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SRH-DECOMPOSITIONS OF CODIMENSION-ONE FOLIATIONS AND THE GODBILLON-VEY CLASSES

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1. Introduction. In this paper we show that the Godbillon-Vey classes of codimension-one foliations with a certain qualitative property are zero.

Since the Godbillon-Vey class was defined in Godbillon-Vey [1], many authors have published studies on it. Thurston [14] proved that the Godbillon-Vey class gives rise to a surjective homomorphism

$$gv\colon \mathscr{F}\Omega_{3,1}^{\infty} \longrightarrow \boldsymbol{R}$$

where $\mathscr{F}\Omega_{3,1}^{\infty}$ is the foliated cobordism group of transversely oriented codimension-one foliations of closed oriented 3-manifolds. The problem to determine its kernel is still open. (See Problem 4 in Lawson [4]). In this point of view it is interesting to investigate what type of foliations are contained in the kernel of gv. Herman [3] proved that a foliation of the 3-torus whose leaves are diffeomorphic to $\boldsymbol{R}^2$ is in the kernel of gv.

On the other hand, the author has been studying the qualitative theory in [8]-[11] and saw that codimension-one foliations with a certain qualitative property admit nice decompositions. By making use of these decompositions, we can compute the Godbillon-Vey classes.

The main result is the following.

THEOREM 1. *Let $\mathscr{F}$ be a transversely orientable codimension-one C^∞ foliation of a closed orientable manifold M. Suppose that the depth $d(\mathscr{F})$ of $\mathscr{F}$ is finite and all holonomy groups of $\mathscr{F}$ are abelian. Then we have*

(1) *If* $\dim M = 3$, *then* $gv(\mathscr{F}) = 0$.

(2) *Let* $\dim M > 3$. *If, for each leaf F of $\mathscr{F}$ whose holonomy group is non-trivial, the cohomology group $H^2_{\mathrm{comp}}(F\colon \boldsymbol{R})$ with compact support is trivial, then* $gv(\mathscr{F}) = 0$.

The author conjectures that the condition in (2) of Theorem 1 is not essential.

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With respect to the problem to investigate the kernel of gv, the following is interesting.

PROBLEM. *Let $\mathscr{F}$ be a transversely-orientable codimension-one foliation of a closed orientable* 3*-manifold M. Suppose that $d(\mathscr{F})$ is finite and all holonomy groups of $\mathscr{F}$ are abelian. Is $\mathscr{F}$ cobordant to zero?*

In § 2 we define SRH-decompositions and in § 3 we give the proof of an existence theorem. In § 4 we state results on the relation between SRH-decompositions and the Godbillon-Vey classes and we give the proof in § 5 for the case of dimension 3 and in § 6 for the case of dimension >3.

FIXED NOTATION. Throughout this paper, $\mathscr{F}$ is a transversely-orientable codimension-one foliation of a closed orientable C^∞ manifold M. We fix a vector field X_0 of M transverse to $\mathscr{F}$ and let $\varphi: M \times \boldsymbol{R} \to M$ be the flow defined by X_0. We work in the C^∞ category and omit the term "C^∞".

2. SRH-decompositions of codimension-one foliations. To clarify the goal of § 2 and § 3 we state an existence theorem of SRH-decompositions before the definition of the terms used there. For the definition of depth see Nishimori [10].

THEOREM 2. *Let $\mathscr{F}$ be a transversely-orientable C^∞ foliation of closed orientable manifold. If the depth $d(\mathscr{F})$ of $\mathscr{F}$ is finite and all holonomy groups of $\mathscr{F}$ are abelian, then $\mathscr{F}$ has an abelian SRH-decomposition whose room-cycles and halls are ventilated.*

Now we begin by introducing some notations as in Nishimori [10], [11]. Let F be a compact manifold with or without boundary and N a transversely-oriented codimension-one compact submanifold of F. Let $C(F, N)$ be the compact manifold obtained from $F - N$ by attaching two copies N_1, N_2 of N as boundary. The suffixes 1, 2 depend on the transverse orientation of N. For a diffeomorphism f: $[0, \delta_1] \to [0, \delta_2]$ with $\delta_1 > \delta_2$ and $f(0) = 0$, we denote by $X(F, N, f)$ the quotient space of $C(F, N) \times [0, \delta_1]$ by the equivalence relation $\sim$ defined by

$$(x_1, t) \sim (x_2, f(t))$$

for $t \in [0, \delta_1]$ and $x_1 \in N_1$, $x_2 \in N_2$ such that $x_1 = x_2$ as elements of N. We denote by $\mathscr{F}(F, N, f)$ the foliation of $X(F, N, f)$ induced by that of $C(F, N) \times [0, \delta_1]$ with leaves $C(F, N) \times \{t\}$, $t \in [0, \delta_1]$.

DEFINITION 1. A subset S of M is called a *staircase* of $\mathscr{F}$ if there are a codimension-zero compact submanifold F of a leaf of $\mathscr{F}$, a codimension-one transversely-oriented closed submanifold N of F with $F - N$ connected, a contraction $f\colon [0, \delta_1] \to [0, \delta_2]$ with $\delta_1 > \delta_2$ and $f(0) = 0$, and an embedding $h\colon X(F, N, f) \to M$ satisfying the following conditions.

(S1) $h(X(F, N, f)) = S$.

(S2) $h(\{x\} \times [0, \delta_1]) \subset \varphi(\{x\} \times \boldsymbol{R})$ for all $x \in F$.

(S3) $h(x, 0) = x$ for all $x \in F$.

(S4) $h(C(F, N) \times \{\delta_1, f(\delta_1), f^2(\delta_1), \cdots\})$ is contained in a leaf of $\mathscr{F}$.

We call $F(S) = F$, $C(S) = h(C(F, N) \times \{\delta_1\})$, $W(S) = h(N_2 \times [\delta_2, \delta_1])$ and $D(S) = h(\partial F \times [0, \delta_1])$ the *floor*, the *ceiling*, the *wall* and the *door* of the staircase S respectively, where N_2 is the copy of N with suffix 2. Note that $\partial S = F(S) \cup C(S) \cup W(S) \cup D(S)$ and that $\mathscr{F}$ is tangent to $F(S) \cup C(S)$ and transverse to $W(S) \cup D(S)$. If $h^*\mathscr{F} = \mathscr{F}(F, N, f)$, we call S *regular*.

DEFINITION 2. A subset R of M is called a *room* of $\mathscr{F}$ if there are a codimension-zero connected compact submanifold F of a leaf of $\mathscr{F}$ and an embedding $h\colon F \times [0, 1] \to M$ such that

(R1) $R = h(F \times [0, 1])$,

(R2) $h(\{x\} \times [0, 1]) \subset \varphi(\{x\} \times \boldsymbol{R})$ and the curves $h|\{x\} \times [0, 1]$ and $\varphi|\{x\} \times \boldsymbol{R}$ have the same direction for all $x \in F$,

(R3) $h(x, 0) = x$ for all $x \in F$,

(R4) $h(F \times \{1\})$ is contained in a leaf of $\mathscr{F}$.

We call $F(R) = F$, $C(R) = h(F \times \{1\})$ and $D(R) = h(\partial F \times [0, 1])$ the *floor*, the *ceiling* and the *door* of the room R respectively. Note that $\partial R = F(R) \cup C(R) \cup D(R)$.

As usual the induced foliation $h^*\mathscr{F}$ defines the "global" holonomy homomorphism

$$\Phi\colon \pi_1(F, x) \longrightarrow \mathrm{Diff}([0, 1])$$

where $\mathrm{Diff}([0, 1])$ is the group of the diffeomorphism of the interval $[0, 1]$. If the image of Φ is trivial or abelian, we call R *trivial* or *abelian* respectively.

DEFINITION 3. A subset H of M is called a *hall* of $\mathscr{F}$ if there are a codimension-zero connected compact submanifold F of a leaf of $\mathscr{F}$ and a diffeomorphism $f\colon D(f) \to R(f)$, where $D(f)$ and $R(f)$ are compact connected submanifolds of F, such that

(H1) $F = D(f) \cup R(f)$,

(H2) for all $x \in D(f)$ there is $t_x > 0$ such that $\varphi(x, t_x) = f(x)$, $\varphi(\{x\} \times (0, t_x)) \cap F = \varnothing$ and

$$H = \{\varphi(x, t) | x \in D(f),\ 0 \leqq t \leqq t_x\}\ .$$

We call $D(H) = \{\varphi(x, t) \mid x \in \partial D(f),\ 0 \leqq t \leqq t_x\}$ the *door* of H. Note that $\partial H = D(H) \cup (D(f) - R(f)) \cup (R(f) - D(f))$.

The induced foliation $\varphi^* \mathscr{F} \mid \{(x, t) \mid x \in D(f),\ 0 \leqq t \leqq t_x\}$ defines the "global" holonomy homomorphism

$$\Phi: \pi_1(D(f), x_0) \longrightarrow \mathrm{Diff}\ ([0, t_{x_0}])$$

for $x_0 \in D(f)$. If the image of Φ is abelian, we call H *abelian*.

DEFINITION 4. A *room-cycle* is the union of a finite sequence $R_1, \cdots, R_l$ of rooms such that $C(R_i) \cap F(R_{i+1}) \neq \varnothing$ for $i = 1, \cdots, l-1$ and $C(R_l) \cap F(R_1) \neq \varnothing$.

REMARK 1. The structures of a room-cycle and a hall are almost the same.

DEFINITION 5. A room-cycle ρ or a hall H is called *ventilated* if the restricted foliation $\mathscr{F}|\rho$ or $\mathscr{F}|H$ has a compact leaf whose holonomy group is trivial, respectively. A room-cycle ρ or a hall H is called *unlocked* if, for all $x \in \rho$ or for all $x \in H$, there are $s < 0$ and $t > 0$ such that $\varphi(x, s) \notin \rho$ and $\varphi(x, t) \notin \rho$ or such that $\varphi(x, s) \notin H$ and $\varphi(x, t) \notin H$ respectively, and otherwise *locked*.

DEFINITION 6. A finite set Δ of subsets of M is called a *quasi-SRH decomposition* of $\mathscr{F}$ if

(1) $M = \bigcup_{A \in \Delta} A$, and Int A 's are disjoint,

(2) $\Delta = \mathscr{S}(\Delta) \cup \mathscr{R}(\Delta) \cup \mathscr{H}(\Delta)$ where $\mathscr{S}(\Delta) = \{A \in \Delta \mid A$ is a regular staircase), $\mathscr{R}(\Delta) = \{A \in \Delta | A$ is a room$\}$, $\mathscr{H}(\Delta) = \{A \in \Delta \mid A$ is a hall$\}$,

(3) $D(A) \subset \bigcup_{S \in \mathscr{S}(\Delta)} W(S)$ for all $A \in \Delta$.

Furthermore if A is abelian for all $A \in \mathscr{R}(\Delta) \cup \mathscr{H}(\Delta)$ we call Δ *abelian*.

PROPOSITION 1. *If $\mathscr{F}$ has an abelian quasi-SRH-decomposition, then all holonomy groups of $\mathscr{F}$ are abelian.*

PROOF. For a leaf F intersecting no elements in $\mathscr{R}(\Delta) \cup \mathscr{H}(\Delta)$ the leaf F contains the floors of just two staircases S_1, S_2 with $F(S_1) \cap F(S_2) \neq \varnothing$ by the condition (3) of Definition 6. Since $\mathscr{F}|\bigcup\{S - F(S)|S \in \mathscr{S}(\Delta)\}$ is without holonomy, the holonomy group of F is isomorphic to $\boldsymbol{Z}$ or $\boldsymbol{Z} \oplus \boldsymbol{Z}$.

For a leaf F intersecting an element $H \in \mathscr{H}(\Delta)$, the intersection $F \cap H$ is connected and $F - H \subset \bigcup\{S - F(S)|S \in \mathscr{S}(\Delta)\}$. Therefore the holonomy group of F is isomorphic to the holonomy group of the leaf $F \cap H$ of the restricted foliation $\mathscr{F}|H$, which is abelian.

