscr{F}$-manifold so that ∂M is a PL-manifold. We define $\mathscr{R}_{\mathscr{F}}(M, \partial M)$ ($\mathscr{R}_{\mathscr{F}}(M, \partial M, x_0)$, $x_0 \in \operatorname{Int}(M)$) to be the set of concordance classes of PL $\mathscr{F}$-resolutions (based at x_0) of M rel(∂M), with concordances to be relative to $\partial M \times I$.

If $f: N \to M$ is a PL $\mathscr{F}$-resolution rel(∂M), then $f \times 1_J : N \times J \to M \times J$ is a PL $\mathscr{F}$-resolution rel($\partial M \times J$), based at $(x_0, 0)$ if f is based at x_0. It follows easily that this construction yields a well-defined map $\varepsilon^* : \mathscr{R}_{\mathscr{F}}(M, \partial M, x_0) \to \mathscr{R}_{\mathscr{F}}(M \times J, \partial M \times J, (x_0, 0))$.

PROPOSITION 5.3. *If $\mathscr{F}$ is a connected, coconnected homotopy class, then $\varepsilon^* : \mathscr{R}_{\mathscr{F}}(M, \partial M, x_0) \to \mathscr{R}_{\mathscr{F}}(M \times J, \partial M \times J, (x_0, 0))$ is injective.*

PROOF. Let $f_i : N_i \to M$, $i = 0, 1$, be based $\mathscr{F}$-resolutions rel(∂M) and $F : P \to M \times J \times I$ a based concordance between them. Assume $M \times J \times I$ is subdivided so that F is simplicial and $M \times J \times \{0, 1\}$, $M \times \{-1, 1\} \times I$, $M \times \{-1, 1\} \times \{0, 1\}$ are full subcomplexes.

Let $W = C_{f_0} \cup M \times I \cup C_{f_1}$, with the obvious identifications. Since $\mathscr{F}$ is a connected, coconnected homotopy class and f_0, f_1 are PL $\mathscr{F}$-resolutions rel(∂M), W is a PL $\mathscr{F}$-manifold with ∂W a PL-manifold. We have $P \cong F^{-1}(M \times J \times 0) \times I \cup P \cup F^{-1}(M \times J \times 1) \times I$, and define $\hat{F} : P \to W \times J$ by $\hat{F} | P = F$ and $\hat{F} | F^{-1}(M \times J \times i)$ equal to the natural projection to C_{f_i}. Clearly, $\hat{F}$ is a based $\mathscr{F}$-resolution rel($\partial W \times J$), and by Theorem 5.1, there exists a based $\mathscr{F}$-resolution $\hat{F}_0 : P_0 \to W \operatorname{rel}(\partial W)$. The map $P_0 \xrightarrow{\hat{F}_0} W \xrightarrow{\pi} M \times I$, where π is the natural projection, is then a based concordance between f_0 and f_1 (rel(∂M)).

We generalize the map ε^* as follows. Let ξ be a PL J^k-bundle over M, and $f: N \to M$ a PL $\mathscr{F}$-resolution rel(∂M) based at x_0. Define $\xi^*(f) = \hat{f} : E(f^*\xi) \to E(\xi)$; clearly $\xi^*(f)$ is a PL $\mathscr{F}$-resolution, rel $E(\xi | \partial M)$, and is based at $i(x_0)$, where $i: M \to E(\xi)$ is the zero-section. It follows that there is a well-defined map $\xi^* : \mathscr{R}_{\mathscr{F}}(M, \partial M, x_0) \to \mathscr{R}_{\mathscr{F}}(E(\xi), E(\xi|\partial M), i(x_0))$.

THEOREM 5.4. *If $\mathscr{F}$ is a connected, coconnected homotopy class, then ξ^* is injective.*

PROOF. Suppose η is a PL J^l-bundle over $E(\xi)$, and let $\eta\circ\xi$ denote the composite bundle over M. Obviously, $(\eta\circ\xi)^*(f) = \eta^*\circ\xi^*(f) : E(f^*(\eta\circ\xi)) = E(f^*\eta) \to E(\eta\circ\xi) = E(\eta)$. Choose η so that $\eta \oplus p^*\xi$ is trivial, where $p: E(\xi) \to M$ is the projection. Then $p^*(\eta\circ\xi) \cong p^*(i^*\eta \oplus \xi) \cong \eta \oplus p^*\xi$ is trivial, and so $\eta\circ\xi \cong \varepsilon^{k+l}$. By Proposition 5.4, $(\eta\circ\xi)^* = \eta^*\circ\xi^*$ is injective, and so ξ^* is injective.

