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ON GROWTH AND DECAY OF SOLUTIONS OF PERTURBED RETARDED LINEAR EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Introduction and statement of certain asymptotic problems. Our purpose in this paper is to compare the solutions of the retarded linear differential equation:

$$\dot{y} = L(y_t) \tag{L}$$

with the solutions of the perturbed equation:

$$\dot{x} = L(x_t) + f(t, x_t) \tag{P}$$

Roughly speaking, we will try to answer the following question: *if the perturbation $f(t, x_t)$ is small, are the solutions of* (P) *"close" to the solutions of* (L), *for large values of t*?

It is clear that we can compare the solutions of the above systems in many ways, depending on what we mean by "close".

Many works have been done in the case where it is required that the difference between the solutions of (L) and the solutions of (P) tends to zero, as t goes to infinity. This can be found, for example in Cooke [2], Evans [5], Kato [8].

In this paper we are interested in the case where the relative error, between the solutions of (L) and the solutions of (P), goes to zero, as t goes to infinity. For ordinary differential equations, results in this direction can be found in Szmidt [11], Onuchic [9], Coppel [3], Brauer and Wong [1], Rodrigues [10], etc. Szmidt uses the topological method of Ważewski and Onuchic [9] uses admissibility theory introduced by Massera and Schäffer. More precisely, we will be concerned with the following problems:

1. THE DIRECT PROBLEM. *Given a solution $y(t)$ of* (L), *$y_t \neq 0$, for all large t, does there exist a family of solutions $x(t)$ of* (P), *such that: $y_t - x_t = o(\|y_t\|)$?*

2. THE CONVERSE PROBLEM. *Given a solution $x(t)$ of* (P), *$x_t \neq 0$, for*

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all large t, does there exist a family of solutions $y(t)$ *of* (L), *such that*: $y_t - x_t = o(||x_t||)$?

For ordinary differential equations, both questions have affirmative answers.

Surprisingly, for retarded equations the direct problem has an affirmative answer, but the coverse one has, in general a negative answer. Hale gave the following counterexample. Let $L = 0$ and let $\dot{x} = -2te^{1-2t}x(t-1)$ be the perturbed equation. Then $x(t) = e^{-t^2}$ is a solution of (P) and there is no solution of the linear equation (L) close to it, in the above sense. However, Hale and Onuchic conjectured that if the Liapunov number of $||x_t||$, that is $\limsup_{t\to\infty} t^{-1}\log||x_t||$, is finite, then the answer to the converse is affirmative.

Chapter 2 contains some basic lemmas which will be important in the analysis of the above problems. Chapter 3 gives a solution of the direct problem. Our approach is related to Hale [6-a]. Chapter 4 gives a solution of the converse problem. As a matter of fact Theorem 4.3 gives this information, Theorem 4.2 is important by itself and in fact it contains the harder part of the chapter. The method used here, for the analysis of the converse problem, when restricted to the case where the delay is zero, is very different from the one used in Coppel [3, Th. 7, p. 104]. The approach used in that reference depends strongly on the fact that the considered ordinary differential equation is defined on a finite dimensional space and it is not clear how to extend it to the case of retarded functional differential equations.

2. **Basic lemmas.** The notation we use is that of Hale [6-c]. Let $r \geqq 0$ and $C = C([-r, 0], R^n)$ with the usual sup norm which will be denoted by $||\cdot||$. We use $|\cdot|$ to denote a norm on R^n. Let L be a continuous linear functional defined on C and suppose that $f(t, \varphi)$ is continuous in $[0, \infty) \times C$. Consider the systems:

$$\text{(L)} \qquad \dot{y} = L(y_t)$$

$$\text{(P)} \qquad \dot{x} = L(x_t) + f(t, x_t)\,.$$

The following lemma gives a characterization of the asymptotic behavior of nonzero solutions of the linear equation.

LEMMA 2.1. *If* $y(t)$ *is a solution of* (L) *such that* $y_t \neq 0$ *for all large t, then there exist a nonnegative integer l and a real number* α, *both uniquely determined, such that*

$$\text{(2.1)} \qquad 0 < \liminf_{t\to\infty} ||y_t||/t^l e^{\alpha t} \leqq \limsup_{t\to\infty} ||y_t||/t^l e^{\alpha t} < \infty$$

PROOF. From Henry [7, Th. 1], it follows that there is a number β such that $y(t)/e^{\beta t}$ does not go to zero. Let $\Lambda = \{\lambda \in \sigma(A) \colon \operatorname{Re} \lambda \geqq \beta\}$, where A denotes the infinitesimal generator of $T(t)$, $\sigma(A)$ is the set of point spectra of A and $T(t)\phi$ indicates the solution of (L) with initial condition ϕ, for $t = 0$. From Hale [6-c, Th. 4.1] it follows that $C = P \oplus Q$, $\dim P < \infty$, P and Q depending on Λ, and there exist positive constants K and γ, such that:

$$\| T(t)\phi^P \| \leqq Ke^{(\beta-\gamma)t} \| \phi^P \| , \quad t \leqq 0$$
$$\| T(t)\phi^Q \| \leqq Ke^{(\beta-\gamma)t} \| \phi^Q \| , \quad t \geqq 0$$

where ϕ^P and ϕ^Q denote the projection onto P and Q.