For a leaf F intersecting $\operatorname{Int} R$ for an element $R \in \mathscr{R}(\varDelta)$ the intersection $F \cap R$ is connected and $F - R \subset \bigcup \{S - F(S) | S \in \mathscr{S}(\varDelta)\}$. Therefore the holonomy group of F is isomorphic to the holonomy group of the leaf $F \cap R$ of $\mathscr{F}|R$, which is abelian.

For a leaf F intersecting $F(R)$ (or $C(R)$) for an element $R \in \mathscr{R}(\varDelta)$, the intersection $F \cap R$ is $F(R)$ (or $C(R)$) and $F \cap R$ is $F(S)$ for an $S \in \mathscr{S}(\varDelta)$ or $C(R')$ (or $F(R')$) for a different $R' \in \mathscr{R}(\varDelta)$. Furthermore $F - R$ is contained in $\bigcup \{S - F(S) | S \in \mathscr{S}(\varDelta)\}$. Therefore in any case the holonomy group of F is abelian. This completes the proof of Proposition 1.

Let $\varDelta = \mathscr{S}(\varDelta) \cup \mathscr{R}(\varDelta) \cup \mathscr{H}(\varDelta)$ be a quasi-SRH-decomposition of $\mathscr{F}$. For $A, B \in \varDelta$ we write $A \leqq B$ if there is a finite sequence $A_0, A_1, \cdots, A_k \in \varDelta$ such that

(1) $A_0 = A$, $A_k = B$,

(2) $W(A_i) \cap D(A_{i+1}) \neq \varnothing$ for $i = 1, \cdots, k-1$

where $W(A_i)$ is considered to be empty if $A_i \in \mathscr{R}(\varDelta) \cup \mathscr{H}(\varDelta)$. Note that $A \in \varDelta$ is maximal if and only if $A \in \mathscr{R}(\varDelta) \cup \mathscr{H}(\varDelta)$.

DEFINITION 7. A finite set $\varDelta$ of subsets of M is called an *SRH-decomposition* if $\varDelta$ is a quasi-SRH-decomposition and $(\varDelta, \leqq)$ is a partially ordered set.

Now all terms in Theorem 2 are defined. We give two examples of SRH-decompositions.

EXAMPLE 1. Let $\mathscr{F}_R$ be the Reeb foliation of S^3. We can take two staircases S_1, S_2 whose floors are the compact leaf of $\mathscr{F}_R$. Then the connected components H_1, H_2 of $\mathrm{Cl}(S^3 - (S_1 \cup S_2))$ are trivial locked halls of $\mathscr{F}_R$. Let $\varDelta = \{S_1, S_2, H_1, H_2\}$. Then $\varDelta$ is an SRH-decomposition.

EXAMPLE 2. Let Σ_2 be the closed orientable surface of genus 2. By Theorem 4 in Nishimori [10] there is a codimension-one foliation $\mathscr{F}$ of $\Sigma_2 \times [0, 1]$ transverse to the last factor $[0, 1]$, with $d(\mathscr{F}) = d$, and with all holonomy groups abelian. In the case $d = 3$ we give an SRH-decomposition of $\mathscr{F}$ as in Figure 1.

Now we give some propositions on SRH-decompositions.

PROPOSITION 2. *If the depth of $\mathscr{F}$ is finite or if all leaves are proper, then a quasi-SRH-decomposition of $\mathscr{F}$ is an SRH-decomposition.**

PROOF. Let $M/\mathscr{F}$ be the set of leaves of $\mathscr{F}$ and for $F_1, F_2 \in M/\mathscr{F}$ let $F_1 \leqq F_2$ if $F_1 \subset \mathrm{Cl}_M(F_2)$. By Proposition 1 in Nishimori [10], the as-

* K. Yano proved that if the depth of $\mathscr{F}$ is finite then all leaves of $\mathscr{F}$ are proper.

$\Delta=\{S_1, S_2, S_3, S_4, H\}$

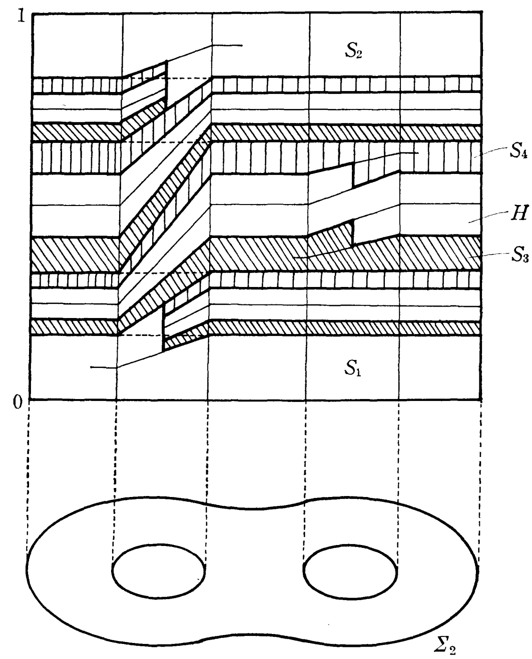

FIGURE 1

sumption of Proposition 2 implies that $(M/\mathscr{F}, \leqq)$ is a partially ordered set. Now suppose that $(\Delta, \leqq)$ is not a partially ordered set for a quasi-SRH-decomposition Δ of $\mathscr{F}$. Then there are two different staircases S_1, S_2 in $\mathscr{S}(\Delta)$ such that $S_1 \leqq S_2$ and $S_2 \leqq S_1$. This implies that $F^*(S_1) \leqq F^*(S_2)$ and $F^*(S_2) \leqq F^*(S_1)$ where $F^*(S_i)$ is the leaf of $\mathscr{F}$ containing the floor $F(S_i)$, $i = 1, 2$. Since

$$F^*(S_i) - F(S_i) \subset \bigcup\{S - F(S) \mid S \in \mathscr{S}(\Delta)\} ,$$

it follows that $F^*(S_1) \neq F^*(S_2)$. Therefore $(M/\mathscr{F}, \leqq)$ is not a partially ordered set, which is a contradiction.

PROPOSITION 3. *Let Δ be an SRH-decomposition of $\mathscr{F}$. Let $\mathscr{S}$ be a subset of $\mathscr{S}(\Delta)$ such that if $S \in \mathscr{S}$ and $S \geqq S' \in \mathscr{S}(\Delta)$ then $S' \in \mathscr{S}$. Then for each leaf F of $\mathscr{F}$, the set $F - \bigcup\{S \mid S \in \mathscr{S}\}$ is connected.*

PROOF. Let $F \in M/\mathscr{F}$ and $p, q \in F - \bigcup\{S \mid S \in \mathscr{S}\}$. We number the elements of $\mathscr{S}$ so that if $S_i \leqq S_j$ then $i \leqq j$. It is sufficient to construct curves c_n: $([0, 1], 0, 1) \to (F - \bigcup_{i=1}^{n} S_i, p, q)$ by induction on n. Since F is connected, there is a curve c_0: $([0, 1], 0, 1) \to (F, p, q)$. Now suppose that c_n is constructed. In the case $c_n([0, 1]) \cap S_{n+1} = \emptyset$, let $c_{n+1} = c_n$. Consider the case $c_n([0, 1]) \cap S_{n+1} \neq \emptyset$. We can write

$$c_n^{-1}(\operatorname{Int} S_{n+1}) = \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda)$$

where (a_λ, b_λ)'s are disjoint. Since S_{n+1} is regular and $c_n | [a_\lambda, b_\lambda]$ is a curve on the same leaf of $\mathscr{F} | S_{n+1}$, we can show that the points $c_n(a_\lambda)$ and $c_n(b_\lambda)$ are on the same leaf of $\mathscr{F} | W(S_{n+1})$. Then we can take curves c_λ: $([a_\lambda, b_\lambda], a_\lambda, b_\lambda) \to (W(S_{n+1}), c_n(a_\lambda), c_n(b_\lambda))$ so that the curve c_n': $[0, 1] \to F$ defined by

(1) $c_n' | [a_\lambda, b_\lambda] = c_\lambda$

(2) $c_n' | [0, 1] - \bigcup_\lambda (a_\lambda, b_\lambda) = c_n | [0, 1] - \bigcup_\lambda (a_\lambda, b_\lambda)$

is continuous. It is easy to modify c_n' and to obtain the desired c_{n+1}. This completes the proof of Proposition 3.

PROPOSITION 4. *For a room R in an SRH-decomposition Δ of $\mathscr{F}$, the floor $F(R)$ and the ceiling $C(R)$ are contained in mutually diffeomorphic leaves of $\mathscr{F}$ and the saturation R^* of R, that is, the union of all leaves of $\mathscr{F}$ intersecting R, is diffeomorphic to $F^*(R) \times [0, 1]$ where $F^*(R)$ is the leaf of $\mathscr{F}$ containing $F(R)$.*

PROOF. We use the notation in Definition 2. We number the staircases in $\mathscr{S}(\Delta)$ so that if $S_i \leqq S_j$ then $i \leqq j$. Let $F_i^*(R)$ be the leaf of $\mathscr{F} | (\bigcup_{\nu \geqq i} S_\nu) \cup R$ containing $F(R)$ and R_i^* the union of all leaves of $\mathscr{F} | (\bigcup_{\nu \geqq i} S_\nu) \cup R$ intersecting R. Then $F_1^*(R) = F^*(R)$ and $R_1^* = R^*$. We construct a diffeomorphism h_i: $F_i^*(R) \times [0, 1] \to R_i^*$ such that

(1) $h_i | F(R) \times [0, 1] = h$

(2) $h_i | F_{i+1}^*(R) \times [0, 1] = h_{i+1}$

by downward induction on i. Suppose that h_{i+1} is defined. If $F^*(R) \cap \operatorname{Int} S_i = \emptyset$ then $F_i^*(R) = F_{i+1}^*(R)$ and $R_i^* = R_{i+1}^*$. In this case let $h_i = h_{i+1}$. Otherwise $S_i \cap R_{i+1}^* = W(S_i) \cap R_{i+1}^*$ by the condition (3) of Definition 6. The intersection $W(S_i) \cap R_{i+1}^*$ consists of a countable number of connected components diffeomorphic to $W(S_i)$. Since $\mathscr{F} | S_i - F(S_i)$ is without holonomy it is easy to extend $h_{i+1} | W(S_i) \cap R_{i+1}^*$ to $h_i | S_i \cap R_i^*$. Thus we have h_i, which completes the proof of Proposition 4.

PROPOSITION 5. *For a hall H in an SRH-decomposition Δ of $\mathscr{F}$, the saturation of H is a fiber bundle over S^1 with fiber F^* where F^* is the leaf of $\mathscr{F}$ containing F in the notation of Definition 3.*

PROOF. We use the notation of Definition 3. By using downward induction as in the proof of Proposition 4, we can extend f: $D(f) \to R(f)$ to a diffeomorphism f^*: $F^* \to F^*$ and find $t_x > 0$ for all $x \in F^*$ such that

(1) $\varphi(\{x\} \times (0, t_x)) \cap F^* = \emptyset$, $\varphi(x, t_x) = f^*(x)$,

(2) $H^* = \{\varphi(x, t) \mid x \in F^*,\ 0 \leqq t \leqq t_x\}$

where H^* is the saturation of H. Therefore H^* has the structure of a fiber bundle over S^1 with the characteristic diffeomorphism f^*.

Now we introduce the term "thinning" of an SRH-decomposition which will be useful in the computation of the Godbillon-Vey classes.

DEFINITION 8. Let S be a staircase of $\mathscr{F}$ and n a non-negative integer. We use the notation of Definition 1. Then $S = h(X(F, N, f))$. The *n-thinning* $S^{(n)}$ of S is the subset $h(C(F, N) \times [0, f^n(\delta_1)]/\sim)$.