In particular, if M is a closed PL manifold and ν_M is the normal bundle of M in some Euclidean space, we have:

COROLLARY 5.6. *$\nu_M^*: \mathscr{R}_{\mathscr{F}}(M, x_0) \to \mathscr{R}(E(\nu_M), x_0)$ is injective.*

The maps ε^* do not appear to be bijective in general, as they are for $\mathscr{F} = \mathscr{H}$, [10], but we do have the following important special case:

PROPOSITION 5.7. *$\mathscr{R}$ $(D^n, 0)$ contains only one element.*

PROOF. Let $f: N \to D^n$ be a PL $\mathscr{F}$-resolution and U a PL n-disc in $\mathring{D}^n$ containing 0 so that $f^{-1}(U) \to U$ is a PL-homeomorphism. Let $D_2^n = \{x \in \boldsymbol{R}^n : ||x|| \leqq 2\}$ and define $F_1, F_2 : D_2^n \times I \to D_2^n \times I$ by

$$F_1(x, t) = \begin{cases} ((1+t)x, t) & ||x|| \leqq 1/2 \\ ((1-t)x + tx/||x||, t) & 1/2 \leqq ||x|| \leqq 1 \\ (x, t) & 1 \leqq ||x|| \end{cases}$$

$$F_2(x, t) = \begin{cases} ((1+t/2)x/||x||, t) & ||x|| \leqq 1 + t/2 \\ (x, t) & ||x|| \geqq 1 + t/2\ . \end{cases}$$

Since $F_1(x, t) = F_2(x, t) = (x, t)$ for $||x|| = 2$, $F_2 \circ F_1$ induces a map $F:(D_2^n/\partial D_2^n)\ I \to (D_2^n/\partial D_2^n) \times I$.

Let $N_0 = N \cup \partial N \times I$ and extend f in the natural way to an $\mathscr{F}$-resolution $f_0 : N_0 \to D_2^n$ $(f_0(x, t) = (1+t)f(x)$ for $x \in \partial N)$. Let $\phi: D_2^n \to D_2^n$ be a PL-homeomorphism taking U to $\{x : ||x|| \leqq 1/2\}$ that is the identity on $D_2^n - \mathring{D}^n$. Let G be the composition

$$(N_0/\partial N \times 1) \times I \xrightarrow{f_0 \times 1_I} (D_2^n/\partial D_2^n) \times I \xrightarrow{\phi \times 1_I} (D_2^n/\partial D_2^n) \times I \xrightarrow{F} (D_2^n/\partial D_2^n) \times I\ .$$

We may assume that G is simplicial on $(N_0 - \partial N \times 1) \times I$ and that $D^n \times I$ is a full subcomplex of $\mathring{D}_2^n \times I$.

Let $W = G^{-1}(D^n \times I)$; by [7], $G|W : W \to D^n \times I$ is a PL $\mathscr{F}$-resolution, and clearly $G^{-1}(0 \times I) \to 0 \times I$, $G^{-1}(D^n \times 1) \to D^n \times 1$ are PL-homeomorphisms and $G|G^{-1}(D^n \times 0) = f$. Therefore, f is based concordant to a PL-homeomorphism $g: G^{-1}(D^n \times 1) \to D^n \times 1$. The proof is completed by noticing that the mapping cylinder of g provides a based concordance from g to 1_{D^n}.

Note that this proof also shows that $\mathscr{R}_{\mathscr{F}}(\mathring{D}^n, 0)$ has cardinality 1. The proof of Proposition 5.7 can be generalized to prove the following relative result, which we leave to the reader.