We can suppose $y_t = T(t)\psi$. The second inequality implies $\| T(t)\psi^P \| / e^{\beta t} \not\rightarrow 0$, $t \rightarrow \infty$. The reference mentioned above also tells that $T(t)\psi^P = \Phi e^{Bt}a$, where Φ is a basis of P and the eigenvalues of B have real part greater than or equal β. Then there exist a nonnegative integer l and a real number $\alpha \geqq \beta$, such that (2.1) is satisfied, since $\| T(t)\psi^Q \| / t^l e^{\alpha t} \rightarrow 0$, as $t \rightarrow \infty$.

REMARK 2.2. The condition (2.1) will be denoted by $\| y_t \| \simeq t^l e^{\alpha t}$.

Let $y(t)$ be a solution of (L) satisfying $\| y_t \| \simeq t^l e^{\alpha t}$. For each $\lambda \in \sigma(A)$ let $k_\lambda \overset{\text{def}}{=} \min \{m \colon \mathscr{N}(A - \lambda I)^{m+1} = \mathscr{N}(A - \lambda I)^m\}$ and

$$N \overset{\text{def}}{=} \max \{k_\lambda \colon \lambda \in \sigma(A) \text{ and } \operatorname{Re} \lambda = \alpha\} , \tag{2.2}$$

where I is the identity operator.

Let $\Lambda = \{\lambda \in \sigma(A) \colon \operatorname{Re} \lambda \geqq \alpha\}$. Following Hale [6-c], this set Λ induces a decomposition $C = P \oplus Q$, where $P = P_\Lambda$, $Q = Q_\Lambda$ and P is a finite dimensional subspace of C.

Let $P_1 = \{\phi \in P \colon \lim_{t \to \infty} \| T(t)\phi \| / t^l e^{\alpha t} = 0\}$.

LEMMA 2.2. *There exists a subspace P_2 of P and projections X^{P_i}: $P \rightarrow P_i$, $i = 1, 2$, such that, $P = P_1 \oplus P_2$, $X^{P_1} + X^{P_2} = I$. Furthermore, there are positive constants M and σ, such that,*

$$\begin{aligned} \| T(t)X^{P_1}T(-s)\phi^P \| &\leqq Mt^{l-1}s^{N-l}e^{\alpha(t-s)} \| \phi^P \| , \quad \sigma \leqq s \leqq t \\ \| T(t)X^{P_2}T(-s)\phi^P \| &\leqq Mt^l s^{N-l-1}e^{\alpha(t-s)} \| \phi^P \| , \quad \sigma \leqq t \leqq s . \end{aligned} \tag{2.3}$$

PROOF. Let us define $C^* = C([0, r], R^{n*})$, where R^{n*} is the n-dimensional vector space of row vectors, and for any α in C^*, ϕ in C, define

$$(\alpha, \phi) = \alpha(0)\phi(0) - \int_{-r}^{0} \int_{0}^{\theta} \alpha(\xi - \theta)[d\eta(\theta)]\phi(\xi)d\xi$$

where η denotes the function of bounded variation which defines the linear function L. Following Hale [6-c, Chap. 7], let A^* be the formal adjoint of A, relative to the bilinear form defined above, and $P = P_\Lambda$, $P_\Lambda^* = P^*$ be the generalized eigenspaces of equation (L) and of the adjoint equation, $\dot{y}(\tau) = -\int_{-r}^{0} y(\tau - \theta)d\eta(\theta)$, respectively, associated with Λ. Let Φ and Ψ be bases for P and P^*, respectively, such that $(\Phi, \Psi) = I$, the identity. From the above reference we get $T(t)\phi^P = \Phi e^{Bt}(\Psi, \phi^P)$. Furthermore the eigenvalues of the matrix B are the elements of Λ and B can be supposed to be in the Jordan canonical form. We claim that we can find projections Z_1, Z_2 in R^q, where $q =$ order of B and positive constants $\bar{M}$ and σ, such that

$$
\begin{aligned}
&|e^{Bt}Z_1e^{-Bs}| \leqq \bar{M}t^{l-1}s^{N-l}e^{\alpha(t-s)}\,, \qquad \sigma \leqq s \leqq t \\
&|e^{Bt}Z_2e^{-Bs}| \leqq \bar{M}t^{l}s^{N-l-1}e^{\alpha(t-s)}\,, \qquad \sigma \leqq t \leqq s\,.
\end{aligned} \tag{2.4}
$$

The result given in (2.4) is partially stated in Coddington-Levinson [4, p. 106]. We present now an outline of the proof. The idea is to decompose B into Jordan's blocks and to obtain (2.4) for each block.