PROPOSITION 6. *Let Δ be an SRH-decomposition and α a non-negative integer valued function on $\mathscr{S}(\Delta)$. Then there are a uniquely defined SRH-decomposition $\Delta^{(\alpha)}$ and a bijection $j^{(\alpha)}\colon \Delta \to \Delta^{(\alpha)}$ such that*

(1) $j^{(\alpha)}(\mathscr{S}(\Delta)) = \mathscr{S}(\Delta^{(\alpha)})$, $j^{(\alpha)}(\mathscr{R}(\Delta)) = \mathscr{R}(\Delta^{(\alpha)})$ *and* $j^{(\alpha)}(\mathscr{H}(\Delta)) = \mathscr{H}(\Delta^{(\alpha)})$,

(2) $j^{(\alpha)}(S) \cap S$ *is the $\alpha(S)$-thinning of S for all* $S \in \mathscr{S}(\Delta)$,

(3) $j^{(\alpha)}(A) \supset A$ *for all* $A \in \mathscr{R}(\Delta) \cup \mathscr{H}(\Delta)$.

PROOF. We construct $j^{(\alpha)}(A)$ for $A \in \Delta$ by induction on the partial order $\leqq$. A minimal element A of Δ is a staircase or a room. In the case $A \in \mathscr{S}(\Delta)$, let $j^{(\alpha)}(A)$ be the $\alpha(S)$-thinning of A. In the case $A \in \mathscr{R}(\Delta)$, let $j^{(\alpha)}(A) = A$.

Consider $S \in \mathscr{S}(\Delta)$ and suppose that $j^{(\alpha)}(S')$ is defined for all $S' < S$. Let $j^{(\alpha)}(S)$ be the union of leaves of

$$\mathscr{F}|\mathrm{Cl}(\bigcup\{S' | S \geqq S' \in \mathscr{S}(\Delta)\} - \bigcup\{j^{(\alpha)}(S') | S > S' \in \mathscr{S}(\Delta)\})$$

intersecting the $\alpha(S)$-thinning of S. Since $\mathscr{F}|\bigcup\{S' - F(S') | S > S' \in \mathscr{S}(\Delta)\}$ is without holonomy, it is easy to see that $j^{(\alpha)}(S)$ is a staircase of $\mathscr{F}$.

Consider $A \in \mathscr{R}(\Delta) \cup \mathscr{H}(\Delta)$ and suppose that $j^{(\alpha)}(S)$ is defined for all $S < A$. Let $j^{(\alpha)}(A)$ be the union of leaves of

$$\mathscr{F} | A \cup \mathrm{Cl}(\bigcup\{S | A > S \in \mathscr{S}(\Delta)\} - \bigcup\{j^{(\alpha)}(S) | A > S \in \mathscr{S}(\Delta)\})$$

intersecting A. Then $j^{(\alpha)}(A)$ is a room or a hall if $A \in \mathscr{R}(\Delta)$ or $A \in \mathscr{H}(\Delta)$ respectively.

Let $\Delta^{(\alpha)} = \{j^{(\alpha)}(A) | A \in \Delta\}$. Then $\Delta^{(\alpha)}$ is an SRH-decomposition with the desired property. We can check the uniqueness by induction and omit the proof.

DEFINITION 9. The SRH-decomposition $\Delta^{(\alpha)}$ in Proposition 4 is called the *α-thinning* of Δ. In the case where α is a constant function with value n, we call it the n-thinning of Δ.

PROPOSITION 7. (1) *The α-thinning of the β-thinning of Δ is the $(\alpha + \beta)$-thinning of Δ.* (2) *Let Δ be an SRH-decomposition and $\mathscr{S}$ a* *subset of* $\mathscr{S}(\varDelta)$. *If a compact subset* K *of* M *does not intersect the leaf* $F^*(S)$ *of* $\mathscr{F}$ *containing* $F(S)$ *for each* $S \in \mathscr{S}$, *then there is a non-negative integer valued function* α *of* $\mathscr{S}(\varDelta)$ *such that* $K \cap (\bigcup_{S\in\mathscr{S}} j^{(\alpha)}(S)) = \emptyset$ *and* $\alpha(S) = 0$ *for all* $S \in \mathscr{S}(\varDelta) - \mathscr{S}$.

PROOF. (1) is clear. (2) We number the elements in $\mathscr{S}$ so that if $S_i \leqq S_j$ then $i \leqq j$. Let $\alpha(S) = 0$ for $S \in \mathscr{S}(\varDelta) - \mathscr{S}$. We define $\alpha(S_i)$ by induction on i. Since $F^*(S_1) \cap K = \emptyset$, there is a positive integer $\alpha(S_1)$ such that the $\alpha(S_1)$-thinning of S_1 does not intersect K. Now suppose that $\alpha(S_1), \cdots, \alpha(S_n)$ are defined. Let β_n be a function of $\mathscr{S}(\varDelta)$ defined by

(a) $\beta_n(S_i) = \alpha(S_i)$, $i = 1, \cdots, n$,

(b) $\beta_n(S) = 0$ for $S \in \mathscr{S}(\varDelta) - \{S_1, \cdots, S_n\}$.

Consider the β_n-thinning of $\varDelta$. Since $F^*(S_{n+1}) \cap K = \emptyset$, there is a positive integer $\alpha(S_{n+1})$ such that the $\alpha(S_{n+1})$-thinning of $j^{(\beta_n)}(S_{n+1})$ does not intersect K.

Note that $j^{(\beta_n)}(S_i) = j^{(\beta_{n+1})}(S_i)$ for $i \leqq n$. Let $l = \#(\mathscr{S})$. Then $\alpha = \beta_l$ is the desired function of $\mathscr{S}(\varDelta)$.

3. The proof of Theorem 2. By Proposition 2, it is sufficient to construct an abelian quasi-SRH-decomposition whose room-cycles and halls are ventilated. Let $d = d(\mathscr{F})$. We may suppose that M is connected.

FIRST STEP. By induction we construct non-empty finite sets $\mathscr{S}_1, \cdots, \mathscr{S}_{d-1}$ of staircases and finite sets $\mathscr{R}_1, \cdots, \mathscr{R}_{d-1}$ of rooms such that

(A1) the interiors of all elements in $\mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{d-1} \cup \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_{d-1}$ are disjoint,

(A2) the door of each element in $\mathscr{S}_i \cup \mathscr{R}_i$ is contained in the wall of a staircase in $\mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{i-1}$,

(A3) the floor of each element in $\mathscr{S}_i \cup \mathscr{R}_i$ is contained in a leaf, of $\mathscr{F}$, of depth i,

(A4) each leaf, of $\mathscr{F}$, of depth i is contained in

$$\bigcup\{A \mid A \in \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_i \cup \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_i\},$$

(A5) $\mathscr{R}_1 \cup \cdots \cup \mathscr{R}_{d-1}$ has no room-cycle.

Let $\mathscr{S}_0 = \emptyset$ and $\mathscr{R}_0 = \emptyset$. Let $0 \leqq k < d-1$ and suppose that $\mathscr{S}_i$ and $\mathscr{R}_i$ are already constructed for all $i \leqq k$. Let $M_k = \bigcup\{A | A \in \mathscr{S}_0 \cup \cdots \cup \mathscr{S}_k \cup \mathscr{R}_0 \cup \cdots \cup \mathscr{R}_k\}$.

LEMMA 3.1. $M - M_k \neq \emptyset$.

PROOF. If $k = 0$ it is clear. Let $k \geqq 1$. The condition (A2) implies that the wall of each staircase in $\mathscr{S}_k$ has no neighborhood, with respect to the topology of M, in M_k. Since $\mathscr{S}_k \neq \emptyset$, it follows that $M - M_k \neq \emptyset$.

LEMMA 3.2. *For a leaf F of the restricted foliation $\mathscr{F}|M - \operatorname{Int} M_k$ we denote by $d_k(F)$ the depth of F with respect to $\mathscr{F}|M - \operatorname{Int} M_k$. Let F^* be the leaf of $\mathscr{F}$ containing F. Then $d(F^*) = d_k(F) + k$.*

PROOF. The condition (A4) implies that $d(G^*) > k$ for each leaf G of $\mathscr{F}|M - \operatorname{Int} M_k$. Therefore $d_k(F) + k \leqq d(F^*)$.

The condition (A2) implies that $\mathrm{Cl}(F^*) \cap M_k \subset \bigcup\{S \mid S \in \mathscr{S}_0 \cup \cdots \cup \mathscr{S}_k\}$. Let $d' = d(F^*)$. Then there are leaves $F_1, \cdots, F_{d'}$ of $\mathscr{F}$ such that

(1) $F_{d'} = F^*$,

(2) $F_i \subset \mathrm{Cl}(F_{i+1}) - F_{i+1}$ for $i = 1, \cdots, d' - 1$.

If a leaf of $\mathscr{F}$ is contained in $\bigcup\{S \mid S \in \mathscr{S}_0 \cup \cdots \cup \mathscr{S}_k\}$ then it is the floor of a staircase in $\mathscr{S}_0 \cup \cdots \cup \mathscr{S}_k$. Therefore $F_1, \cdots, F_k$ are the floors of staircases in $\mathscr{S}_0 \cup \cdots \cup \mathscr{S}_k$ and $F_{k+1}, \cdots, F_{d'}$ are not contained in $\bigcup\{S \mid S \in \mathscr{S}_0 \cup \cdots \cup \mathscr{S}_k\}$. It follows that $d' - k \leqq d_k(F)$. Therefore $d' - k = d_k(F)$. This completes the proof of Lemma 3.2.

Since a connected component of $M - \operatorname{Int} M_k$ contains the wall of a staircase in $\mathscr{S}_k$, the set $M - \operatorname{Int} M_k$ has a finite number of connected components. Let K be one of them. By Lemma 3.1 and Lemma 3.2, there are leaves F_1, F_2 of $\mathscr{F}|K$ such that F_1 is compact and $\mathrm{Cl}(F_2) \supset F_1$. Since the holonomy group of F_1 is abelian, there is a staircase S_1 with $F(S_1) = F_1$ and with $C(S_1) \subset F_2$ by Theorem 1 in Nishimori [9]. By the proof of Lemma 9 in [10], for each $x \in K$ there is a neighborhood $U(x)$ of x in K satisfying one of the following.

(I) $U(x)$ intersects no compact leaf of $\mathscr{F}|K$.

(II) $U(x)$ intersects just one compact leaf of $\mathscr{F}|K$.

(III) There is an abelian room $R(x)$ such that $D(R(x)) \subset \partial M_k$, $R(x) \cap \operatorname{Int} S_1 = \emptyset$ and $R(x)$ contains all compact leaves of $\mathscr{F}|K$ intersecting $U(x)$.

Since $K - \operatorname{Int} S_1$ is compact, there are $x_1, \cdots, x_a \in K - \operatorname{Int} S_1$ such that $U(x_1) \cup \cdots \cup U(x_a) \supset M - \operatorname{Int} S_1$. By renumbering x_i's if necessary, we can suppose that $U(x_1), \cdots, U(x_b)$ are of type (III). Let $\{L_\lambda \mid \lambda \in \Lambda\}$ be the set of connected components of

$$L = \bigcup_{i=1}^{b} R(x_i) - \bigcup_{i=1}^{b} F(R(x_i)) - \bigcup_{i=1}^{b} C(R(x_i)) .$$

Then for each L_λ the closure $\mathrm{Cl}(L_\lambda)$ is an abelian room and $\bigcup_{\lambda \in \Lambda} \mathrm{Cl}(L_\lambda) = \mathrm{Cl}(L) = \bigcup_{i=1}^{b} R(x_i)$. Let $\mathscr{R}'_{k+1} = \{\mathrm{Cl}(L_\lambda) \mid \lambda \in \Lambda\}$ and $\mathscr{R}_{k+1}$ the union of the $\mathscr{R}'_{k+1}$'s for all connected components K of $M - \operatorname{Int} M_k$. Then (A1) and (A2) are clearly satisfied. The floor of the room $\mathrm{Cl}(L_\lambda)$ is a compact leaf of $\mathscr{F}|K$ and then it is contained in a leaf of $\mathscr{F}$ of depth $1 + k$

by Lemma 3.2. Thus (A3) is satisfied.

