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EXISTENCE OF PERIODIC SOLUTIONS OF PERIODIC NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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1. Introduction. In a recent paper by Busenberg and Cooke [1], the scalar differential-delay equation

$$x'(t) = p(t)x(t-h)(1-x(t)) - cx(t)\ , \quad t \geqq 0\ , \tag{1.1}$$

was studied, where c and h are positive constants and $p(t)$ is a positive continuous periodic function of minimal period $\omega > 0$. Equation (1.1) is modelled on the proportion of infectious persons with a communicable disease carried by a vector. The motivation for studying this model is to explain periodic outbreaks of certain infectious diseases. See also [3], [4], or [5] for discussions of similar models. In [1], it was shown that a positive periodic solution exists if c is less than a certain threshold value c_h, and that no positive periodic solution exists if c is greater than or equal to c_h.

In this paper, using the same ideas as in [1], we obtain similar results for a system more general than (1.1) (see Section 3). In Section 4, we shall show the existence of periodic solutions of (1.1) as a corollary to our results, and moreover, we shall discuss further applications of the results obtained in Section 3. Particularly, Equation (4.3) has some connection with the scalar differential-delay equation $x'(t) = -\sin x(t-h)$, studied by the author in [6].

We thank the referee for pointing out that Busenberg and Cooke [2] also generalized their results to a system more general than (1.1).

2. Notations and assumptions. Throughout this paper, we shall employ the same notations as in [1], but we repeat those for convenience. Let I and R denote the intervals $0 \leqq t < \infty$, and $-\infty < t < \infty$, respectively. For any continuous periodic function $u(t)$ defined on I, the symbols $\underline{u}$ and $\bar{u}$ will denote $\min_{t\in I} u(t)$ and $\max_{t\in I} u(t)$, respectively. For a given $h > 0$, $\mathscr{C}^h$ denotes the space of continuous functions mapping the interval $[-h, 0]$ into R. The norm we use is the supremum norm in all cases. $\mathscr{C}^h_+$ will denote the set of $\phi \in \mathscr{C}^h$ such that $\phi(\theta) \geqq 0$ for $\theta \in [-h, 0]$. For any continuous function $x(u)$ defined on $-h \leqq u < A$, $A > 0$ and any fixed t, $0 \leqq t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t-h, t]$, i.e., x_t is an element of $\mathscr{C}^h$ defined by $x_t(\theta) = x(t+\theta)$, $-h \leqq \theta \leqq 0$.

Consider a scalar nonlinear functional differential equation

$$x'(t) = p(t)f(t, x_t) - c(t)x(t), \quad t \geqq 0 , \tag{2.1}$$

where $p(t)$ and $c(t)$ are continuous periodic functions of period $\omega > 0$, and moreover, $p(t)$ is positive. We define the sets C^h_ω, K, and K_r by

$$C^h_\omega = \{x: [-h, \infty) \to R, \ x(t) \text{ is continuous and periodic of period } \omega > 0\} ,$$

$$K = \{x \in C^h_\omega: x(t) \geqq 0 \quad \text{for all} \quad t \in [-h, \infty)\} ,$$

and

$$K_r = \{x \in C^h_\omega: 0 \leqq x(t) \leqq r \quad \text{for all} \quad t \in [-h, \infty)\} ,$$

respectively. We make the following assumptions.

(H 1) $f(t, \phi)$ is a scalar functional which is defined on $I \times \mathscr{C}^h$, is continuous in (t, ϕ), satisfies $f(t, \phi) \geqq 0$ for $\phi \in \mathscr{C}^h_+$ and $f(t+\omega, \phi) = f(t, \phi)$ for all $t \in I$, $\phi \in \mathscr{C}^h$.

(H 2) For $M(r) = \sup\{f(t, \phi): t \in I, \ 0 \leqq \phi(\theta) \leqq r \ \text{ for } \ \theta \in [-h, 0]\}$, $\lim_{r\to\infty} M(r)/r = 0$.

(H 3) There exists a continuous functional $\Lambda(t, \phi)$, which is linear in ϕ, and the following conditions hold:

(i) $a\int_0^\omega x(t)dt \leqq \int_0^\omega \Lambda(t, x_t)dt \leqq b\int_0^\omega x(t)dt$ for $x \in K$, where a, b are positive constants.