PROPOSITION 5.8. *Let $f: M \to J^{m-1}$ be a PL $\mathscr{F}$-resolution based at 0, and $F: W \to J^m$ a based concordance between f and itself. Then there is a PL $\mathscr{F}$-resolution $G: V \to J^{m+1}$ so that $\partial V = W \cup (M \times \{-1, 1\}) \times J \cup M \times J$ and $G|M \times J = f \times 1_J$, $G|W = F$, $G|M \times \{\pm 1\} \times J = f \times 1_J$ and $G|G^{-1}(0 \times J^2): G^{-1}(0 \times J^2) \to 0 \times J^2$ is a PL-homeomorphism.*

6. The classification theorem. Throughout this section, assume that $\mathscr{F}$ is a fixed connected, coconnected homotopy class. All spaces will be assumed to be pointed (the basepoint of X will be denoted x_0), maps basepoint preserving and homotopies relative to the basepoint. Let $\mathscr{S}et_0$ denote the category of pointed sets.

Let $\mathscr{M}_n$ denote the category of compact, pointed PL n-manifolds and codimension 0 embeddings. Define a contravariant functor $\mathscr{R}_{\mathscr{F}}^{(n)}: \mathscr{M}_n \to \mathscr{S}et_0$ as follows: If $(M, m_0) \in \mathrm{Ob}(\mathscr{M}_n)$, let $\mathscr{R}_{\mathscr{F}}^{(n)}(M, m_0) = \mathscr{R}_{\mathscr{F}}(M, m_0)$, with basepoint 1_M (usually denoted 1). If $(i: N \to M) \in \mathrm{Mor}(\mathscr{M}_n)$, let $i^*: \mathscr{R}_{\mathscr{F}}^{(n)}(M, m_0) \to \mathscr{R}_{\mathscr{F}}^{(n)}(N, n_0)$ be defined by $i^*(f): i^{-1} \circ f|f^{-1}(i(N)): f^{-1}(i(N)) \to N$ where M is subdivided so that f is simplicial and $i(N)$ is a full subcomplex. If $F: W \to M \times I$ is a (simplicial) concordance between f_0 and f_1, then $(i \times 1_I)^{-1} \circ F|F^{-1}(i(N) \times I)$ is a concordance between $i^*(f_0)$ and $i^*(f_1)$, so that i^* is well-defined.

To get our classification theorem, we must first stabilize $\mathscr{R}_{\mathscr{F}}^{(n)}$. For $s \geqq 0$, let $F_s(M, m_0) = \mathscr{R}_{\mathscr{F}}^{(n+s)}(M \times J^s, (m_0, 0))$. Define an equivalence relation $\sim$ on $F(M, m_0) = \coprod_{s \in Z_+} F_s(M, m_0)$ by $\alpha \sim \beta$, $\alpha \in F_s(M, m_0)$, $\beta \in F_t(M, m_0)$, if there exists an $r \geqq \max\{s, t\}$ so that $(\varepsilon^{r-t})^* \alpha = (\varepsilon^{r-s})^* \beta$. (Here ε^k denotes the appropriate trivial bundle. See §5.) Define $\hat{\mathscr{R}}_{\mathscr{F}}^{(n)}(M, m_0) = F(M, m_0)/\sim$.

LEMMA 6.1. *A morphism $i: N \to M$ of $\mathscr{M}_n$ induces a map i^*: $\hat{\mathscr{R}}_{\mathscr{F}}^{(n)}(M, m_0) \to \hat{\mathscr{R}}_{\mathscr{F}}^{(n)}(N, n_0)$, depending only on the isotopy class of i.*

PROOF. Let ξ be a PL J^k-bundle over M, and $\hat{i}: E(\xi|N) \to E(\xi)$. It is easy to verify that $\hat{i}^* \circ \xi^* = (\xi|N)^* \circ i^*: \mathscr{R}_{\mathscr{F}}^{(n)}(M, m_0) \to \mathscr{R}_{\mathscr{F}}^{(n)}(E(i^*\xi), e_0)$, and it follows that $i^*: \hat{\mathscr{R}}_{\mathscr{F}}^{(n)}(M, m_0) \to \hat{\mathscr{R}}_{\mathscr{F}}^{(n)}(N, n_0)$ is well-defined.