LEMMA 3.3. *$\mathscr{R}_{k+1}$ has no room-cycle.*

PROOF. Suppose that $\mathscr{R}_{k+1}$ has a room-cycle ρ. Then $\mathscr{R}'_{k+1}$ has a room-cycle for a connected component K of $M - \operatorname{Int} M_k$. Since each connected component of $\partial\rho$ is without boundary and is contained in ∂M_k, it is a connected component of ∂M_k. Therefore $\partial\rho \subset \partial K$ and ρ is a closed open subset of K, which implies that $\rho = K$. On the other hand since $R(x_i) \cap S_1 = \varnothing$ for all i, it follows that $\rho \cap S_1 = \varnothing$. This is a contradiction.

By Lemma 3.3 the condition (A5) is satisfied.

Now we construct $\mathscr{S}_{k+1}$. The restricted foliation $\mathscr{F}\,|K - \operatorname{Int}(\bigcup_{i=1}^{b} R(x_i))$ has a finite number of compact leaves. Since all holonomy groups of the compact leaves are abelian, by Theorem 1 in [9] for each compact leaf F of $\mathscr{F}\,|K - \operatorname{Int}(\bigcup_{i=1}^{b} R(x_i))$ we can take a staircase whose floor is F and whose door is contained in ∂M_k if F is in the boundary of $\bigcup_{i=1}^{b} R(x_i)$ and otherwise two staircases. We denote by $\mathscr{S}'_{k+1}$ the set of such staircases and by $\mathscr{S}_{k+1}$ the union of $\mathscr{S}'_{k+1}$'s for all connected components K of $M - \operatorname{Int} M_k$. Clearly $\mathscr{S}_{k+1}$ satisfies the conditions (A1), (A2) and (A3). By Proposition 3 and Lemma 3.2 for each leaf F^* of $\mathscr{F}$ of depth $k+1$ the intersection $F^* \cap (M - \operatorname{Int} M_k)$ is empty or a compact leaf of $\mathscr{F}\,|M - \operatorname{Int} M_k$. Therefore the sets $\mathscr{S}_1, \cdots, \mathscr{S}_{k+1}, \mathscr{R}_1, \cdots, \mathscr{R}_{k+1}$ satisfy the condition (A4).

SECOND STEP. By Lemma 3.2 all leaves of the restricted foliation $\mathscr{F}|M - \operatorname{Int} M_{d-1}$ have trivial holonomy groups and then the leaves are all compact. As in the First step the set $M - \operatorname{Int} M_{d-1}$ has a finite number of connected components. Let K be one of them. Let $F_1, \cdots, F_l$ be the leaves of $\mathscr{F}\,|K$ intersecting the ceiling $C(S)$ for a staircases in $\mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{d-1}$. Each connected component K_λ of $K - (F_1 \cup \cdots \cup F_l)$ is diffeomorphic to $F \times (0, 1)$ for a submanifold F of one of F_i's and $\mathscr{F}\,|K_\lambda$ is a product foliation. If $K - (F_1 \cup \cdots \cup F_l)$ is connected then $l = 1$ and K is a trivial hall. Let $\mathscr{H}$ be the set of such halls. If $K - (F_1 \cup \cdots \cup F_l)$ is not connected then the closure of K_λ is a trivial room and K is a ventilated room-cycle. Let $\mathscr{R}_d$ be the set of such rooms where K varies.

Now let $\varDelta = \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{d-1} \cup \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_d \cup \mathscr{H}$. Clearly the set $\varDelta$ satisfies the conditions (1)-(3) of Definition 6 and $\varDelta$ is a quasi-SRH-decomposition. By the construction of $\mathscr{R}_i$ each room-cycle ρ in $\varDelta$ consists of rooms in $\mathscr{R}_d$, hence ρ is ventilated. Each hall in $\varDelta$ is trivial, hence ventilated. This completes the proof of Theorem 2.

4. The relation between SRH-decompositions and the Godbillon-Vey classes. In this section we state the results of computation of the Godbillon-Vey classes by using SRH-decompositions.

THEOREM 3. *Let* $\dim M = 3$. *If* $\mathscr{F}$ *has an abelian SRH-decomposition whose room-cycles and halls are ventilated or unlocked, then the Godbillon-Vey class* $gv(\mathscr{F})$ *of* $\mathscr{F}$ *is zero.*

THEOREM 4. *Let* $\dim M > 3$. *If* $\mathscr{F}$ *has a ventilated SRH-decomposition and, for each leaf* F *of* $\mathscr{F}$ *whose holonomy group is non-trivial, the cohomology group* $H^2_{\mathrm{comp}}(F: \boldsymbol{R})$ *with compact support is trivial, then* $gv(\mathscr{F}) = 0$.

THEOREM 5. *Let* $\dim M > 3$. *If* $\mathscr{F}$ *has an SRH-decomposition and, for each leaf* F *of* $\mathscr{F}$ *whose holonomy group is non-trivial, the cohomology group* $H^i_{\mathrm{comp}}(F: \boldsymbol{R})$ *with compact support are trivial for* $i = 2, 3$, *then* $gv(\mathscr{F}) = 0$.

Now Theorem 1 follows from Theorems 2, 3 and 4.

We recall the Herman's theorem and strengthen it, whose proof suggests the proof of Theorem 3.

THEOREM 6 (Herman [3]). *Let* $\mathscr{F}$ *be a codimension-one foliation of the 3-torus* $S^1 \times S^1 \times S^1$ *transverse to the last factor. Then* $gv(\mathscr{F}) = 0$.

THEOREM 7. *Let* Σ_g *be a closed orientable surface of genus* g. *Let* $\mathscr{F}$ *be a codimension-one foliation of* $\Sigma_g \times S^1$ *transverse to the last factor* S^1. *The foliation* $\mathscr{F}$ *defines the "global" holonomy homomorphism* $\Phi: \pi_1(\Sigma_g) \to \mathrm{Diff}(S^1)$. *If the image of* Φ *is abelian, then* $gv(\mathscr{F}) = 0$.

PROOF OF THEOREM 7. Let $p: \Sigma_g \times S^1 \to \Sigma_g$ be the projection. We choose circles $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ in Σ_g such that α_i and β_i intersect at one point for $i = 1, \cdots, g$ and any other pair of the circles do not intersect. Let $T(\alpha_1), \cdots, T(\alpha_g), T(\beta_1), \cdots, T(\beta_g)$ be small closed tubular neighborhoods of $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$. Since the image of Φ is abelian, the restricted foliation $\mathscr{F} | \Sigma_g \times S^1 - p^{-1}(\bigcup_{i=1}^g (\alpha_i \cup \beta_i))$ is isomorphic to a product foliation of $(\Sigma_g - \bigcup_{i=1}^g (\alpha_i \cup \beta_i)) \times S^1$. We can construct a non-singular 1-form ω of $\Sigma_g \times S^1$ such that

(1) $T\mathscr{F} = \{v \in TM \mid \omega(v) = 0\}$,

(2) $\mathrm{Supp}(d\omega) \subset \bigcup_{i=1}^g \mathrm{Int}(T(\alpha_i) \cup T(\beta_i))$

where $\mathrm{Supp}(d\omega)$ is the support of $d\omega$. Then there is a 1-form η of $\Sigma_g \times S^1$ such that $d\omega = \eta \wedge \omega$ and $\mathrm{Supp}\, \eta \subset \bigcup_{i=1}^g \mathrm{Int}(T(\alpha_i) \cup T(\beta_i))$. Therefore the Godbillon-Vey number

$$\int_{\Sigma_g \times S^1} \eta \wedge d\eta$$

of $\mathscr{F}$ equals to $\sum_{i=1}^{g}\int_{T(\alpha_i)\cup T(\beta_i)}\eta\wedge d\eta$.

In order to compute $\int_{T(\alpha_i)\cup T(\beta_i)}\eta\wedge d\eta$, we attach a quadrangle Q to $T(\alpha_i)\cup T(\beta_i)$ so that we obtain the 2-torus $S^1\times S^1$. Since the foliation $\mathscr{F}|p^{-1}(T(\alpha_i)\cup T(\beta_i)-\alpha_i\cup\beta_i)$ is isomorphic to a product foliation, we can extend it to a foliation $\mathscr{F}_i$ on the 3-torus $S^1\times S^1\times S^1$ and construct a non-singular 1-form ω_i of $S^1\times S^1\times S^1$ such that $T\mathscr{F}_i=\{v\in T(S^1\times S^1\times S^1)|\ \omega_i(v)=0\}$ and $\omega_i|p^{-1}(T(\alpha_i)\cup T(\beta_i))=\omega|p^{-1}(T(\alpha_i)\cup T(\beta_i))$. Let η_i be the 1-form defined by

(1) $\eta_i|S^1\times S^1\times S^1-p^{-1}(T(\alpha_i)\cup T(\beta_i))=0$,

(2) $\eta_i|T(\alpha_i)\cup T(\beta_i)=\eta|T(\alpha_i)\cup T(\beta_i)$.

Then $d\omega_i=\eta_i\wedge\omega_i$ and $\int_{S^1\times S^1\times S^1}\eta_i\wedge d\eta_i=\int_{T(\alpha_i)\cup T(\beta_i)}\eta\wedge d\eta$. By Theorem 6 the Godbillon-Vey number $\int_{S^1\times S^1\times S^1}\eta_i\wedge d\eta_i$ of $\mathscr{F}_i$ is zero. Therefore

$$gv(\mathscr{F})[\Sigma_g\times S^1]=\int_{\Sigma_g\times S^1}\eta\wedge d\eta=0$$

and then $gv(\mathscr{F})=0$, which completes the proof of Theorem 7.

5. **The proof of Theorem 3.** Let $\mathscr{F}$ be a transversely-orientable codimension-one foliation of a closed orientable 3-manifold M and $\varDelta$ an abelian SRH-decomposition of $\mathscr{F}$ whose room-cycles and halls are ventilated or unlocked. Recall that X_0 is a vector field of M transverse to $\mathscr{F}$ and φ is the flow defined by X_0.

FIRST STEP. We can suppose that for each staircase S in $\varDelta$ the ceiling $C(S)$ has trivial holonomy, by taking 1-thinning of $\varDelta$ if necessary. Let $\mathscr{K}$ be the set of connected components of $M-\mathrm{Int}(\bigcup\{S|S\in\mathscr{S}(\varDelta)\})$.

LEMMA 5.1. *Let $K\in\mathscr{K}$. Then K is one of the following;*

(I) *a hall,*

(II) *a room-cycle,*

(III) *the union of a sequence of rooms sandwiched by two staircases.*

PROOF. Suppose that K contains a hall H in $\varDelta$. By the definition of SRH-decompositions the boundary ∂H of H is contained in $\bigcup\{\partial S|S\in\mathscr{S}(\varDelta)\}$. This implies that $\partial H\subset\partial K$ and then that H is a closed open subset of K. Therefore $H=K$ and the case (I) occurs.

Suppose that K contains no hall in $\varDelta$. Let R be a room in $\varDelta$ contained in K. By Proposition 4 the union R^* of all leaves of $\mathscr{F}$ intersecting R is diffeomorphic to $F^*(R)\times[0,1]$. Note that $R^*-R\subset\mathrm{Int}$

$(\bigcup\{S \mid S \in \mathcal{S}(\Delta)\})$. This implies that, for rooms R_1, R_2 in Δ contained in K, if $C(R_1) \cap C(R_2) \neq \emptyset$ then $R_1 = R_2$, and if $F(R_1) \cap F(R_2) \neq \emptyset$ then $R_1 = R_2$. Suppose that K is not a room-cycle in Δ. Then we can number the rooms in Δ contained in K so that

$$C(R_i) \cap F(R_{i+1}) \neq \emptyset \quad \text{for} \quad i = 1, \cdots, l-1$$

where $R_1, \cdots, R_l$ are the numbered rooms. It is easy to see that $F(R_1) \cap F(S) \neq \emptyset$ and $C(R_l) \cap F(S') \neq \emptyset$ for some $S, S' \in \mathcal{S}(\Delta)$. This is the case (III), which completes the proof of Lemma 5.1.