(ii) $\{\Lambda(t, \phi) - f(t, \phi)\}/|\phi| \to 0$ uniformly in t as $\phi \in \mathscr{C}^h_+$ tends to 0.

(H 4) $\Lambda(t, \phi) > f(t, \phi)$ for all $t \in I$, $\phi \in \mathscr{C}^h_+$ such that $\phi(\theta) > 0$ for $\theta \in [-h, 0]$.

Busenberg and Cooke imposed certain conditions on their equation, and considered the existence of periodic solutions $x(t)$ in $0 < x < 1$, since they considered in [2] an equation which is modelled on the proportion of infectious persons. But our equation (2.1) does not necessarily be modelled on the proportion of infectious persons. Thus the above conditions are different from those in [2]. We allow $c(t)$ with $\underline{c} < 0$ and consider the existence of periodic solutions $x(t)$ which is not necessarily in $0 < x < 1$.

3. Existence of positive periodic solutions. In this section we shall discuss the existence of positive periodic solutions of (2.1) by modifying the manner in [1]. For $x \in C^h_\omega$, define the nonlinear operator G by

$$(Gx)(t) = x(t) + f(t, x_t) , \quad \text{if} \quad t \geqq 0 ,$$

$$(Gx)(t) = (Gx)(k\omega + t) , \quad \text{if} \quad -h \leqq \theta < 0 ,$$

and define the linear operator L by

$$(Lx)(t) = e^{-\gamma(t)}\left\{(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega e^{\gamma(s)}p(s)x(s)ds + \int_0^t e^{\gamma(s)}p(s)x(s)ds\right\} , \quad \text{if} \quad t \geqq 0 ,$$
$$(Lx)(t) = (Lx)(k\omega + t) , \qquad \text{if} \quad -h \leqq t < 0 ,$$

where $\gamma(t) = \int_0^t (c(s) + p(s))ds$ with the assumption $\gamma(\omega) > 0$ and k is the smallest positive integer such that $k\omega > h$. Since G takes C_ω^h into C_ω^h, we can define the operator N on C_ω^h by $(Nx)(t) = ((L\circ G)x)(t)$ for $t \in [-h, \infty)$.

By the definition of N and Assumption (H 1), if $x(t) \geqq 0$ for all $t \in [-h, \infty)$, then $x(t) + f(t, x_t) \geqq 0$ for all $t \geqq 0$, and consequently $(Nx)(t) \geqq 0$ for all $t \geqq 0$. Moreover, the following lemmas hold.

LEMMA 3.1. *Let $f(t, \phi)$ satisfy Assumptions* (H 1) *and* (H 2). *Then the operator N has the following properties*:

(i) *For any $c = c(t)$ satisfying $\gamma(\omega) > 0$ and the following condition*

$$(3.1) \quad (e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega c(s)e^{\gamma(s)}ds + \int_0^t c(s)e^{\gamma(s)}ds > 0 , \qquad \textit{for all} \quad t \geqq 0 ,$$

there exist $r = r(c)$ and $\alpha = \alpha(c) \in (0, 1)$ such that N takes K_r into $K_{\alpha r}$.

(ii) *$x(t)$ is a periodic solution* (*of period ω*) *of* (2.1) *such that $0 \leqq x(t) \leqq r^*$ for $t \in [-h, \infty)$ if and only if $Nx = x$ and $x \in K_{r^*}$.*

PROOF. First we show that $x \in C_\omega^h$ implies $Lx \in C_\omega^h$ and consequently $Nx \in C_\omega^h$. We only need to prove that $(Lx)(t)$ is continuous and periodic of period ω for $t > 0$. Clearly, $(Lx)(t)$ is continuous on $t > 0$ by the definition of $(Lx)(t)$ on $t \geqq 0$. Since we have $\gamma(t) - \gamma(t + \omega) = -\gamma(\omega)$ by the periodicity of $c(t)$ and $p(t)$, we obtain