Let $B = \operatorname{diag}(B_1, B_2)$, where the eigenvalues of B_1 have real part greater than α and the eigenvalues of B_2 have real part α.

For the part corresponding to B_1 it is easier to get the projections and in fact we can get estimates better than (2.4).

Let us now suppose that J is a Jordan's block of B_2, of order s. We construct the projection Z_1^J such that $|e^{Jt}Z_1^J| = o(t^l e^{\alpha t})$, that is,

$$
e^{Jt}Z_1^J = e^{\lambda t}\begin{bmatrix} 1 & & & & 0 \\ \vdots & \ddots & & & \\ t^{l-1}/(l-1)! & & \ddots & & \\ \vdots & & & \ddots & \\ t^{s-1}/(s-1)! & \cdots & t^{l-1}/(l-1)! & \cdots & 1 \end{bmatrix}\begin{bmatrix} 0 & & & 0 \\ & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & \underbrace{0 \quad}_{l\ \text{columns}} & & 1 \end{bmatrix}.
$$

If we take $Z_2^J = I - Z_1^J$, a straightforward calculation gives estimates like (2.4) for Z_1^J and Z_2^J. Using those projections we get Z_1 and Z_2.

The next step is to define X^{P_1} and X^{P_2}. Let $\Phi = (\phi_1, \cdots, \phi_q)$ be the considered basis of P. We define,

$$
(X^{P_i}\phi_1, \cdots, X^{P_i}\phi_q) \stackrel{\text{def}}{=} \Phi Z_i\,, \qquad i = 1, 2\,.
$$

The above definition implies that

$$
T(t)X^{P_i}T(-s)\phi^P = T(t)X^{P_i}\Phi e^{-Bs}(\Psi, \phi^P) = T(t)\Phi Z_i e^{-Bs}(\Psi, \phi^P) = \Phi e^{Bt}Z_i e^{-Bs}(\Psi, \phi^P)
$$

for $i = 1, 2$ and the proof is complete.

LEMMA 2.3. *Let $\rho(t)$ and $\beta(t)$ be nonnegative continuous functions defined for $t \geqq 0$, such that $\int_0^\infty \rho(s)ds < \infty$ and $\beta(t)$ is a decreasing func-*

tion. If $u(t) \geqq 0$ is a bounded continuous function for $t \geqq 0$ and satisfies,

$$u(t) \leqq \beta(t) + \int_t^\infty \rho(s)u(s)ds\,, \qquad \text{if} \quad t \geqq t_0 \geqq 0\,,$$

then,

$$u(t) \leqq \beta(t) \exp\left[\int_t^\infty \rho(s)ds\right], \qquad \text{for} \quad t \geqq t_0\,.$$

The proof of the above lemma is straightforward and it is similar to the proof of Gronwall's inequality.

The following integral inequality plays an important role on the analysis of the converse problem.

LEMMA 2.4. *Let ρ, $g \in L_1([0, \infty), R)$ be nonnegative continuous functions. Let $\gamma(t) > 0$ be a decreasing continuous function, for $t \geqq \sigma$ and σ sufficiently large, in such a way that, $\beta = \int_\sigma^\infty g(s)ds + \int_\sigma^\infty \rho(s)ds < 1$. Suppose that $u(t)$ is a nonnegative continuous function such that $\gamma(t)u(t)$ is bounded and*

$$u(t) \leqq K + \int_\sigma^t u(s)\rho(s)ds + \frac{1}{\gamma(t)}\int_t^\infty \gamma(s)u(s)g(s)ds$$

for $t \geqq \sigma$, where K is a constant. Then,

$$u(t) \leqq \frac{K}{1-\beta} \exp\left(\frac{1}{1-\beta}\int_t^\infty g(s)ds\right).$$

PROOF. Let $v(t) \overset{\text{def}}{=} \max_{\sigma \leqq s \leqq t} u(s)$. Then $v(t)$ is an increasing continuous function, such that, $u(t) \leqq v(t)$ and $\gamma(t)v(t)$ is bounded for $t \geqq 0$. For a given $t \geqq \sigma$, there exists $t_1 \in [\sigma, t]$ satisfying $v(t) = u(t_1)$. This implies,

$$v(t) \leqq K + \int_\sigma^{t_1} v(s)\rho(s)ds + \frac{1}{\gamma(t_1)}\int_{t_1}^\infty \gamma(s)v(s)g(s)ds\,.$$

But

$$\begin{aligned}\int_{t_1}^\infty \gamma(s)v(s)g(s)ds &= \int_{t_1}^t \gamma(s)v(s)g(s)ds + \int_t^\infty \gamma(s)v(s)g(s)ds \\ &\leqq \gamma(t_1)v(t)\int_\sigma^\infty g(s)ds + \int_t^\infty \gamma(s)v(s)g(s)ds\,.\end{aligned}$$

Combining the above inequalities we get

$$v(t) \leqq K + v(t)\left[\int_\sigma^\infty \rho(s)ds + \int_\sigma^\infty g(s)ds\right] + \frac{1}{\gamma(t)}\int_t^\infty \gamma(s)v(s)g(s)ds\,.$$

Then,

$$\gamma(t)v(t) \leqq \frac{1}{1-\beta}\left[K\gamma(t) + \int_t^\infty \gamma(s)v(s)g(s)ds\right].$$

Using Lemma 2.3 we get

$$\gamma(t)v(t) \leqq \frac{K}{1-\beta}\gamma(t)\exp\left(\frac{1}{1-\beta}\int_t^\infty g(s)ds\right)$$

and this completes the proof.