Let $F: N \times I \to M \times I$ be a simplicial isotopy between i_0 and i_1. If $f: W \to M$ is a simplicial based $\mathscr{F}$-resolution, then clearly $(f \times 1_I)|(f \times 1_I)^{-1}(F(N \times I))$ is a concordance between $i_0^*(f)$ and $i_1^*(f)$ and the result follows.

Let $\mathscr{M}_n^0$ be the category with objects the same as $\mathscr{M}_n$, but with morphisms replaced by isotopy classes of embeddings. Define a covariant functor $T_n : \mathscr{M}_n^0 \to \mathscr{M}_{n+1}^0$ by $T_n(M, m_0) = (M \times J, (m_0, 0))$, $T_n(i) = i \times 1_J$. This defines a direct system of categories, and we let $\mathscr{M} = \varinjlim \mathscr{M}_n^0$.

By Lemma 6.1, $\hat{\mathscr{R}}_{\mathscr{F}}^{(n)}$ defines a contravariant functor $\mathscr{M}_n^0 \to \mathscr{S}ets_0$, and clearly, $\hat{\mathscr{R}}_{\mathscr{F}}^{(n+1)} \circ T_n = \hat{\mathscr{R}}_{\mathscr{F}}^{(n)}$. Therefore, there is an induced functor $\hat{\mathscr{R}}_{\mathscr{F}} : \mathscr{M} \to \mathscr{S}ets_0$.

Let $\mathscr{C}_0$ denote the category of finite pointed simplicial complexes and simplicial maps. Let $\mathscr{N} : \mathscr{C}_0 \to \mathscr{M}$ be the regular neighborhood functor, as in [25], sending an object X to the regular neighborhood of X under some embedding of X into the interior of some cube I^n, and a simplicial map $f: Y \to X$ to a codimension 0 embedding defined as follows: Let $\tilde{M}_f$ be the reduced mapping cylinder of f, identifying (y_0, t) with x_0. Let M be a regular neighborhood of $\tilde{M}_f$ in I^n and N a regular neighborhood of Y in M. Then $M \searrow \tilde{M}_f \searrow X$ and so M represents $\mathscr{N}(X)$ by [12]. We define $N(f)$ to be the isotopy class of the inclusion $N \subset M$.

Define a contravariant functor $H: \mathscr{C}_0 \to \mathscr{S}ets_0$ by $H = \hat{\mathscr{R}}_{\mathscr{F}} \circ \mathscr{N}$. We first show that H satisfies the axioms of Brown [6].

PROPOSITION 6.2 (Homotopy axiom). *Let $f, g : Y \to X$ be homotopic. Then $H(f) = H(g)$.*

PROOF. Let $F: Y \times I \to X$ be a homotopy between f and g, and define $\hat{F} : Y \times I \to X \times I$ by $\hat{F}(y, t) = (F(y, t), t)$. Let $\tilde{M}_{\hat{F}}$ be $M_{\hat{F}}$ with $((y_0, t), s)$ identified with (x_0, t). Embed $\tilde{M}_f$, $\tilde{M}_g$ as subcomplexes of $I^m \times 0$, $I^m \times 1$, and assume m is large enough so that these embeddings extend to an embedding of $\tilde{M}_{\hat{F}}$ in I^{m+1}. Let W be a regular neighborhood of $\tilde{M}_F$ in I^{m+1}, and V a regular neighborhood of $Y \times I$ in W.

An element α of $H(X)$ is represented by a based $\mathscr{F}$-resolution $\phi : N \to W$, since W represents $\mathscr{N}(X \times I) = \mathscr{N}(X)$, and $\phi|\phi^{-1}(W \cap I^m \times i)$ represents $H(f)\alpha$ if $i = 0$ and $H(g)\alpha$ if $i = 1$. The concordance $\phi|\phi^{-1}(V): \phi^{-1}(V) \to V$ then shows that $H(f)\alpha = H(g)\alpha$, since $V \cap (I^m \times i)$ is a regular neighborhood of Y and we may choose m large enough so that $V \cong (V \cap (I^m \times i)) \times I$.