Let K^* be the saturation of K with respect to $\mathcal{F}$. In the cases (I) and (II) the set K^* is a fiber bundle over circle and in the case (III) a fiber bundle over an interval.

From now we are going to find a subset s of K^* (or the union of K^* and the sandwiching staircases S_1, S_2 in the case (III)) such that $\mathcal{F} \mid K^* - s$ (or $\mathcal{F} \mid (K^* \cup S_1 \cup S_2) - s$ respectively) is without holonomy. We call such s a *holonomy-killing slit*.

DEFINITION 10. For a compact orientable surface Σ_g of genus g with or without boundary, a set Γ of $2g$ circles $\alpha_1, \cdots, \alpha_{2g}$ in Σ_g is called a *basic system* of circles in Σ_g if, for $i < j$, the intersection $\alpha_i \cap \alpha_j$ is one point in the case $i + 1 = j = 2k$ for some $k \in \{1, \cdots, g\}$ and otherwise $\alpha_i \cap \alpha_j$ is empty.

Now consider the case (I). Let H be a hall. By the assumption H is ventilated or unlocked. We use notations in Definition 3. Then

$$H = \{\varphi(x, t) \mid x \in D(f),\ 0 \leqq t \leqq t_x\}\ .$$

(I-1). Suppose that H is ventilated. There is a compact leaf G of $\mathcal{F} \mid H$ with trivial holonomy group. There is $0 < s_x < t_x$ for each $x \in D(f)$ such that $G = \{\varphi(x, s_x) \mid x \in D(f)\}$. Since $\mathcal{F} \mid H^* - H$ is without holonomy where H^* is the saturation of H with respect to $\mathcal{F}$, there is $r_x < 0$ for each $x \in D(f)$ such that $\varphi(\{x\} \times (r_x, 0)) \cap G^* = \emptyset$ and $\varphi(x, r_x) \in G^*$ where G^* is the leaf of $\mathcal{F}$ containing G, and there is $u_x > 0$ for each $x \in R(f)$ such that $\varphi(\{x\} \times (0, u_x)) \cap G^* = \emptyset$ and $\varphi(x, u_x) \in G^*$. Let $\tilde{H} = H \cup \{\varphi(x, t) \mid x \in D(f) - R(f),\ r_x \leqq t \leqq 0\} \cup \{\varphi(x, t) \mid x \in R(f) - D(f),\ 0 \leqq t \leqq u_x\}$ and $\tilde{D}(f) = G \cup \{\varphi(x, s_x) \mid x \in D(f) - R(f)\}$. Then $\tilde{H}$ is a hall. Note that $\tilde{H} \subset j^{(1)}(H) \in \Delta^{(1)}$. Now $\tilde{D}(f)$ is a compact 2-manifold with boundary. Choose a basic system Γ of circles in $\tilde{D}(f)$ and, for $\alpha \in \Gamma$, let

$$\bar{\alpha}(\varepsilon) = \{\varphi(x, t) \mid x \in \alpha,\ \varepsilon < t < s_x - r_x - \varepsilon\}\ .$$

Let $s(H, \varepsilon) = \bigcup\{\bar{\alpha}(\varepsilon) \mid \alpha \in \Gamma\}$. Since the hall $\tilde{H}$ is also abelian and the holonomy group of $\tilde{D}(f)$ is trivial, the foliation $\mathscr{F}|H - s(H, \varepsilon)$ is without holonomy for all sufficiently small $\varepsilon > 0$. Therefore $s(H, \varepsilon)$ is a holonomy-killing slit.

(I-2). Suppose that H is unlocked. Then there is a positive integer n such that $F \subset \bigcup_{i=0}^{n} f^i(D(f) - R(f))$. We take a basic system Γ_1 of circles in $\mathrm{Cl}(D(f) - R(f))$. Furthermore we take a set Γ_2 of circles in $(D(f) - R(f)) \cup f(D(f) - R(f))$ such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{f(\alpha) | \alpha \in \Gamma_1\}$ is a basic system of circles in $(D(f) - R(f)) \cup f(\mathrm{Cl}(D(f) - R(f)))$. For each $x \in D(f)$ there is $s_x > 0$ such that $\varphi(x, s_x) = f^n(x)$. For $\alpha \in \Gamma_1 \cup \Gamma_2$ let

$$\bar{\alpha}(\varepsilon) = \{\varphi(x, t) | x \in \alpha, \varepsilon < t < s_x - \varepsilon\} .$$

Let $s(H, \varepsilon) = \bigcup\{\bar{\alpha}(\varepsilon) | \alpha \in \Gamma_1 \cup \Gamma_2\}$. Then the foliation $\mathscr{F}|H^* - s(H, \varepsilon)$ is without holonomy for all sufficiently small $\varepsilon > 0$. Therofore $s(H, \varepsilon)$ is a holonomy-killing slit.

Now consider the case (II) in Lemma 5.1. Let $\rho = R_1 \cup \cdots \cup R_l$ be a room-cycle in Δ. We may suppose that

$$C(R_i) \cap F(R_{i+1}) \neq \emptyset \quad \text{for} \quad i = 1, \cdots, l-1 ,$$
$$C(R_l) \cap F(R_1) \neq \emptyset .$$

By Proposition 4 the saturation R_i^* of R_i with respect to $\mathscr{F}$ is diffeomorphic to $F^*(R_i) \times [0, 1]$. Therefore $\bigcup_{i=1}^{l} R_i^*$ is a fiber bundle over S^1. For each $x \in F^*(R_1)$ there is $t_x > 0$ such that $\varphi(\{x\} \times (0, t_x)) \cap F^*(R_1) = \emptyset$ and $\varphi(x, t_x) \in F^*(R_1)$. Let

$$D = \left\{x \in F^*(R_1) \,|\, \varphi(\{x\} \times [0, t_x]) \cap \left(\bigcup_{i=1}^{l} R_i\right) \neq \emptyset\right\} ,$$

and

$$H = \{\varphi(x, t) | x \in D, 0 \leqq t \leqq t_x\} .$$

By Proposition 7 there is a positive integer n such that H intersects no staircase in the n-thinning of Δ. Note that H is a hall. Then we can apply the arguments in the case (I) and we have a holonomy-killing slit $s(\rho, \varepsilon)$ in $\bigcup_{i=1}^{l} R_i^*$.

Consider the case (III). Let $K = R_1 \cup \cdots \cup R_l$ where

$$C(R_i) \cap F(R_{i+1}) \neq \emptyset \quad \text{for} \quad i = 1, \cdots, l-1$$
$$F(S_1) \cap F(R_1) \neq \emptyset , \quad F(S_2) \cap C(R_l) \neq \emptyset$$
$$R_1, \cdots, R_l \in \mathscr{R}(\Delta) , \quad S_1, S_2 \in \mathscr{S}(\Delta) .$$

Let R_i^* be the saturation of R_i with respect to $\mathscr{F}$ as before and let S_i^* be the saturation of S_i with respect to the foliation

$$\mathscr{F} | \bigcup\{S \in \mathscr{S}(\Delta) | S \leqq S_i\} .$$

Then by Proposition 4 the subset $\hat{K} = \mathrm{Int}(S_1^* \cup R_1^* \cup \cdots \cup R_l^* \cup S_2^*)$ is diffeomorphic to $F^*(R_1) \times (0, 1)$. For each $x \in F^*(R_1)$ there are $s_x < 0$ and $t_x > 0$ such that

(1) $\varphi(\{x\} \times (s_x, t_x)) \subset \hat{K}$,

(2) $\varphi(x, s_x) \in F(S_1)$, $\varphi(x, t_x) \in F(S_2)$.

It is easy to see that there is a compact submanifold $\tilde{F}$ of $F^*(R_1)$ such that $\tilde{K} = \{\varphi(x, t) \mid x \in \tilde{F}, s_x < t < t_x\}$ contains K. We take a basic system Γ of circles in F. For $\alpha \in \Gamma$ let

$$\bar{\alpha}(\varepsilon) = \{\varphi(x, t) \mid x \in \alpha, s_x - \varepsilon < t < t_x + \varepsilon\} .$$

Let $s(K, \varepsilon) = \bigcup\{\bar{\alpha}(\varepsilon) \mid \alpha \in \Gamma\}$. Since $\mathscr{F} \mid S_1 - F(S_1)$ and $\mathscr{F} \mid S_2 - F(S_2)$ are without holonomy and all R_i's are abelian, the foliation $\mathscr{F} \mid \hat{K} - s(K, \varepsilon)$ is without holonomy for all sufficiently small $\varepsilon > 0$.

By Proposition 7 for a sufficiently large integer n the n-thinning $\Delta^{(n)}$ of Δ satisfies that each $S \in \mathscr{S}(\Delta^{(n)})$ intersects no circles in the basic systems taken in the above argument in the cases (I) and (II). By a similar argument as the proof of Proposition 7 we may suppose that each $\tilde{F}$ in the above argument in the case (III) is contained in some $S \in \mathscr{S}(\Delta^{(n)})$.

Let $S_1, S_2 \in \mathscr{S}(\Delta^{(n)})$ satisfy $F(S_1) \cap F(S_2) \neq \emptyset$. We call such a pair an *adjacent pair* of staircases and denote by $\mathscr{P}$ the set of adjacent pairs. Note that $G = F(S_1) \cup F(S_2)$ is connected and that $F^*(S_1) - G$ and $F^*(S_2) - G$ have no holonomy. Take a basic system Γ of circles in G. For each $x \in G$ there are $s_x < 0$ and $t_x > 0$ such that

(1) $\varphi(\{x\} \times (s_x, t_x)) \subset \mathrm{Int}(S_1 \cup S_2)$

(2) $\varphi(x, s_x), \varphi(x, t_x) \in C(S_1) \cup C(S_2)$.

For $\alpha \in \Gamma$ let

$$\bar{\alpha}(\varepsilon) = \{\varphi(x, t) \mid x \in \alpha_i, s_x + \varepsilon < t < t_x - \varepsilon\}$$

and $s(\{S_1, S_2\}, \varepsilon) = \bigcup\{\bar{\alpha}(\varepsilon) \mid \alpha \in \Gamma\}$. Then $\mathscr{F} \mid (S_1 \cup S_2) - s(\{S_1, S_2\}, \varepsilon)$ is without holonomy for all sufficiently small $\varepsilon > 0$.

Let $\Sigma(\varepsilon)$ be the set of all holonomy-killing slits constructed above and let

$$M(\varepsilon) = M - \bigcup\{s \mid s \in \Sigma(\varepsilon)\} .$$

Then $\mathscr{F} \mid M(\varepsilon)$ is without holonomy for all sufficiently small $\varepsilon > 0$. We fix such ε from now.

SECOND STEP. We are going to construct a vector field X on $M(\varepsilon)$ such that

$$(*)\quad \begin{cases} \text{the local 1-parameter group generated by } X \text{ preserves } \mathscr{F} \text{ and} \\ \text{the orbits of } X \text{ are the orbits of } X_0 | M(\varepsilon)\ . \end{cases}$$

We construct $X|A \cap M(\varepsilon)$ for $A \in \Delta^{(n)}$ by induction on the partial order $\leqq$.