3. A solution of the direct problem. The next theorem gives a solution of the above problem.

THEOREM 3.1. *Let $y(t)$ be a solution of* (L) *with Liapunov number α. Let S be the subspace of C defined by $\{\phi \in C: \lim_{t\to\infty} T(t)\phi/\|y_t\| = 0\}$. Let N be defined as in* (2.2). *Suppose f satisfies: $f(t, 0) = 0$, $|f(t, \phi) - f(t, \psi)| \leqq h(t)\|\phi - \psi\|$, for $t \geqq 0$, ϕ and ψ in C, where h is continuous and satisfies $\int_0^\infty t^{N-1}h(t)dt < \infty$.*

Then there exist a subset Y_S of C and a real number $\sigma \geqq 0$, such that:

(a) *for each $\phi \in Y_S$ the solution $x_t = x_t(\sigma, \phi)$, of* (P), *satisfies*:

$$\lim_{t\to\infty} \|y_t - x_t\|/\|y_t\| = 0 \tag{3.1}$$

(b) *there is a homeomorphism $W: S \to Y_S$, whose inverse is the restriction to Y_S of a projection from C onto S.*

PROOF. Lemma 2.1 implies that there is a nonnegative integer l, such that, $\|y_t\| \simeq t^l e^{\alpha t}$. If we put $x = y + z$, finding solutions of (P) satisfying (3.1) is equivalent to finding solutions $z(t)$ of

$$\dot{z} = L(z_t) + f(t, y_t + z_t) \overset{\text{def}}{=} L(z_t) + F(t, z_t) \tag{3.2}$$

satisfying $z_t = o(t^l e^{\alpha t})$.

Let $\Lambda = \{\lambda \in \sigma(A): \operatorname{Re}\lambda \geqq \alpha\}$ and $C = P \oplus Q$, as in Lemma 2.2. Let $P_1 = \{\phi \in P: T(t)\phi = o(t^l e^{\alpha t})\}$. It is easy to see that $S = Q \oplus P_1$. Lemma 2.2 implies that there is a subspace P_2, such that, the estimates (2.3) hold, for a sufficiently large σ.

Let E be the space of functions $g \in C([\sigma, \infty), C)$, such that $g(t) = o(t^l e^{\alpha t})$. If g is in E, we define $\|g\|_E = \sup_{t\geqq\sigma} \|g(t)\|/t^l e^{\alpha t}$. It is possible to prove that, in E, the equation (3.2) with initial condition $z_\sigma = \phi$ is equivalent to the integral equation:

$$\begin{aligned} z_t = T(t-\sigma)\phi^S &+ \int_\sigma^t T(t-s)X_0^Q F(s, z_s)ds \\ &+ \int_\sigma^t T(t)X^{P_1}T(-s)X_0^P F(s, z_s)ds - \int_t^\infty T(t)X^{P_2}T(-s)X_0^P F(s, z_s)ds\,, \end{aligned} \tag{3.3}$$

where X_0^P and X_0^Q are given as in Hale [6-c, p. 186], X^{P_1}, X^{P_2} are obtained in Lemma 2.2 and

$$\phi^S = \phi + \int_\sigma^\infty T(\sigma)X^{P_2}T(-s)X_0^P F(s, z_s)ds .$$

Before solving equation (3.3), where ϕ^S is an arbitrary element of S, we will first solve the following equation in E:

$$g(t) = T(t-\sigma)\phi^S + \int_\sigma^t T(t-s)X_0^Q F(s, g(s))ds + \int_\sigma^t T(t)X^{P_1}T(-s)X_0^P F(s, g(s))ds - \int_t^\infty T(t)X^{P_2}T(-s)X_0^P F(s, g(s))ds . \tag{3.4}$$

If $g(t)$ is a solution of (3.4), then z_t is a solution of (3.3) where z_t is defined by $z_\sigma = g(\sigma)$, $z(t) = g(t)(0)$, for $t \geqq \sigma$.