$$\begin{aligned}
(Lx)(t + \omega) &= e^{-\gamma(t+\omega)}\left\{(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega e^{\gamma(s)}p(s)x(s)ds + \int_0^{t+\omega} e^{\gamma(s)}p(s)x(s)\,ds\right\} \\
&= e^{-\gamma(t)}e^{-\gamma(\omega)}\left\{e^{\gamma(\omega)}(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega e^{\gamma(s)}p(s)x(s)ds + \int_0^t e^{\gamma(s+\omega)}p(s)x(s)ds\right\} \\
&= e^{-\gamma(t)}\left\{(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega e^{\gamma(s)}p(s)x(s)ds + \int_0^t e^{\gamma(s)}p(s)x(s)ds\right\} \\
&= (Lx)(t)
\end{aligned}$$

for $t > 0$, and consequently $Lx \in C_\omega^h$.

Secondly, we show that for any $c = c(t)$ satisfying $\gamma(\omega) > 0$ and (3.1), there exist $r = r(c) > 0$ and $\alpha = \alpha(c) \in (0, 1)$ such that N takes K_r into $K_{\alpha r}$. Since we have

$$0 \leqq x(s) + f(s, x_s) \leqq r + M(r) \quad \text{for all} \quad s \geqq 0 ,$$

for $x \in K_r$, it follows that

$$0 \leqq (Nx)(t) \leqq (r + M(r))e^{-\gamma(t)}\left\{(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega e^{\gamma(s)}p(s)ds + \int_0^t e^{\gamma(s)}p(s)ds\right\}$$
$$\leqq (r + M(r))e^{-\gamma(t)}\left\{(e^{\gamma(\omega)} - 1)^{-1}[e^{\gamma(s)}]_{s=0}^{s=\omega} + [e^{\gamma(s)}]_{s=0}^{s=t}\right.$$
$$\left. - (e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega c(s)e^{\gamma(s)}ds - \int_0^t c(s)e^{\gamma(s)}ds\right\}$$
$$\leqq (r + M(r))\left[1 - e^{-\gamma(t)}\left\{(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega c(s)e^{\gamma(s)}ds + \int_0^t c(s)^{\gamma(s)}ds\right\}\right],$$

for $t \geqq 0$. If we define $d(t)$ by

$$d(t) = e^{-\gamma(t)}\left\{(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega c(s)e^{\gamma(s)}ds + \int_0^t c(s)e^{\gamma(s)}ds\right\}, \quad t \geqq 0,$$

then $d(t)$ is positive by the assumption and is continuous. Moreover we easily see that $d(t + \omega) = d(t)$ for $t \geqq 0$ by an argument similar to the one we used above to show that $x \in C_\omega^h$ implies $Lx \in C_\omega^h$. Thus, we have $m(c) = \min_{t \geqq 0} d(t) > 0$, while $m(c) \leqq d(0) < 1$. Now let $\alpha = \alpha(c) = (2 - m(c))/2$ and $r = r(c)$ be numbers such that $M(r)/r \leqq m(c)/\{2(1 - m(c))\}$. Then we have $0 < \alpha < 1$, and $0 \leqq (Nx)(t) \leqq (1 + M(r)/r)(1 - m(c))r \leqq \alpha r$. This completes the proof of (i).

There is no difficulty in proving the part (ii). Actually, the same explanations as in the proof of [1, Lemma 2 b)] are enough. The following lemma corresponds to [1, Lemma 4].

LEMMA 3.2. *Let $f(t, \phi)$ satisfy Assumptions* (H 1), (H 2), *and* (H 3), *and let $N: C_\omega^h \to C_\omega^h$ be as in Lemma* 3.1. *Then N is completely continuous and has a Fréchet derivative at* 0 *with respect to K given by*

$$(N'(0)x)(t) = e^{-\gamma(t)}\left[(e^{\gamma(\omega)} - 1)^{-1}\int_0^\omega e^{\gamma(s)}p(s)\{x(s) + \Lambda(s, x_s)\}ds\right.$$
$$\left. + \int_0^t e^{\gamma(s)}p(s)\{x(s) + \Lambda(s, x_s)\}ds\right], \quad t \geqq 0,$$

and

$$(N'(0)x)(t) = (N'(0)x)(k\omega + t), \quad k\omega > h, \quad \text{for} \quad t \in [-h, 0).$$

The operator $N'(0)$ on C_ω^h is compact.