Let $H = j^{(n)}(H') \in \mathscr{H}(\Delta^{(n)})$. Then H is maximal with respect to the partial order $\leqq$. Let H^* be the saturation of H with respect to $\mathscr{F}$ and let $s(H', \varepsilon)$ be the holonomy-killing slit constructed in the First step. Note that $s(H', \varepsilon) \subset H$. The foliation $\mathscr{F}|H^* - s(H', \varepsilon)$ is without holonomy. Take a C^∞ map $c: R \to H^* - s(H', \varepsilon)$ such that

(1) c is transverse to $\mathscr{F}$,

(2) $c(t+1) = c(t)$ for all $t \in R$,

(3) $c(t_1)$ and $c(t_2)$ are on the same leaf of $\mathscr{F} | H^* - s(H', \varepsilon)$ if and only if $t_1 - t_2 \in \boldsymbol{Z}$.

Now let $x \in H^* - s(H', \varepsilon)$. Choose a neighborhood U of x in $H^* - s(H', \varepsilon)$ and a number $\delta > 0$ such that $\varphi(U \times [-\delta, \delta])$ does not intersect some leaf of $\mathscr{F} | H^* - s(H', \varepsilon)$. Let $(y, t) \in U \times [-\delta, \delta]$ and let $u \in R$ satisfy that $c(u)$ and y are on the same leaf of $\mathscr{F} | H^* - s(H', \varepsilon)$. Then there is unique $\tau \in (-1, 1)$ such that $\tau t \geqq 0$ and $c(u + \tau)$ and $\varphi(y, t)$ are on the same leaf of $\mathscr{F} | H^* - s(H', \varepsilon)$. Let $\tau = f(y, t)$. Then we have a C^∞ map $f: U \times [-\delta, \delta] \to (-1, 1)$. Let

$$X(x) = \left(\frac{\partial f}{\partial t} \bigg|_{(x,0)} \right) \cdot X_0(x)\ .$$

It is easy to see that $X(x)$ gives rise to the desired vector field X on $H - s(H', \varepsilon)$ satisfying the condition corresponding to $(*)$.

Let $R = j^{(n)}(R') \in \mathscr{R}(\Delta^{(n)})$. Then R is maximal. Consider the case where R' is contained in a room-cycle ρ. By the argument looking for the holonomy-killing slit $s(\rho, \varepsilon)$, we can work in the same way as in the case of a hall and we have an adequate vector field on $\rho^* - s(\rho, \varepsilon)$. We take its restriction to $R - s(\rho, \varepsilon)$ as the desired vector field X there. We make the construction for all rooms contained in ρ at the same time.

Consider the case R' is contained in $K \in \mathscr{K}$ of the case (III) in Lemma 5.1. We describe K as in the argument looking for the holonomy-killing slit $s(K, \varepsilon)$ in the First step:

(1) $K = R_1 \cup \cdots \cup R_l$ where $R_i \in \mathscr{R}(\Delta)$,

(2) $C(R_i) \cap F(R_{i+1}) \neq \varnothing$ for $i = 1, \cdots, l-1$,

(3) $F(R_1) = F(S_1)$, $C(R_l) = F(S_2)$ for some $S_1, S_2 \in \mathscr{S}(\Delta)$.

Let $K^+ = j^{(n)}(S_1) \cup j^{(n)}(R_1) \cup \cdots \cup j^{(n)}(R_l) \cup j^{(n)}(S_2)$. By the assumption of the induction

$$X\Big|\bigcup_{i=1}^{2}(C(j^{(n)}(S_i))\cup W(j^{(n)}(S_i)))$$

is already defined. We are going to construct X for $K^+ - s(K, \varepsilon)$.

Choose a line segment L in $K^+ - s(K, \varepsilon)$ such that L is transverse to $\mathscr{F}$ and $\partial L \subset \bigcup_{i=1}^{2} C(j^{(n)}(S_i))$. We see that each leaf of $\mathscr{F}|K^+ - s(K, \varepsilon)$ intersecting $W(j^{(n)}(S_1)) \cup W(j^{(n)}(S_2))$ intersects L at a finite number of points. Let

$$L_i = \left\{x \in L \,\middle|\, \begin{array}{l}\text{The leaf of } \mathscr{F}|K^+ - s(K, \varepsilon) \text{ passing } x\\ \text{intersects } W(j^{(n)}(S_i))\ .\end{array}\right\}$$

$i = 1, 2$. Let $W_i = \varphi(\{w_i\} \times [0, \tau_i])$ be an orbit of $X_0|W(j^{(n)}(S_i))$ and let $\hat{W}_1 = \varphi(\{w_1\} \times (-\delta, \tau_1))$ and $\hat{W}_2 = \varphi(\{w_2\} \times (0, \tau_2 + \delta))$ for a sufficiently small $\delta > 0$. For each $x \in L_i$ the leaf of $\mathscr{F}|K^+ - s(K, \varepsilon)$ passing x intersects $\hat{W}_i$ at one point, say $\xi(x)$. For a sufficiently small neighborhood U_x of x in L_i there is a C^∞ map $\eta_x\colon U_x \to \hat{W}_i$ such that

(1) $\eta_x(x) = \xi(x)$,

(2) for each $y \in L_i$ the points y and $\eta_x(y)$ is on the same leaf of $\mathscr{F}$.

We denote by ψ the local 1-parameter group defined by $X|\hat{W}_1 \cup \hat{W}_2$. Then there is a C^∞ map $\tau_x\colon U_x \to \boldsymbol{R}$ such that $\tau_x(x) = 0$ and $\eta_x(y) = \psi(\xi(x), \tau_x(y))$. It is easy to construct a C^∞ map $e\colon L \to \boldsymbol{R}$ such that

$$e(y) - e(x) = \tau_x(y)$$

for each $x \in L$ and each $y \in U_x$.

Let $z \in K^+ - s(K, \varepsilon)$. For a sufficiently small neighborhood V_z of z and a number $\delta_z > 0$ the set $\varphi(V_z \times [-\delta_z, \delta_z])$ is contained in $K^+ - s(K, \varepsilon)$. There is a C^∞ map $g\colon V_z \times [-\delta_z, \delta_z] \to \boldsymbol{R}$ such that w and $g(w)$ are on the same leaf of $\mathscr{F}|K^+ - s(K, \varepsilon)$ for each $w \in V_z$. We define a C^∞ map $f\colon V_z \times [-\delta_z, \delta_z] \to \boldsymbol{R}$ by the equation

$$f(w, t) = e(g(\varphi(w, t))) - e(g(w))$$

where $w \in V_z$, $t \in [-\delta_z, \delta_z]$. Let

$$X(z) = \left(\frac{\partial f}{\partial t}\Big|_{(z,0)}\right)\cdot X_0(z)\ .$$

Then $X(z)$ gives rise to an adequate vector field on $K^+ - s(K, \varepsilon)$.

Now consider an adjacent pair $(S_1, S_2) \in \mathscr{P}$. We can construct an adequate vector field on $(S_1 \cup S_2) - s(\{S_1, S_2\}, \varepsilon)$ in the similar way as above. Thus we have a vector field X on $M(\varepsilon)$ satisfying the condition $(*)$.

THIRD STEP. The goal of this step is to decompose the Godbillon-Vey number of $\mathscr{F}$ to a sum of integrals over neighborhoods of the holonomy-killing slits $s \in \Sigma(\varepsilon)$.

We take disjoint compact regular neighborhoods $N(s)$ of the slits $s \in \Sigma(\varepsilon)$. Let $\hat{M}(\varepsilon) = M - \bigcup\{\operatorname{Int} N(s) \mid s \in \Sigma(\varepsilon)\}$. There is a non-singular vector field Y on M such that

(1) Y is transverse to $\mathscr{F}$,

(2) $Y = X$ on a neighborhood of $\hat{M}(\varepsilon)$.

We denote by ω the C^∞ 1-form on M defined by

(1) $T\mathscr{F} = \{v \in TM \mid \omega(v) = 0\}$,

(2) $\omega(Y)$ is the constant function with value 1.

We use ω for computing the Godbillon-Vey number of $\mathscr{F}$.

LEMMA 5.2. *$d\omega = 0$ on a neighborhood of $\hat{M}(\varepsilon)$.*

PROOF. Let Z_1, Z_2 be vector fields on a neighborhood of $\hat{M}(\varepsilon)$ tangent to $\mathscr{F}$. Then in the formula

$$2d\omega(Y, Z_1) = Y\omega(Z_1) - Z_1\omega(Y) - \omega([Y, Z_1]) ,$$

$\omega(Z_1) = 0$ and $\omega(Y)$ is constant. Furthermore $[Y, Z_1] = 0$ by the construction of Y. Therefore $d\omega(Y, Z_1) = 0$. In the formula

$$2d\omega(Z_1, Z_2) = Z_1\omega(Z_2) - Z_2\omega(Z_1) - \omega([Z_1, Z_2]) ,$$

$\omega(Z_1) = 0$ and $\omega(Z_2) = 0$. Furthermore $[Z_1, Z_2]$ is tangent to $\mathscr{F}$ by the Frobenius theorem and so $\omega([Z_1, Z_2]) = 0$. Therefore $d\omega(Z_1, Z_2) = 0$. This completes the proof of Lemma 5.2.

By using Lemma 5.2 we can construct a C^∞ 1-form η such that $d\omega = \eta \wedge \omega$ on M and $\eta = 0$ on a neighborhood of $\hat{M}(\varepsilon)$. Then

$$gv(\mathscr{F})[M] = \int_M \eta \wedge d\eta = \sum_{s \in \Sigma(\varepsilon)} \int_{N(s)} \eta \wedge d\eta .$$

THE LAST STEP. Now we compute $gv[s] = \int_{N(s)} \eta \wedge d\eta$.

Consider the case $s = s(K, \varepsilon)$ where $K \in \mathscr{K}$ is a hall or a room-cycle. Let $P = S^1 \times S^1 - \operatorname{Int} D^2$ and $I = [0, 1]$. For each connected component C of $N(s)$ we have a diffeomorphism $h: P \times I \to C$. Furthermore for a neighborhood V of the boundary $\partial(P \times I)$ of $P \times I$ we may suppose that the leaves of $(h|V)^*\mathscr{F}$ are

$$(P \times \{t\}) \cap V$$

where $t \in I$. By attaching the foliation on $D^2 \times I$ whose leaves are $D^2 \times \{t\}$ where $t \in I$ we have a foliation $\mathscr{F}_1$ on $S^1 \times S^1 \times I$. Furthermore by identifying $S^1 \times S^1 \times \{0\}$ and $S^1 \times S^1 \times \{1\}$ we have a foliation $\mathscr{F}_2$ on $S^1 \times S^1 \times S^1$. It is easy to construct a non-singular 1-form ω_1 on $S^1 \times$

$S^1 \times I$ such that

(1) $T\mathscr{F}_1 = \{v \in T(D^2 \times I) \mid \omega_1(v) = 0\}$,

(2) $\omega_1 | P \times I = h^*\omega$,

(3) $d\omega_1 | D^2 \times I = 0$.

Furthermore it is easy to construct a 1-form η_1 on $S^1 \times S^1 \times I$ such that

(1) $d\omega_1 = \eta_1 \wedge \omega_1$ on $S^1 \times S^1 \times I$

(2) $\eta_1 | P \times I = h^*\eta$,

(3) $\eta_1 | D^2 \times I = 0$.

We may suppose that ω_1 and η_1 define consistently 1-forms ω_2 and η_2 on $S^1 \times S^1 \times S^1$ respectively. Then

$$\begin{aligned}\int_C \eta \wedge d\eta &= \int_{P\times I} h^*\eta \wedge d(h^*\eta) = \int_{S^1\times S^1\times I} \eta_1 \wedge d\eta_1 \\ &= \int_{S^1\times S^1\times S^1} \eta_2 \wedge d\eta_2 = gv(\mathscr{F}_2)[S^1 \times S^1 \times S^1] \,.\end{aligned}$$

By Herman's theorem we see $gv(\mathscr{F}_2) = 0$. Therefore $gv[s] = 0$.