We will solve equation (3.4) via contraction principle. If $g \in E$, let $(Ug)(t)$ be the second member of (3.4). Of course, U depends on ϕ^S. From Hale [6-c, p. 181] it follows that $\|T(t)\phi^Q\| \leqq Ke^{(\alpha-\gamma)t}\|\phi^Q\|$, for $t \geqq 0$, where K and γ are positive numbers. If we combine this estimate with the ones given in Lemma 2.2 we get for g and w in E:

$$\|(Ug)(t)\| \leqq M_1 t^l e^{\alpha t}\Big[o(1) + e^{-\gamma t}\int_\sigma^t e^{\gamma s}h(s)ds + \frac{1}{t}\int_\sigma^t s^N h(s)ds + \int_t^\infty s^{N-1}h(s)ds\Big] , \tag{3.5}$$

$$\|(Ug)(t) - (Uw)(t)\| \leqq M_2 t^l e^{\alpha t}\int_\sigma^\infty s^{N-1}h(s)ds\, \|g - w\|_E , \tag{3.6}$$

where M_1, M_2 are positive constants and M_2 does not depend on ϕ^S.

Using the above estimates and the uniform contraction principle (see Hale [6-b, Th. 3.2]), we prove that there exists a unique fixed point $g = g_{\phi^S}$ depending continuously on ϕ^S, if σ is large enough to satisfy $M_2\int_\sigma^\infty s^{N-1}h(s)ds < 1$.

Consider now the function

$$W(\phi^S) \stackrel{\text{def}}{=} \phi^S - \int_\sigma^\infty T(\sigma)X^{P_2}T(-s)X_0^P F(s, g_{\phi^S}(s))ds$$

defined on S. Let $Y_S \stackrel{\text{def}}{=} W(S)$. If we consider the continuous projection,

$$X^S \stackrel{\text{def}}{=} X_0^Q + T(\sigma)X^{P_1}T(-\sigma)X_0^P$$

from C onto S, it is not difficult to see that $X^S|_{Y_S}$ is the inverse of W. Then W is a homeomorphism and the proof is complete.

4. A solution of the converse problem. The following theorem shows us how to relate the Liapunov numbers of solutions of the perturbed equation (P) with the spectrum of the infinitesimal generator A.

THEOREM 4.1. *Let $x(t)$ be a solution of* (P) *with Liapunov number* $\mu \in R$. *We assume that* $|f(t, \phi)| \leqq h(t)\|\phi\|$ *for all* $t \geqq 0$ *and* ϕ *in* C, *where* $f(t, \phi)$, $h(t)$ *are continuous and* $\int_0^\infty h(s)ds < \infty$. *Then there exists* $\lambda \in \sigma(A)$ *such that* $\operatorname{Re} \lambda = \mu$.

PROOF. Suppose that $\operatorname{Re} \lambda \neq \mu$ for all $\lambda \in \sigma(A)$. Let $\Lambda = \{\lambda \in \sigma(A): \operatorname{Re} \lambda > \mu\}$. Then there exist positive numbers ε and K such that $C = P \oplus Q$ and

$$\begin{aligned} \|T(t)\phi^P\| &\leqq Ke^{(\mu+\varepsilon)t}\|\phi^P\|, \qquad t \leqq 0 \\ \|T(t)\phi^Q\| &\leqq Ke^{(\mu-\varepsilon)t}\|\phi^Q\|, \qquad t \geqq 0 . \end{aligned} \tag{4.1}$$

The variation of constant formula gives,

$$x_t = T(t-\sigma)\phi + \int_\sigma^t T(t-s)X_0^Q f(s, x_s)ds - \int_t^\infty T(t-s)X_0^P f(s, x_s)ds$$

which combined with the above inequalities implies,

$$\begin{aligned} \|T(t-\sigma)\phi\|e^{-(\mu+\varepsilon)t} \leqq \|x_t\|e^{-(\mu+\varepsilon)t} &+ Ke^{-2\varepsilon t}\int_\sigma^t e^{2\varepsilon s}h(s)\|x_s\|e^{-(\mu+\varepsilon)s}ds \\ &+ K\int_t^\infty h(s)\|x_s\|e^{-(\mu+\varepsilon)s}ds , \qquad t \geqq \sigma \end{aligned}$$

As a consequence of the assumption that the Liapunov number of $\|x_t\|$ is μ we get that the second member of the above inequality goes to zero as $t \to \infty$. But this implies that $\phi^P = 0$. Using again (4.1) we obtain

$$\begin{aligned} \|x_t\|e^{-(\mu-\varepsilon)t} \leqq K\Big[1 &+ \int_\sigma^t h(s)\|x_s\|e^{-(\mu-\varepsilon)s}ds \\ &+ e^{2\varepsilon t}\int_t^\infty e^{-2\varepsilon s}h(s)\|x_s\|e^{-(\mu-\varepsilon)s}ds\Big], \qquad t \geqq \sigma . \end{aligned}$$

Lemma 2.4 implies that $\|x_t\|e^{-(\mu-\varepsilon)t}$ is bounded, for $t \geqq \sigma$. But this contradicts the fact that the Liapunov number of $\|x_t\|$ is μ. The proof is complete.