Now we need to study some of the spectral properties of the operator $N'(0)$. Throughout this paper, the relation $y \geqq x$ for $x, y \in C_\omega^h$ will mean $y - x \in K$. Also, $y > x$ will mean $y - x$ is in the interior of K.

LEMMA 3.3. *Let $f(t, \phi)$ and $N'(0)$ be as in Lemma* 3.2. *Then, there exists a $c_h \in C_\omega^h$ such that if $c = c_h$, there exists $x \in K$ with $N'(0)x = x \neq 0$. Also, if $c < c_h$, then $N'(0)x = x \neq 0$ implies $x \notin K$; moreover, there* *exist* $y \in K \setminus \{0\}$ *and* $\alpha > 1$ *such that* $N'(0)y = \alpha y$. *On the other hand, if* $c > c_h$, *the spectral radius of* $N'(0)$ *is less than one. Finally, such a* c_h *satisfies* $[a\underline{p}, b\bar{p}] \cap [\underline{c}_h, \bar{c}_h] \neq \emptyset$.

The proof of the lemma is similar to that of [1, Lemma 5]. If $N_c'(0)$ denotes the operator of Lemma 3.2 for a particular c, and if $\rho(N_c'(0))$ denotes the spectral radius of the operator $N_c'(0)$, it can be shown that $\rho(N_c'(0))$ is a continuous decreasing function of c.

The verification of $[a\underline{p}, b\bar{p}] \cap [\underline{c}_h, \bar{c}_h] \neq \emptyset$ is as follows. Let $x \in K \setminus \{0\}$ satisfy $N'(0)x = \lambda x$, $\lambda > 0$. Then $x(t)$ is periodic and satisfies

$$x'(t) = \{p(t)(1/\lambda - 1) - c(t)\}x(t) + (1/\lambda)p(t)A(t, x_t) . \tag{3.2}$$

If we put

$$I_c = \int_0^\omega [\{p(t)(1/\lambda - 1) - c(t)\}x(t) + (1/\lambda)p(t)A(t, x_t)]dt ,$$

then we obtain $I_c = 0$ by integrating (3.2) from 0 to ω. If $\lambda < 1$, then we have

$$0 = I_c \geqq \underline{p}\{(1 + a)/\lambda - 1\} \int_0^\omega x(t)dt - \int_0^\omega c(t)x(t)dt .$$

For $c(t) = a\underline{p}$, this yields the following contradiction

$$0 \geqq \underline{p}\{(1 + a)/\lambda - (1 + a)\} \int_0^\omega x(t)dt > 0 .$$

On the other hand, if $\lambda > 1$, then we obtain

$$0 = I_c \leqq \{(b\bar{p})/\lambda\} \int_0^\omega x(t)dt - \int_0^\omega c(t)x(t)dt ,$$

which is impossible if $c(t) = b\bar{p}$. Therefore, we have $\rho(N_{a\underline{p}}'(0)) \geqq 1 \geqq \rho(N_{b\bar{p}}'(0))$. Hence, if $c_h = c_h(t)$ exists such that $N_{c_h}'(0)x = x$, $x \in K \setminus \{0\}$, then we have $[a\underline{p}, b\bar{p}] \cap [\underline{c}_h, \bar{c}_h] \neq \emptyset$.

REMARK. It is easily seen from the above fact that for any $d \in C_\omega^h$ with $\underline{d} > 0$, we can choose $c_h = rd$ for some positive number r.

Combining the above results, we have the following Theorem 3.1 on the existence of a positive periodic solution of (2.1), which can be proved as in the case of [1, Lemma 6] by means of the following theorem in Schmitt [7, Corollary 4.10].

THEOREM S. *Let* E *be a real Banach space, let* K *be a cone in* E, *and let* D *be an open, bounded, nonempty neighborhood of zero in* E. *Let* $\bar{D}$ *be the closure of* D *and let* $N: K \cap \bar{D} \to K$ *be a completely continuous* *operator such that* $N(0)=0$ *and that* N *has a Fréchet derivative* $N'(0)$ *with respect to* K. *Assume the following are satisfied:*

(i) *All solutions* $x \in K$ *of* $x = \lambda N x$, $0 < \lambda < 1$, *satisfy* $x \notin \partial D$.