Consider the case $s = s(\{S_1, S_2\}, \varepsilon)$ where $(S_1, S_2) \in \mathscr{P}$. We represent S_i as

$$h^{(i)}(X(F^{(i)}, N^{(i)}, f^{(i)}: [0, \delta_1^{(i)}] \longrightarrow [0, \delta_2^{(i)}]))$$

as in Definition 1. Extend $f^{(i)}$ to a diffeomorphism

$$\bar{f}^{(i)}: [0, 3\delta_1^{(i)}] \longrightarrow [0, 3\delta_1^{(i)}]$$

such that $f^{(i)}|[2\delta_1^{(i)}, 3\delta_1^{(i)}]$ is the identity map. By using $\bar{f}^{(i)}$ instead of $f^{(i)}$ we construct a manifold $\bar{S}_i$ diffeomorphic to $F^{(i)} \times [0, 3\delta_1^{(i)}]$. We may consider S_i as a subset of $\bar{S}_i$. We extend the slit s naturally to $\bar{s} \subset \bar{S}_1 \cup \bar{S}_2$ such that $\mathscr{F} | \bar{S}_i - S_i - \bar{s}$ is without holonomy. Furthermore we extend $N(s)$ to a compact regular neighborhood $N(\bar{s})$ of $\bar{s}$. See Figure 2.

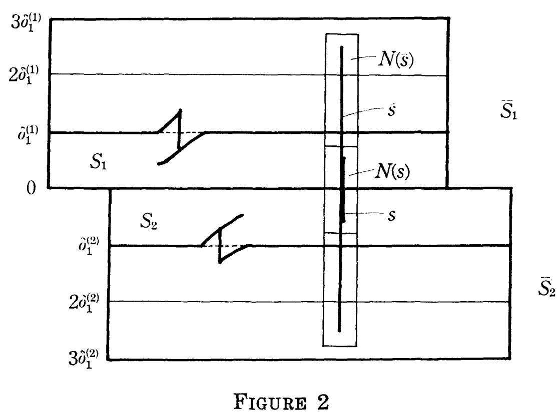

FIGURE 2

Then the vector field $Y|S_i$ constructed in the Third step extends to a vector field $Y|\bar{S}_i$ on $\bar{S}_i$ which, on a neighborhood of $\bar{S}_i - N(\bar{s})$, preserves the foliation $\mathscr{F}^{(i)}$ on $\bar{S}_i$ defined by $\bar{f}^{(i)}$. Furthermore $\omega|S_i$ extends to a non-singular 1-form $\omega^{(i)}$ on $\bar{S}_i$ such that

(1) $T\mathscr{F}^{(i)} = \{v \in T\bar{S}_i | \omega^{(i)}(v) = 0\}$,

(2) $d\omega^{(i)} = 0$ on a neighborhood of $\bar{S}_i - N(\bar{s})$,

and $\eta|S_i$ extends to a 1-form $\eta^{(i)}$ on $\bar{S}_i$ such that

(1) $d\omega^{(i)} = \eta^{(i)} \wedge \omega^{(i)}$ on $\bar{S}_i$,

(2) $\eta^{(i)} = 0$ on a neighborhood of $\bar{S}_i - N(\bar{s})$.

Let $\bar{\eta}$ be the 1-form on $\bar{S}_1 \cup \bar{S}_2$ such that $\bar{\eta}|\bar{S}_i = \eta^{(i)}$. Then

$$gv(\mathscr{F}^{(1)} \cup \mathscr{F}^{(2)})[\bar{S}_1 \cup \bar{S}_2] = \int_{\bar{S}_1 \cup \bar{S}_2} \bar{\eta} \wedge d\bar{\eta} = \int_{N(\bar{s})} \bar{\eta} \wedge d\bar{\eta} = 0$$

since the argument in the case of a hall or a room-cycle is valid also for $N(\bar{s})$. Therefore

$$\int_{N(s)} \eta \wedge d\eta + \int_{\bar{S}_1 - S_1} \eta^{(1)} \wedge d\eta^{(1)} + \int_{\bar{S}_2 - S_2} \eta^{(2)} \wedge d\eta^{(2)} = 0\ .$$

Now we compute $\int_{\bar{S}_i - S_i} \eta^{(i)} \wedge d\eta^{(i)}$. Let Q_i be the quotient space of $\boldsymbol{R} \times [0, 3\delta_1^{(i)}]$ by the equivalence relation $\sim$ defined by

$$(r, t) \sim (r + 4, f^{(i)}(t))$$

for all $r \in R$ and all $t \in [0, 3\delta_1^{(i)}]$. Let $\mathscr{G}^{(i)}$ be the foliation on Q_i induced from one on $\boldsymbol{R} \times [0, 3\delta_1^{(i)}]$ whose leaves are $\boldsymbol{R} \times \{t\}$ where $t \in [0, 3\delta_1^{(i)}]$. It is easy to see that there is a non-singular 1-form $\omega_0^{(i)}$ on Q_i such that

(1) $T\mathscr{G}^{(i)} = \{v \in TQ_i | \omega_0^{(i)}(v) = 0\}$,

(2) $d\omega_0^{(i)} = 0$ on a neighborhood of

$$\begin{aligned} Z_i = [-3, -1] \times [0, 3\delta_1^{(i)}] \cup [-1, 1] \times \{3\delta_1^{(i)}\} \cup [-1, 0] \times \{\delta_1^{(i)}\} \cup \\ \{0\} \times [f^{(i)}(\delta_1^{(i)}), \delta_1^{(i)}] \cup [0, 1] \times \{f^{(i)}(\delta_1^{(i)})\}\ . \end{aligned}$$

Furthermore there is a 1-form $\eta_0^{(i)}$ on Q_i such that

(1) $d\omega_0^{(i)} = \eta_0^{(i)} \wedge \omega_0^{(i)}$ on Q_i,

(2) $\eta_0^{(i)} = 0$ on a neighborhood of Z_i.

Then for a neighborhood B_i of $N^{(i)} \times [0, 3\delta_1^{(i)}]$ in $\bar{S}_i$ there is a C^∞ map $\alpha\colon B_i \to Q_i$ such that

(1) $\mathscr{F}|B_i = \alpha^*\mathscr{G}^{(i)}$,

(2) $\alpha(W(S_i)) = \{0\} \times [\delta_2^{(i)}, \delta_1^{(i)}]$.

Since $\mathscr{F}|\bar{S}_i - S_i - N \times [\delta_2^{(i)}, 3\delta_1^{(i)}]$ is without holonomy, there is a non-singular 1-form $\omega_*^{(i)}$ on $\bar{S}_i - S_i$ such that

(1) $\omega_*^{(i)} = \alpha^*\omega_0^{(i)}$ on $B_i \cap (\bar{S}_i - S_i)$,

(2) $T\mathscr{F}^{(i)} = \{v \in T\bar{S}_i | \omega_*^{(i)}(v) = 0\}$ on $\bar{S}_i - S_i$,

(3) $d\omega_*^{(i)} = 0$ on $\bar{S}_i - S_i - B_i$.

Let $\eta_*^{(i)}$ be the 1-form on $\bar{S}_i - S_i$ such that

(1) $\eta_*^{(i)} = \alpha^*\eta_0^{(i)}$ on $B_i \cap (\bar{S}_i - S_i)$,

(2) $\eta_*^{(i)} = 0$ on $\bar{S}_i - S_i - B_i$.

Then $d\omega_*^{(i)} = \eta_*^{(i)} \wedge \omega_*^{(i)}$ on $\bar{S}_i - S_i$.

LEMMA 5.3. *For* $i = 1, 2$,

$$\int_{\bar{S}_i - S_i} \eta^{(i)} \wedge d\eta^{(i)} = \int_{\bar{S}_i - S_i} \eta_*^{(i)} \wedge d\eta_*^{(i)} .$$

PROOF. Since $T\mathscr{F}^{(i)} \mid \bar{S}_i - S_i = \{v \in T(\bar{S}_i - S_i) \mid \omega^{(i)}(v) = 0\} = \{v \in T(\bar{S}_i - S_i) \mid \omega_*^{(i)}(v) = 0\}$, there is a positive function g of $\bar{S}_i - S_i$ with $\omega_*^{(i)} = g\omega^{(i)}$. Since

$$\begin{aligned} d\omega_*^{(i)} &= dg \wedge \omega^{(i)} + gd\omega^{(i)} = dg \wedge \omega^{(i)} + g\eta^{(i)} \wedge \omega^{(i)} \\ &= (d \log g + \eta^{(i)}) \wedge g\omega^{(i)} , \end{aligned}$$

we have $\eta_*^{(i)} = \eta^{(i)} + d \log g + h\omega^{(i)}$ for some function h. Then

$$\begin{aligned} \eta_* \wedge d\eta_*^{(i)} &= (\eta^{(i)} + d \log g + h\omega^{(i)}) \wedge (d\eta^{(i)} + dh \wedge \omega^{(i)} + h\eta^{(i)} \wedge \omega^{(i)}) \\ &= \eta^{(i)} \wedge d\eta^{(i)} + \eta^{(i)} \wedge dh \wedge \omega^{(i)} + d \log g \wedge d\eta^{(i)} \\ &\quad + d \log g \wedge dh \wedge \omega^{(i)} + d \log g \wedge h\eta^{(i)} \wedge \omega^{(i)} \\ &\quad + h\omega^{(i)} \wedge d\eta^{(i)} \\ &= \eta^{(i)} \wedge d\eta^{(i)} - d(hd\omega^{(i)}) + d(\log g\, d\eta^{(i)}) \\ &\quad + d(\log g\, d(h\omega^{(i)})) + h\omega^{(i)} \wedge d\eta^{(i)} . \end{aligned}$$

Since $0 = dd\omega^{(i)} = d(\eta^{(i)} \wedge \omega^{(i)}) = d\eta^{(i)} \wedge \omega^{(i)} - \eta^{(i)} \wedge \eta^{(i)} \wedge \omega^{(i)}$, we have $d\eta^{(i)} \wedge \omega^{(i)} = 0$. Therefore

$$\eta_*^{(i)} \wedge d\eta_*^{(i)} = \eta^{(i)} \wedge d\eta^{(i)} + d(-hd\omega^{(i)} + \log g\, d\eta^{(i)} + \log g\, d(h\omega^{(i)})) .$$

Since $\eta_*^{(i)} = \eta^{(i)} = 0$ on a neighborhood V of the boundary of $\bar{S}_i - \operatorname{Int} S_i$, we have $d \log g + h\omega^{(i)} = 0$ on V. Then $d(h\omega^{(i)}) = -dd \log g = 0$ on V. Therefore $-hd\omega^{(i)} + \log g\, d\eta^{(i)} + \log g\, d(h\omega^{(i)}) = 0$ on V. This implies that $[\eta_*^{(i)} \wedge d\eta_*^{(i)}] = [\eta^{(i)} \wedge d\eta^{(i)}]$ in $H^3_{\mathrm{comp}}(\operatorname{Int}(\bar{S}_i - S_i); \boldsymbol{R})$. Hence the integrals coincide.

Since Q_i is a 2-manifold, we see that the 3-form $\eta_0^{(i)} \wedge d\eta_0^{(i)}$ vanishes. Therefore $\eta_*^{(i)} \wedge d\eta_*^{(i)} = 0$. So we see

$$gv[s] = \int_{N(s)} \eta \wedge d\eta = -\sum_{i=1}^{2} \int_{\bar{S}_i - S_i} \eta^{(i)} \wedge d\eta^{(i)} = 0 .$$

Now we consider the case $s = s(K, \varepsilon)$ where K is the union of a sequence of rooms sandwiched by two staircases. In this case the argument for adjacent pairs of staircases is vaild and we see $gv[s]=0$.

Therefore $gv(\mathscr{F})[M]=\sum_{s\in\Sigma(\varepsilon)}gv[s]=0$ and so $gv(\mathscr{F})=0$, which completes the proof of Theorem 3.

6. The proof of Theorem 4 **and Theorem** 5. Let M be a closed orientable C^∞ manifold of dimension >3. Let $\mathscr{F}$ be a foliation of M and Δ an SRH-decomposition satisfying the condition of Theorem 4 or Theorem 5.