The next theorem gives a more precise information about the growth or decay of the solutions of the perturbed equation.

THEOREM 4.2. *Let $x(t)$ be a solution of* (P) *with Liapunov number* $\mu \in R$. *Suppose* $|f(t, \phi)| \leqq h(t)\|\phi\|$, *for all* $t \geqq 0$ *and* ϕ *in* C, *where* $f(t, \phi)$, $h(t)$ *are continuous* $\int_0^\infty t^{N-1}h(t)dt < \infty$, *and N is defined as in* (2.2).

Then there exists a nonnegative integer l, such that,

$$0 < \liminf_{t\to\infty} ||x_t||/t^l e^{\mu t} \leqq \limsup_{t\to\infty} ||x_t||/t^l e^{\mu t} < \infty .$$

PROOF. Let $\Lambda = \{\lambda \in \sigma(A): \operatorname{Re}(\lambda) > \mu\}$. This set induces a decomposition $C = S \oplus \tilde{Q}$, where S has finite dimension and there exist positive numbers k_1 and ε such that,

$$\begin{aligned} &||T(t)\psi^{\tilde{Q}}|| \leqq k_1 t^{N-1} e^{\mu t} ||\psi^{\tilde{Q}}||, \qquad t \geqq 0 \\ &||T(t)\psi^S|| \leqq k_1 e^{(\mu+\varepsilon)t} ||\psi^S|| , \qquad t \leqq 0 . \end{aligned} \tag{4.2}$$

The variation of constant formula gives,

$$x_t = T(t-\sigma)\phi + \int_\sigma^t T(t-s)X_0^{\tilde{Q}} f(s, x_s)ds - \int_t^\infty T(t-s)X_0^S f(s, x_s)ds \tag{4.3}$$

where $X_0^{\tilde{Q}}$, X_0^S are given in Hale [6-c, p. 186]. From (4.2) and (4.3) we get, for a convenient constant k,

$$\begin{aligned} e^{-(\mu+\varepsilon)t} ||T(t-\sigma)\phi|| \leqq ||x_t|| e^{-(\mu+\varepsilon)t} &+ k e^{-\varepsilon t/2} \int_\sigma^t h(s) ||x_s|| e^{-(\mu+\varepsilon/2)s} ds \\ &+ k \int_t^\infty h(s) ||x_s|| e^{-(\mu+\varepsilon)s} ds . \end{aligned}$$

The Liapunov number of $||x_t||$ being μ implies that the second member of the above inequality goes to zero, as $t \to \infty$ and thus $e^{-(\mu+\varepsilon)t}||T(t-\sigma)\phi|| \to 0$, as $t \to \infty$. Then $\phi^S = 0$ and $||T(t-\sigma)\phi|| = O(t^{N-1}e^{\mu t})$, for large values of t. From (4.2) and (4.3) we get,

$$\begin{aligned} t^{-(N-1)} e^{-\mu t} ||x_t|| \leqq k_1 \Big[1 &+ \int_\sigma^t s^{N-1} h(s) s^{-(N-1)} e^{-\mu s} ||x_s|| sd \Big] \\ &+ k_1 t^{-(N-1)} e^{\varepsilon t} \int_t^\infty e^{-\varepsilon s} s^{N-1} h(s) s^{-(N-1)} e^{-\mu s} ||x_s|| ds . \end{aligned}$$

If we let $u(t) \stackrel{\text{def}}{=} t^{-(N-1)} e^{-\mu t} ||x_t||$, $\gamma(t) \stackrel{\text{def}}{=} e^{-\varepsilon t} t^{N-1}$, Lemma 2.4 shows that $u(t)$ is bounded.

Let $l \stackrel{\text{def}}{=} \min\{n \geqq 0: ||x_t|| = O(t^n e^{\mu t})$, for large values of $t\}$. Let $\Lambda_P = \{\lambda \in \sigma(A): \operatorname{Re}\lambda = \mu\}$, $\Lambda_Q = \{\lambda \in \sigma(A): \operatorname{Re}\lambda < \mu\}$. As before $C = P \oplus S \oplus Q$. If we let $P_1 \stackrel{\text{def}}{=} \{\psi \in P: ||T(t)\psi|| = o(t^l e^{\mu t})\}$, there exists a subspace P_2 of P and there exist positive constants K and ε, such that:

$$\begin{aligned} &||T(t)X^{P_1}T(-s)\psi^P|| \leqq K t^{l-1} s^{N-l} e^{\mu(t-s)} ||\psi^P|| , \quad \sigma \leqq s \leqq t \\ &||T(t)X^{P_2}T(-s)\psi^P|| \leqq K t^l s^{N-l-1} e^{\mu(t-s)} ||\psi^P|| , \quad \sigma \leqq t \leqq s \\ &||T(t-s)\psi^Q|| \leqq K e^{(\mu-\varepsilon)(t-s)} ||\psi^Q|| , \quad \sigma \leqq s \leqq t \\ &||T(t-s)\psi^S|| \leqq K e^{(\mu+\varepsilon)(t-s)} ||\psi^S|| , \quad \sigma \leqq t \leqq s \end{aligned} \tag{4.4}$$