(ii) *There exist* $y \in K$, $\|y\| = 1$, *and* $\alpha > 1$ *such that* $N'(0)y = \alpha y$; *and* $N'(0)x = x$ *for* $x \neq 0$ *implies* $x \notin K$.

Then N *has a nontrivial fixed point in* $K \cap \bar{D}$.

THEOREM 3.1. *Let* $f(t, \phi)$ *satisfy Assumptions* (H 1), (H 2), and (H 3), *and let* $c = c(t)$ *satisfy* $\gamma(\omega) > 0$ *and* (3.1). *Then there exists a* $c_h \in C_\omega^h$ *such that* $[a\underline{p}, b\bar{p}] \cap [\underline{c}_h, \bar{c}_h] \neq \emptyset$ *and the following hold.*

(i) *If* $c < c_h$, *there exists a positive periodic solution of* (2.1); *and moreover, if the solutions of* (2.1) *are ultimately bounded for bound* r_0, *then the positive periodic solution* $x(t)$ *satisfies* $0 < x(t) < r_0$ *for all* $t \geqq -h$.

(ii) *If* $f(t, \phi)$ *satisfies Assumption* (H 4) *also, and if* $c \geqq c_h$, *then* (2.1) *has no positive periodic solutions.*

In the above, we say that the solutions of (2.1) are ultimately bounded for bound $r_0 > 0$, if there exists $T > 0$ such that for every solution $x(t)$ of Equation (2.1) with the initial conditions (t_0, ϕ), $|x(t)| < r_0$ for all $t \geqq t_0 + T$, where r_0 is independent of the particular solution while T may depend on each solution.

REMARK. Since it is not difficult to find a $c \in C_\omega^h$ which satisfies the conditions $\underline{c} < 0$, $\gamma(\omega) > 0$, (3.1), and $c < c_h$, there can exist a positive periodic solution of (2.1) even if $\underline{c} < 0$.

4. Applications. Consider the following differential-delay equation,

$$x'(t) = p(t)F(x(t), x(t-h)) - cx(t)\,, \quad t \geqq 0\,, \tag{4.1}$$

where c, h and $p(t)$ are the same as in (1.1) and where $F(x, y)$ is a function on $R \times R$ defined by

$$F(x, y) = \begin{cases} (1-x)y\,, & 0 \leqq x \leqq 1\,, \quad 0 \leqq y \leqq 1\,, \\ 1 - x\,, & 0 \leqq x \leqq 1\,, \quad y > 1\,, \\ \min(1, y)\,, & x < 0\,, \quad y > 0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then $f(t, \phi) = F(\phi(0), \phi(-h))$ satisfies (H 1). Also, (H 2) holds, since $M(r)$ in (H 2) satisfies $M(r) \leqq 1$. Moreover (H 3) and (H 4) are satisfied with $A(t, \phi) = \phi(-h)$ and $a = b = 1$. Therefore, by Theorem 3.1, there exist a constant $c_h \in [\underline{p}, \bar{p}]$ such that if $c \geqq c_h$, (4.1) has no positive periodic solution and that if $0 < c < c_h$, (4.1) has a nontrivial positive periodic solution $x(t)$. Moreover, since we have $p(t)F(x, y) - cx < 0$ for $x \geqq 1$,

$y \in R$, and $p(t)F(x, y) - cx > 0$ for $x < 0$, $y \in R$, the solutions of (4.1) are ultimately bounded for bound 1, and consequently $0 < x(t) < 1$ for all $t \geqq -h$. Thus $x(t)$ satisfies the differential-delay equation (1.1), and hence, we have the following results as a corollary to Theorem 3.1.

COROLLARY 4.1. *There exists a constant* $c_h \in [\underline{p}, \bar{p}]$ *such that the following hold.*

(i) *If* $c \geqq c_h$, (1.1) *has no positive periodic solution in* $0 < x < 1$.

(ii) *If* $0 < c < c_h$, *there exists a nontrivial positive periodic solution of* (1.1) *in* $0 < x < 1$.