Combining this with Proposition 5.7, we have:

COROLLARY 6.3. *If X is contractible, then $H(X) = \{1\}$.*

Let Z be a finite simplicial complex, which is the union of two subcomplexes X, Y, with basepoint in the subcomplex $A = X \cap Y$. Let $i_1 : A \to X$, $i_2 : A \to Y$, $j : X \to Z$, $k : Y \to Z$ be the inclusions.

PROPOSITION 6.4 (Mayer-Vietoris axiom). *Let $\alpha \in H(X)$, $\beta \in H(Y)$ and suppose $H(i_1)\alpha = H(i_2)\beta$. Then there exists an element $\gamma \in H(Z)$ with $H(j)\gamma = \alpha$ and $H(k)\gamma = \beta$.*

PROOF. Let Z be embedded as a full subcomplex of the interior of I^m, and let M, N be the regular neighborhoods of X, Y as constructed in Proposition 1.3. Then $P = M \cap N$, $Q = M \cup N$ are regular neighborhoods of A, Z. Let α, β be represented by simplicial based $\mathscr{F}$-resolutions $f : U \to M$, $g : V \to N$. Since $H(i_1)\alpha = H(i_2)\beta$, there is a concordance $F : W \to P \times I$ between $f | f^{-1}(P)$ and $g | g^{-1}(P)$ by Theorem 5.1.

Define $T = U \times I \cup W \cup V \times I/F^{-1}(z_0 \times I)$, identifying $f^{-1}(P) \times 1$ with $F^{-1}(P \times 0)$ and $g^{-1}(P) \times 0$ with $F^{-1}(P \times 1)$, and $\hat{Q} = M \times I \cup P \times I \cup N \times I/z_0 \times I$ with similar identifications (regarding $z_0 \times I \subset P \times I$). The functions $f \times 1_I$, F and $g \times 1_I$ then define a based $\mathscr{F}$-resolution $G : T \to \hat{Q}$.

Let $h : \hat{Q} \to Q \times I$ be a basepoint-preserving PL-homeomorphism taking $M \times I \cup P \times [0, 1/2]$ onto $M \times I$. Define $\gamma \in H(Z)$ to be the equivalence class of $h \circ G$.

We show that $H(j)\gamma = \alpha$. By construction, $H(j)\gamma$ is represented by $\phi = h \circ G | G^{-1}(M \times I \cup P \times [0, 1/2])$ and it is easily checked that the composition $U \times I^2 \cup F^{-1}(P \times [0, 1/2]) \times I \xrightarrow{f \times 1_{I^2} \cup F \times 1_I} M \times I^2 \cup P \times [0, 1/2] \times I \xrightarrow{h \times I} M \times I$ is a concordance between $f \times 1_I$ and ϕ. Similarly, $H(k)\gamma = \beta$.

This theorem and Corollary 6.3 imply the following result, which we will need in order to show that $H(X)$ has a natural abelian group structure. For convenience, we let $\alpha | Y$ denote $H(i)\alpha$ if $i : Y \subset X$ and $\alpha \in H(X)$.

COROLLARY 6.5. *Suppose $Z = X \cup cA$, $X \cap cA = A$, and $\alpha \in H(X)$. Then there exists $\gamma \in H(Z)$ with $\gamma | X = \alpha$ if and only if $\alpha | A = 1$.*

Let X and Y be objects of $\mathscr{C}_0$ and let $i : X \to X \vee Y$, $j : Y \to X \vee Y$ be the inclusions. Define $H(i, j) : H(X \vee Y) \to H(X) \times H(Y)$ by $H(i, j)\alpha = (\alpha | X, \alpha | Y)$.

PROPOSITION 6.6 (Wedge axiom). *$H(i, j) : H(X \vee Y) \to H(X) \times H(Y)$ is a bijection.*

PROOF. We construct an inverse $\Phi: H(X) \times H(Y) \to H(X \vee Y)$ to $H(i, j)$. Consider $X \times y_0, x_0 \times Y \subset X \vee Y$. Since $X \times y_0 \cap x_0 \times Y = (x_0, y_0)$, $H(X \times y_0 \cap x_0 \times Y) = \{1\}$ by Corollary 6.3. Let $\alpha \in H(X)$, $\beta \in H(Y)$. Then by Proposition 6.4, there exists a $\gamma \in H(X \vee Y)$ with $H(i)\gamma = \alpha$, $H(j)\gamma = \beta$. Pick one of these to be $\Phi(\alpha, \beta)$. Clearly $H(i, j) \circ \Phi$ is the identity.