As in (3.3) we can prove that, there is ψ in C, such that x_t can be written in the form

$$(4.5)\qquad x_t = T(t-\sigma)\psi + \int_\sigma^t T(t-s)X_0^Q f(s, x_s)ds$$
$$+ \int_\sigma^t T(t)X^{P_1}T(-s)X_0^P f(s, x_s)ds - \int_t^\infty T(t)X^{P_2}T(-s)X_0^P f(s, x_s)ds$$
$$- \int_t^\infty T(t-s)X_0^S f(s, x_s)ds\ ,$$

where $\psi = \phi^{\tilde{Q}} + \int_\sigma^\infty T(\sigma)X^{P_2}T(-s)X_0^P f(s, x_s)ds.$

Below, we show the convergence of the above integrals. Then,

$$(4.6)\qquad \|T(t-\sigma)\psi\| \leqq O(t^l e^{\mu t}) + Ke^{(\mu-\varepsilon)t}\int_\sigma^t e^{-(\mu-\varepsilon)s}\|x_s\|h(s)ds$$
$$+ Kt^{l-1}e^{\mu t}\int_\sigma^t s^{N-l}e^{-\mu s}\|x_s\|h(s)ds$$
$$+ Kt^l e^{\mu t}\int_t^\infty s^{N-l-1}e^{-\mu s}\|x_s\|h(s)ds + Ke^{\mu t}e^{\varepsilon t}\int_t^\infty e^{-\varepsilon s}e^{-\mu s}\|x_s\|h(s)ds$$
$$\leqq O(t^l e^{\mu t}) + K\Big[t^l e^{\mu t}e^{-\varepsilon t}\int_\sigma^t e^{\varepsilon s}h(s)s^{-l}e^{-\mu s}\|x_s\|ds$$
$$+ t^l e^{\mu t}\frac{1}{t}\int_\sigma^t s^N h(s)s^{-l}e^{-\mu s}\|x_s\|ds + t^l e^{\mu t}\int_t^\infty s^{N-1}h(s)s^{-l}e^{-\mu s}\|x_s\|ds$$
$$+ t^l e^{\mu t}\Big(t^{-l}\int_t^\infty s^l h(s)s^{-l}e^{-\mu s}\|x_s\|ds\Big)\Big]$$
$$= O(t^l e^{\mu t})\ .$$

This implies that $\|T(t-\sigma)\psi\| = O(t^l e^{\mu t})$. Now, we claim that $t^{-l}e^{-\mu t}T(t-\sigma)\psi$ does not go to zero, as $t\to\infty$. Let us suppose this is not the case. For $l \geqq 1$, by Lemma 2.1, we must have $T(t-\sigma)\psi = O(t^{l-1}e^{\mu t})$.

A procedure similar to (4.6) gives us:

$$t^{-(l-1)}e^{-\mu t}\|x_t\| \leqq K + Ke^{-\varepsilon t}\int_\sigma^t e^{\varepsilon s}s^l h(s)s^{-l}e^{-\mu s}\|x_s\|ds$$
$$+ K\int_\sigma^t s^{N-1}h(s)s^{-(l-1)}e^{-\mu s}\|x_s\|ds$$
$$+ Kt\int_t^\infty s^{N-2}h(s)s^{-(l-1)}e^{-\mu s}\|x_s\|ds$$
$$+ Ke^{\varepsilon t}\int_t^\infty e^{-\varepsilon s}s^l h(s)s^{-l}e^{-\mu s}\|x_s\|ds\ ,\quad t\geqq\sigma\ .$$

Then there is a constant M such that

$$t^{-(l-1)}e^{-\mu t}||x_t|| \leqq M + K\left[\int_\sigma^t s^{N-1}h(s)s^{-(l-1)}e^{-\mu s}||x_s||ds \right.$$
$$\left. + t\int_t^\infty s^{N-2}s^{-(l-1)}e^{-\mu s}||x_s||h(s)ds\right], \qquad t \geqq \sigma .$$

If we let $u(t) \overset{\text{def}}{=} t^{-(l-1)}e^{-\mu t}||x_t||$ and $\gamma(t) \overset{\text{def}}{=} t^{-1}$, using Lemma 2.4 we get $||x_t||t^{-(l-1)}e^{-\mu t}$ bounded, which contradicts the definition of l.