REMARK. Equation (1.1) is the one studied by Busenberg and Cooke in [1]. In Corollary 4.1, (i) and (ii) correspond to Theorem 1 a) and b) in [1], respectively. But Theorem 1 b) contains the uniqueness of positive periodic solutions of (1.1) with a fixed c. Moreover, Theorem 1 c) asserts that the map $c \mapsto x_c$ is continuous and monotone, where x_c is the positive periodic solution. Unfortunately, we do not know whether these results continue to hold for (2.1).

Next, let $F(t, x, y)$ be a function on $I \times R \times R$ defined by

$$F(t, x, y) = \begin{cases} a(t)x + b(t)y - xy\ , & t \in I\ , \quad 0 \leqq x \leqq b(t)\ , \quad 0 \leqq y \leqq a(t)\ , \\ a(t)x\ , & t \in I\ , \quad 0 \leqq x \leqq b(t)\ , \quad y < 0\ , \\ b(t)y\ , & t \in I\ , \quad x < 0\ , \quad 0 \leqq y \leqq a(t)\ , \\ 0\ , & t \in I\ , \quad x < 0\ , \quad y < 0\ , \\ a(t)b(t)\ , & \text{otherwise}\ , \end{cases}$$

where $a(t)$ and $b(t)$ are continuous nonnegative ω-periodic functions with $\underline{a} + \underline{b} > 0$. Consider the differential-delay equation

$$x'(t) = p(t)F(t, x(t), x(t-h)) - c(t)x(t)\ , \qquad t \geqq 0\ , \tag{4.2}$$

where h, $p(t)$ and $c(t)$ are the same as in (2.1). The functional $f(t, \phi) = F(t, \phi(0), \phi(-h))$ satisfies (H 1). Since $M(r)$ in (H 2) satisfies $M(r) \leqq \overline{ab}$, (H 2) holds. Also, (H 3) and (H 4) are clearly satisfied with $\Lambda(t, \phi) = a(t)\phi(0) + b(t)\phi(-h)$, $a = \underline{a} + \underline{b}$, and $b = \bar{a} + \bar{b}$. Thus, by Theorem 3.1, we have the following corollary.

COROLLARY 4.2. *Let* $c = c(t)$ *satisfy* $\gamma(\omega) > 0$ *and* (3.1). *Then there exists a* $c_h \in C_\omega^h$ *such that if* $c \geqq c_h$, (4.2) *has no positive periodic solutions, and if* $c < c_h$, (4.2) *has a positive periodic solution. Moreover,* $[a\underline{p}, b\bar{p}] \cap [\underline{c}_h, \bar{c}_h] \neq \emptyset$.

Finally, consider the differential-delay equation

$$x'(t) = p(t)F(x(t-h)) - c(t)x(t)\ , \qquad t \geqq 0\ , \tag{4.3}$$

where h, $p(t)$ and $c(t)$ are the same as in (2.1), and where $F(y)$ is a function on R defined by

$$F(y)=\begin{cases}\sin y\,, & |y|\leqq \pi\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

Then $f(t,\phi)=F(\phi(-h))$ clearly satisfies (H 1), (H 2), (H 3), and (H 4) with $\Lambda(t,\phi)=\phi(-h)$ and $a=b=1$. Thus we have the following corollary by Theorem 3.1.

COROLLARY 4.3. *Let $c=c(t)$ satisfy $\gamma(\omega)>0$ and* (3.1). *Then there exists a $c_h\in C_\omega^h$ such that if $c\geqq c_h$,* (4.3) *has no positive periodic solution and that if $c<c_h$,* (4.3) *has a positive periodic solution. Moreover, $[\underline{p},\bar{p}]\cap[\underline{c}_h,\bar{c}_h]\neq\varnothing$. Particularly, if c is a constant and if $\bar{p}/\pi\underline{p}$, then there exists a positive constant $c_h\in[\underline{p},\bar{p}]$ such that if $\bar{p}/\pi<c<c_h$,* (4.3) *has a positive periodic solution $x(t)$ which satisfies $0<x(t)<\pi$ for all $t\geqq -h$.*

This corollary is a direct consequence of Theorem 3.1 and the fact that if c is a constant, and if $\bar{p}<c\pi$, then the solutions of (4.3) are ultimately bounded for bound π.

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