Suppose $\gamma \in H(X \vee Y)$ is represented by $G : T \to \hat{Q}$ with $f = g$, $F = f \times 1_I$ (using notation as in the proof of Proposition 6.4). Then $H(i, j) \circ \Phi(\gamma)$ is represented by $G' : T' \to Q'$, where $G' = G$ on $G^{-1}(M \times I \cup N \times I)$ and is some concordance F on W. Since P is a regular neighborhood of a point, $P \cong J^m$, and G and G' are concordant by Proposition 5.8.

By Propositions 6.2, 6.4 and 6.6, we have:

THEOREM 6.7. *H is a homotopy functor.*

Since H is not defined for all simplicial complexes, we cannot apply the results of [6]. Let $\mathscr{A}$ be the category of abelian groups. We show that H is a functor from $\mathscr{C}_0$ to $\mathscr{A}$, which implies H is representable by [1].

To do this, we need still another condition on our class $\mathscr{F}$. We say that $\mathscr{F}$ is *representable* if $\mathscr{F}$ is a connected, coconnected homotopy class satisfying the following *product axiom*: If Δ_1, Δ_2 are $\mathscr{F}$-pseudodiscs, then $\Delta_1 \times \Delta_2$ is an $\mathscr{F}$-pseudodisc.

If $\mathscr{F}$ is representable, then we may define a product $H(X) \times H(X) \to H(X)$ as follow: Let $\alpha_1, \alpha_2 \in H(X)$ be represented by based $\mathscr{F}$-resolutions $f_i : N_i \to M$, $i = 1, 2$. It follows from the product axiom that $f_1 \times f_2 : N_1 \times N_2 \to M \times M$ is a based $\mathscr{F}$-resolution. Since M is compact, $M \times M \searrow M \times X \searrow X \times X$, and so $M \times M$ represents a regular neighborhood of $X \times X$. Therefore $f_1 \times f_2$ defines an element $\alpha_1 \times \alpha_2 \in H(X \times X)$, which is easily seen to be well-defined. We now let $\alpha_1 \cdot \alpha_2 \in H(X)$ be $\Delta^*(\alpha_1 \times \alpha_2)$ where $\Delta : X \to X \times X$ is the diagonal. This product is natural and 1 acts as an identity.

THEOREM 6.8. *$H(X)$ has a natural abelian group structure.*

PROOF. We need only show that every element in $H(X)$ has an inverse. Suppose first that X is the reduced suspension of a complex Y. Let $\nu: X \to X \vee X$ be the standard comultiplication and $T: X \to X$ the standard reflection. Define $f: X \vee X \to X$ to be 1_X on the first copy of X and T on the second, and let $p_i: X \vee X \to X$, $i = 1, 2$, be the composition $X \vee X \subset X \times X \xrightarrow{\pi_i} X$. For $\alpha, \beta \in H(X)$, let $\alpha \vee \beta \in H(X \vee X)$ be $H(p_1)(\alpha) \cdot H(p_2)(\beta)$.

First notice that $\alpha \vee H(T)\alpha = H(f)\alpha$ by Proposition 6.6 since

$$\begin{aligned} H(i, j)(\alpha \vee H(T)\alpha) &= H(i, j)(H(p_1)(\alpha)\cdot H(p_2\circ T)(\alpha)) \\ &= H(i, j)\circ H(p_1)(\alpha)\cdot H(i, j)\circ H(p_2)\circ H(T)(\alpha) \\ &= H(p_1\circ i)(\alpha)\cdot H(p_2\circ j)\circ H(T)(\alpha) \\ &= (\alpha, 1)\cdot(1, H(T)(\alpha)) = (\alpha, H(T)(\alpha)) \\ &= H(i, j)\circ H(f)(\alpha)\ . \end{aligned}$$

Since $f\circ\nu$ is null-homotopic, we have $H(\nu)(\alpha \vee H(T)\alpha) = 1$ by Propositions 5.7 and 6.2. Therefore

$$\begin{aligned} 1 = H(\nu)(\alpha \vee H(T)\alpha) &= H(\nu)(H(p_1)(\alpha)\cdot H(p_2)\circ H(T)(\alpha)) \quad \text{(as above)} \\ &= H(p_1\circ\nu)(\alpha)\cdot H(p_2\circ\nu)\circ H(T)(\alpha) = \alpha\cdot H(T)\alpha\ , \end{aligned}$$

so that $\alpha^{-1} = H(T)\alpha$.