If $l = 0$, we have $P_1 = \{0\}$, $P_2 = P$, $||x_t|| = O(e^{\mu t})$ and $x_t = T(t-\sigma)\psi + \int_\sigma^t T(t-s)X_0^Q f(s, x_s)ds - \int_t^\infty T(t-s)X_0^{P+S}f(s, x_s)ds$. If $\varepsilon > 0$ is sufficiently small, there exists a positive constant K_1, such that

$$||T(t-s)\phi^{P+S}|| \leqq K_1 e^{(\mu+\varepsilon)(t-s)}||\phi^{P+S}|| , \qquad t \leqq s$$

and that $||T(t-\sigma)\psi|| \leqq K_1 e^{(\mu-\varepsilon)t}||\psi||$, since $T(t-\sigma)\psi/e^{\mu t} \to 0$ by hypothesis.

If we combine this result with (4.4) we get, for a convenient constant K_2,

$$||x_t|| \leqq K_2\left[e^{(\mu-\varepsilon)t} + \int_\sigma^t e^{(\mu-\varepsilon)(t-s)}h(s)||x_s||ds + \int_t^\infty e^{(\mu+\varepsilon)(t-s)}h(s)||x_s||ds\right], \qquad t \geqq \sigma .$$

Then,

$$e^{-(\mu-\varepsilon)t}||x_t|| \leqq K_2\left[1 + \int_\sigma^t h(s)e^{-(\mu-\varepsilon)s}||x_s||ds\right]$$
$$+ K_2 e^{2\varepsilon t}\int_t^\infty e^{-(\mu+\varepsilon)s}h(s)||x_s||ds , \qquad t \geqq \sigma .$$

If we let $u(t) \overset{\text{def}}{=} ||x_t||e^{-(\mu-\varepsilon)t}$ and $\gamma(t) \overset{\text{def}}{=} e^{-2\varepsilon t}$, Lemma 2.4 implies that $u(t)$ is bounded, which contradicts the fact that the Liapunov number of $||x_t||$ is μ.

Our conclusion is that $T(t-\sigma)\psi/t^l e^{\mu t} \nrightarrow 0$, as $t \to \infty$. So, there is a positive constant c_1, such that

$$t^{-l}e^{-\mu t}||T(t-\sigma)\psi|| \geqq c_1 > 0 \tag{4.7}$$

for all large t. This is a consequence of Lemma 2.1.

Using (4.5), (4.4) and (4.7), we get,

$$t^{-l}e^{-\mu t}||x_t|| \geqq c_1 - K\left[e^{-\varepsilon t}\int_\sigma^t e^{\varepsilon s}h(s)s^{-l}e^{-\mu s}||x_s||ds + \frac{1}{t}\int_\sigma^t s^N h(s)s^{-l}e^{-\mu s}||x_s||ds\right.$$
$$\left. + \int_t^\infty s^{N-1}h(s)s^{-l}e^{-\mu s}||x_s||ds + e^{\varepsilon s}\int_t^\infty e^{-\varepsilon s}s^l h(s)s^{-l}e^{-\mu s}||x_s||ds\right]$$
$$= c_1 + o(1) , \qquad \text{as} \quad t \to \infty .$$

Then, $\liminf_{t\to\infty} t^{-l}e^{-\mu t}||x_t|| > 0$ and this completes the proof.

The next theorem gives answer to the converse problem.

THEOREM 4.3. *Suppose the solution x_t of* (P) *and the perturbation $f(t, \phi)$ satisfy the conditions of Theorem* 4.2. *Then there exists a solution $y(t)$ of* (L), *such that,*

$$x_t - y_t = o(\|x_t\|) , \qquad as \quad t \to \infty .$$

PROOF. Theorem 4.2 implies that there is a nonnegative integer l, such that $\|x_t\| \simeq t^l e^{\mu t}$ for large values of t. From (4.4) and (4.5) we get:

$$\begin{aligned} t^{-l}e^{-\mu t}\|x_t - T(t-\sigma)\psi\| \leqq K\Big[& t^{-l}e^{-\varepsilon t}\int_\sigma^t e^{\varepsilon s}h(s)s^{-l}e^{-\mu s}\|x_s\|ds \\ & + \frac{1}{t}\int_\sigma^t s^N h(s)s^{-l}e^{-\mu s}\|x_s\|ds + \int_t^\infty s^{N-1}h(s)s^{-l}e^{-\mu s}\|x_s\|ds \\ & + e^{\varepsilon t}\int_t^\infty e^{-\varepsilon s}s^l h(s)s^{-l}e^{-\mu s}\|x_s\|ds\Big] \to 0 , \qquad t \to \infty . \end{aligned}$$

If we let $y_t = T(t-\sigma)\psi$, we have $\|x_t - y_t\|/\|x_t\| \to 0$, as $t \to \infty$ and the proof is complete.

REMARK 4.1. In fact, in general, we can get a result more general than the above one in the following sense: under the conditions of Theorem 4.3, we can get a family of solutions $T(t)\phi$ of (L), for ϕ varying in a translate of a subspace S of C, such that $\operatorname{codim} S < \infty$. In order to get this result we suffice to pick up the solution y_t obtained in Theorem 4.3 and apply Theorem 3.1, considering the perturbation as the zero function.

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