Let $\mathscr{K}$ be the set of connected components of $M-\bigcup\{\mathrm{Int}\,S\,|\,S\in\mathscr{S}(\Delta)\}$ and $\mathscr{P}$ the set of adjacent pairs of the staircases in Δ. Let

$$\begin{aligned}\tilde{\mathscr{K}}=\{K\in\mathscr{K}\,|\,K&\text{ in the cases (I), (II) of Lemma 5.1}\}\cup\{K_+\,|\,K\in\mathscr{K}\\&\text{in the case (III) of Lemma 5.1}\}\end{aligned}$$

where K_+ be the union of K and the sandwiching staircases and let

$$\tilde{\mathscr{P}}=\{S_1\cup S_2\,|\,(S_1,S_2)\in\mathscr{P}\}\ .$$

We can number the elements of the union $\mathscr{U}$ of $\tilde{\mathscr{K}}\cup\tilde{\mathscr{P}}$ so that

(1) $\mathscr{U}=\{K_1,\cdots,K_k\}$,

(2) if $A_j\in\Delta$ is contained in $K_{i(j)}$ for $j=1,2$, and $A_1\leqq A_2$ then $i(1)\leqq i(2)$.

We are going to construct a vector field X on M transverse to $\mathscr{F}$, a non-singular 1-form ω on M with $T\mathscr{F}=\{v\in TM|\,\omega(v)=0\}$, a 1-form η on M with $d\omega=\eta\wedge\omega$, and a 2-form ξ on M with $d\xi=\eta\wedge d\eta$, which implies that $gv(\mathscr{F})=[\eta\wedge d\eta]=0$.

By downward induction we construct non-negative integer valued functions α_i of $\mathscr{S}(\Delta)$ such that

$$\alpha_i(S)=\alpha_{i+1}(S)\quad\text{if}\quad S\subset K_j\in\mathscr{U}\quad\text{and}\quad j>i\ ,$$

and we construct X,ω,η,ξ on $j^{(\alpha_i)}(K_i)$ in each step where $j^{(\alpha_i)}(K_i)=\bigcup_{\nu=1}^{l}j^{(\alpha_i)}(A_\nu)$ for $K_i=A_1\cup\cdots\cup A_l\in\mathscr{U}$, $A_\nu\in\Delta$.

Suppose that all are defined for $i\geqq n+1$.

(I). Let $K_n\in\mathscr{U}$ be a hall H. Note that we can construct X,ω,η,ξ without restriction since H is maximal in $(\Delta,\leqq)$. We describe H as in Definition 3. Then we have $F,f\colon D(f)\to R(f)$, $t_x>0$ for each $x\in D(f)$ such that

$$H=\{\varphi(x,t)\,|\,x\in D(f),\ 0\leqq t\leqq t_x\}\ .$$

In the case where H is trivial, we can take X on $j^{(\alpha_{n+1})}(H)$ such that the local 1-parameter group defined by X preserves $\mathscr{F}|j^{(\alpha_{n+1})}(H)$. Then we can take ω on $j^{(\alpha_{n+1})}(H)$ with $d\omega=0$. Let $\alpha_n=\alpha_{n+1}$ and

$\eta = 0, \xi = 0$ on $j^{(\alpha_n)}(H)$.

Consider the case where H is non-trivial. Let H^* be the saturation of H with respect to $\mathscr{F}$ and F^* the leaf of $\mathscr{F}$ containing F. Recall that H^* is a fiber bundle over S^1 with fiber F^* and $\mathscr{F}|H^* - H$ is without holonomy. We can construct a non-singular vector field X on H^* such that

(1) each orbit of X is contained in an orbit of X_0,

(2) in the case of Theorem 4, the local 1-parameter group defined by $X|(H^* - H) \cup T$ preserves $\mathscr{F}$ where T is a tubular neighborhood of a leaf F_1 of $\mathscr{F}|H$ with trivial holonomy, or

(2′) in the case of Theorem 5, the local 1-parameter group defined by $X|H^* - H$ preserves $\mathscr{F}$.

We denote by ω the non-singular 1-form on H^* such that

(1) $T\mathscr{F}|H^* = \{v \in TH^* | \omega(v) = 0\}$,

(2) $\omega(X) \equiv 1$.

Then we see $d\omega = 0$ on $(H^* - H) \cup T$ or on $H^* - H$ respectively.

Consider the case of Theorem 4. We consider $\eta \wedge d\eta$ as a 3-form on $H^* - F_1^*$ with compact support. Since $H^* - F_1^*$ is diffeomorphic to $F_1^* \times \boldsymbol{R}$ and $H^2_{\text{comp}}(F_1^*; \boldsymbol{R}) = 0$, we see

$$H^3_{\text{comp}}(H^* - F_1^*; \boldsymbol{R}) \cong \sum_{p+q=3} H^p_{\text{comp}}(F_1^*; \boldsymbol{R}) \otimes H^q_{\text{comp}}(\boldsymbol{R}; \boldsymbol{R}) \cong 0 .$$

Therefore there is a 2-form ξ on $H^* - F_1^*$ with compact support satisfying $d\xi = \eta \wedge d\eta$. We can consider ξ as a 2-form on H^*.

Consider the case of Theorem 5. In the Wang exact sequence

$$\cdots \longrightarrow H^2_{\text{comp}}(F_1^*; \boldsymbol{R}) \longrightarrow H^3_{\text{comp}}(H^*; \boldsymbol{R}) \longrightarrow H^3_{\text{comp}}(F_1^*; \boldsymbol{R}) \xrightarrow{a_* - \text{id}_*} H^3_{\text{comp}}(F_1^*; \boldsymbol{R}) \longrightarrow \cdots$$

where $a: F_1^* \to F_1^*$ is the characteristic diffeomorphism of the bundle, the groups $H^2_{\text{comp}}(F_1^*; \boldsymbol{R})$ and $H^3_{\text{comp}}(F_1^*; \boldsymbol{R})$ are trivial by the assumption. Therefore $H^3_{\text{comp}}(H^*; \boldsymbol{R})$ is trivial and there is a 2-form ξ on H^* with compact support satisfying $d\xi = \eta \wedge d\eta$.

Let $\mathscr{S} = \{S \in \mathscr{S}(\varDelta^{(\alpha_{n+1})}) | S \subset H^*\}$. Note that the support of ξ intersects $F^*(S)$ for no $S \in \mathscr{S}$. By Proposition 7 there is a function $\alpha: \mathscr{S}(\varDelta^{(\alpha_{n+1})}) \to \{0, 1, 2, \cdots\}$ such that

(1) $\operatorname{supp} \xi \cap (\bigcup \{j^{(\alpha)}(S) | S \in \mathscr{S}(\varDelta^{(\alpha_{n+1})})\}) = \emptyset$,

(2) $\alpha(S) = 0$ for $S \in \mathscr{S}(\varDelta^{(\alpha_{n+1})}) - \mathscr{S}$.

Let $\alpha_n = \alpha \circ j^{(\alpha_{n+1})} + \alpha_{n+1}$. We fix X, ω, η, ξ on $j^{(\alpha_n)}(H)$.

(II). Let K_n be a room-cycle ρ. We can treat ρ in the same way as halls and we have α_n and X, ω, η, ξ on $j^{(\alpha_n)}(K_n)$.

(III) Consider the case where $K_n = K_+ = K \cup S_1 \cup S_2$, that is, the union of rooms and the sandwiching staircases S_1, S_2. Let K_+^* be the saturation of $j^{(\alpha_{n+1})}(K_+)$ with respect to

$$\mathscr{F}|j^{(\alpha_{n+1})}(K_+) \cup (\bigcup\{j^{(\alpha_{n+1})}(S) \mid S \in \mathscr{S}(\varDelta),\ S < S_1 \text{ or } \ S < S_2\}) .$$

Let U be a neighborhood of the boundary $\partial(j^{(\alpha_{n+1})}(K_+))$ in K_+^*. We take U sufficiently small so that $\mathscr{F}|(K_+^* - j^{(\alpha_{n+1})}(K_+)) \cup U$ is without holonomy. As in § 5 we have a vector field X on K_+^* such that

(1) X coincides with X already defined in a neighborhood of $\bigcup_{i=1}^{2}(C(j^{(\alpha_{n+1})}(S_i)) \cup W(j^{(\alpha_{n+1})}(S_i)))$.

(2) each orbit of X is contained in an orbit of φ.

(3) the local 1-parameter group defined by $X|(K_+^* - j^{(\alpha_{n+1})}(K_+)) \cup U$ preserves $\mathscr{F}$.

Let ω be the 1-form on K_+^* defined by

(1) $\mathscr{F}|K_+^* = \{v \in TK_+^* \mid \omega(v) = 0\}$,

(2) $\omega(X) \equiv 1$.

Then $d\omega = 0$ on $(K_+^* - j^{(\alpha_{n+1})}(K_+)) \cup U$. There is a 1-form η on K_+^* such that

(1) $d\omega = \eta \wedge \omega$,

(2) $\eta = 0$ on $(K_+^* - j^{(\alpha_{n+1})}(K_+)) \cup U$.

We consider $\eta \wedge d\eta$ as a 3-form on Int K_+^* with compact support. Since Int K_+^* is diffeomorphic to $F^*(S_1) \times \boldsymbol{R}$ and $H^2_{\mathrm{comp}}(F^*(S_1); \boldsymbol{R})$ is trivial by the assumption, we see

$$H^3_{\mathrm{comp}}(K_+^*; \boldsymbol{R}) \cong \sum_{p+q=3} H^p_{\mathrm{comp}}(F^*(S_1); \boldsymbol{R}) \otimes H^q_{\mathrm{comp}}(\boldsymbol{R}; \boldsymbol{R}) \cong 0 .$$

Therefore there is a 2-form ξ on Int K_+^* with compact support satisfying $d\xi = \eta \wedge d\eta$. We can consider ξ as a 2-form on K_+^*. By Proposition 7 there is a function $\alpha: \mathscr{S}(\varDelta^{(\alpha_{n+1})}) \to \{0, 1, 2, \cdots\}$ such that

(1) $\operatorname{supp} \xi \cap (\bigcup\{j^{(\alpha)}(S) \mid S \in \mathscr{S}(\varDelta^{(\alpha_{n+1})})\}) = \varnothing$,

(2) $\alpha(S) = 0$ for $S \notin \{S' \in \mathscr{S}(\varDelta^{(\alpha_{n+1})}) \mid S' < j^{(\alpha_{n+1})}(S_1) \text{ or } S' < j^{(\alpha_{n+1})}(S_2)\}$.

Let $\alpha_n = \alpha \circ j^{(\alpha_{n+1})} + \alpha_{n+1}$. We fix X, ω, η, ξ on $j^{(\alpha_{n+1})}(K_+)$.

(IV). Now consider the case where K_n is the union of an adjacent pair of staircases S_1, S_2. Let K_n^* be the saturation of $j^{(\alpha_{n+1})}(S_1) \cup j^{(\alpha_{n+1})}(S_2)$ with respect to

$$\mathscr{F}|\bigcup\{S \in \mathscr{S}(\varDelta^{(\alpha_{n+1})}) \mid S \leqq j^{(\alpha_{n+1})}(S_1) \quad \text{or} \quad S \leqq j^{(\alpha_{n+1})}(S_2)\} .$$

By the similar argument as in the case (III) we have X, ω, η, ξ on K_n^* and $\alpha: \mathscr{S}(\varDelta^{(\alpha_{n+1})}) \to \{0, 1, 2, \cdots\}$. Let $\alpha_n = \alpha \circ j^{(\alpha_{n+1})} + \alpha_{n+1}$. We fix X, ω, η, ξ on $j^{(\alpha_{n+1})}(K_n)$.

By (I)-(IV) we have X, ω, η, ξ on M, which completes the proof of Theorem 4 and Theorem 5.

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