Now assume that X is a finite simplicial complex of dimension n. By the case above and Proposition 6.2, $H(X)$ is a group if $n = 0$ or 1. Assume the result for complexes of dimension $n - 1$.

Let $\alpha \in H(X)$ and $i\colon X^{(n-1)} \to X$ the inclusion of the $(n - 1)$-skeleton. By the induction hypothesis, there is an element $\beta_0 \in H(X^{(n-1)})$ so that $H(i)(\alpha)\cdot\beta_0 = 1$. Let σ be an n-simplex. By Corollary 6.3, $H(i)(\alpha)|\dot{\sigma} = 1$ and so $\beta_0|\dot{\sigma} = (H(i)(\alpha)\cdot\beta_0)|\dot{\sigma} = 1$. Therefore, by Corollary 6.5, β_0 may be extended over σ. Doing this for each n-simplex, there exists a $\beta_1 \in H(X)$ so that $H(i)(\beta_1) = \beta_0$.

Let $Y = X \cup c(X^{(n-1)})$ and $k : X \to Y$ the inclusion. By Corollary 6.5, and the suspension case, there exists an invertible $\gamma \in H(Y)$ so that $H(k)\gamma = \alpha\cdot\beta_1$, since $H(i)(\alpha\cdot\beta_1) = 1$. Define $\beta = \beta_1\cdot H(k)(\gamma^{-1})$. We have $\alpha\cdot\beta = \alpha\cdot\beta_1\cdot(H(k)(\gamma))^{-1} = 1$, and so $H(X)$ is a group.

This method of constructing inverses is due to Milnor [22]. By Theorems 6.7 and 6.8 and Adams [1], we have:

THEOREM 6.9. *Let $\mathscr{F}$ be a representable class. Then there is a weak H-space $R_{\mathscr{F}}$ and a natural equivalence $H(X) \cong [X, \mathscr{R}_{\mathscr{F}}]$.*

Let M be a closed PL-manifold, $x_0 \in M$, and $\mathscr{F}$ a representable class. Define a map $\Phi : \mathscr{R}_{\mathscr{F}}(M, x_0) \to [M, R_{\mathscr{F}}]$ by the composition $\mathscr{R}_{\mathscr{F}}(M, x_0) \xrightarrow{\nu_M^*} \mathscr{R}_{\mathscr{F}}(E(\nu_M), e_0) \to \hat{\mathscr{R}}_{\mathscr{F}}(E(\nu_M), e_0) \to H(M, x_0) \to [M, R_{\mathscr{F}}]$.

CLASSIFICATION THEOREM. *$\Phi : \mathscr{R}_{\mathscr{F}}(M, x_0) \to [M, R_{\mathscr{F}}]$ is injective.*

PROOF. By Corollary 5.6 and Theorem 6.9, Φ is the composition of injective maps.

Appendix. **Λ-acyclic resolutions.** A compact n-dimensional polyhedron M is called a *Λ-homology manifold* if $\tilde{H}_*(Lk(x, M); \Lambda) \cong \tilde{H}_*(S^{n-1}; \Lambda)$ or 0 for every $x \in M$; $\partial M = \{x \in M : \tilde{H}_*(Lk(x, M); \Lambda) = 0\}$ is an unbounded Λ-homology manifold of dimension $n - 1$.

LEMMA 1. *M is a PL $\mathscr{H}_K$-manifold if and only if M is a Λ-homology manifold.*

PROOF. Clearly every $\mathscr{H}_K$-manifold is a Λ-homology manifold. For the converse, assume $\partial M = \emptyset$, the bounded case being similar. We have M is a PL $\mathscr{H}_K$-manifold if and only if $H_*(Lk(\alpha, M); \Lambda) \cong H_*(S^{n-i-1}; \Lambda)$ for each i-simplex α, if and only if $H_*(Lk(x, M); \Lambda) \cong H_*(S^{n-1}; \Lambda)$ for each $x \in M$, since $S^{i-1} * Lk(\alpha^i, M) \cong Lk(b_\sigma, M')$.

A proper PL-surjection $f: N \to M$ between Λ-homology n-manifolds is called a *Λ-acyclic resolution* if $\tilde{H}_*(f^{-1}(x); \Lambda) = 0$ for each $x \in M$ and $f | \partial N: \partial N \to \partial M$ is also a Λ-acyclic resolution.

LEMMA 2. *f is a strong $\mathscr{H}_K$-resolution if and only if f is a Λ-acyclic resolution.*

The proof is immediate from Proposition 5.4 of [7] since any Λ-acyclic, PL $\mathscr{H}_K$-manifold is an $\mathscr{H}_K$-pseudodisc.

LEMMA 3. *$\mathscr{H}_K$ is representable.*

The proof is clear from Mayer-Vietoris and Poincaré duality.

Define $\mathscr{R}_K = \mathscr{R}_{\mathscr{H}_K}$ and $R_K = R_{\mathscr{H}_K}$, and let $BH(K)$ denote the classifying space for stable Λ-homology cobordism bundles of [3], [4]. Let $f: N \to M$ be a Λ-acyclic resolution between PL-manifolds.

LEMMA 4. *T_N and $f^* T_M$ are stably isomorphic as Λ-homology cobordism bundles.*

PROOF. Let $p: N \times I \to C_f$ be the projection. By Lemmas 2 and 3, and Proposition 2.5, C_f is a Λ-homology manifold, and $p^* T_{C_f}$ defines the desired isomorphism between $T_N \oplus \varepsilon^1$ and $f^* T_M \oplus \varepsilon^1$.

Therefore f defines an element of $[N, H_K/\widetilde{PL}]$. Define $T(f)$ to be the image of $[f^* T_M]$ under the maps $[N, H_K/\widetilde{PL}] \to [N_K, (H_K/\widetilde{PL})_K] \overset{(f^\#)^{-1}}{\cong} [M_K, (H_K/\widetilde{PL})_K] = [M, (H_K/\widetilde{PL})_K]$. It is easily checked that T induces a natural transformation $H \to [\ , (H_K/PL)_K]$ (as functors to $\mathscr{M}$). Therefore:

PROPOSITION 5. *There is a map $\phi: R_K \to (H_K/PL)_K$ of weak H-spaces.*

PROPOSITION 6. *$\pi_n(R_K)$ contains a subgroup G isomorphic to ψ_n^K so that $\phi_\# | G: G \to \pi_n(H_K/\widetilde{PL}) \otimes \Lambda \cong \psi_n^K \otimes \Lambda$ is the localization map.*

PROOF. Define $f: \psi_n^K \to \pi_n(R_K)$ by $f[\Sigma] = \Phi(g)$, where $g: \Sigma \to S^n$ is the map of Lemma 2.1; f is well-defined by an argument similar to the one used in Proposition 2.4 of [21]. It is clear that f is a homomorphism (using the usual homotopy group multiplication on $\pi_n(R_K)$), and f is injective by the Classification theorem. The remainder of the proof follows from the proof of Theorem 3.6 of [3].

COROLLARY 7. *$R_K \simeq H_K/\widetilde{PL}$ if and only if $K = \varnothing$.*

PROOF. If $K = \varnothing$, then it follows immediately from [10] that $R_K \simeq H_K/\widetilde{PL}$. If $K \neq \varnothing$, then ψ_{11}^K, and so $\pi_{11}(R_K)$, contains a subgroup isomorphic to $\Lambda/\boldsymbol{Z} \oplus \Lambda/\boldsymbol{Z}$ by [3] and the proposition. But by Theorem 6.1 of [4], $\pi_{11}(H_K/PL)$ is isomorphic to $\Lambda/\boldsymbol{Z} \oplus A$ where A is a finite group, so that $\pi_{11}(R_K) \not\cong \pi_{11}(H_K/\widetilde{PL})$.

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