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# LIAPUNOV'S SECOND METHOD IN FUNCTIONAL DIFFERENTIAL EQUATIONS

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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**1. Introduction.** For the theory of stability in differential equations, Liapunov's second method may be the most important. In the case of functional differential equations, there were also many attempts to establish various kinds of Liapunov type theorems, see e.g. [1]~[6], [8], [10]~[18]. Among them, there are three main ideas, one based on Liapunov functionals (cf. [13]), one by Razumikhin [15] and one by Barnea [1]. The idea based on Liapunov functionals is the most perfect in the sense that we can develop it in the way parallel to that for ordinary differential equations including the establishment of necessary and sufficient conditions of stability. However, the construction of such a Liapunov functional is very hard for concrete problems. This difficulty stimulates the development of the ideas by Razumikhin and by Barnea (cf. [2], [5], [6], [14], [17], [18]). The author also gave some extensions of the ideas in a unified way combining both, see [10], [11]. In this paper we give several results in the same direction. For the examples, refer to [12].

Recently, Burton [3] presented a kind of stability theorems. Noting that actually his result is deeply related with the choice of the phase space and that in a concrete problem there are several possibilities for the choice of the phase space, we state our results for functional differential equations on an abstract phase space discussed in [9].

**2. Admissible phase space.** Let $(X, \|\cdot\|_X)$, or simply $X$, be a linear space of $R^n$-valued functions on $(-\infty, 0]$ with a semi-norm $\|\cdot\|_X$, and denote by $X_\tau$ the space of functions $\phi(s)$ on $(-\infty, 0]$ which are continuous on $[-\tau, 0]$ and satisfying $\phi_{-\tau} \in X$ for given $X$ and $\tau \geqq 0$, where and henceforth $\phi_t$ denotes the function on $(-\infty, 0]$ defined by $\phi_t(s) = \phi(t+s)$.

The space $(X, \|\cdot\|_X)$ is said to be *admissible*, if the following are satisfied: For any $\tau \geqq 0$ and any $\phi \in X_\tau$

($a_1$) $\phi_t \in X$ for all $t \in [-\tau, 0]$, especially, $\phi_0 = \phi \in X$,

($a_2$) $\phi_t$ is continuous in $t \in [-\tau, 0]$,

(a$_3$) $\mu\|\phi(0)\| \leqq \|\phi\|_X \leqq K(\tau)\sup_{-\tau\leqq s\leqq 0}\|\phi(s)\| + M(\tau)\|\phi_{-\tau}\|_X$,
where $\mu > 0$ is a constant and $K(s)$, $M(s)$ are continuous functions. In the above, $(X, \|\cdot\|_X)$ is said to have *a fading memory*, if $K(s) = K$ is a constant and $M(s) \to 0$ as $s \to \infty$. We note that $(X_\tau, \|\cdot\|_{X_\tau})$ again is admissible for $\tau \geqq 0$, where $\|\phi\|_{X_\tau} = \sup_{-\tau\leqq s\leqq 0}\|\phi_s\|_X$, while $R^n$ is an admissible space with the semi-norm defined by $\|\phi\|_{R^n} = \|\phi(0)\|$. For convenience, we introduce an order relation $X < Y$ in the set of admissible spaces, which means $X \subset Y$ and $\|\phi\|_Y \leqq \gamma\|\phi\|_X$ $(\phi \in X)$ for a constant $\gamma$. Therefore, the condition (a$_3$) means that $X_\tau < X < R^n$.

As typical and important examples of admissible spaces, we will bring to mind the spaces $C_h^\gamma$ of functions $\phi$, continuous on $[-h, 0]$ (or on $(-\infty, 0]$ with the property that $e^{\gamma s}\phi(s)$ has a limit as $s \to -\infty$ if $h = \infty$) and $M_h^\gamma$ of functions $\phi$, measurable on $(-h, 0]$, such that $e^{\gamma s}\|\phi(s)\|$ are integrable on $(-h, 0]$, where $\gamma$, $h$ are constants satisfying $0 \leqq \gamma < \infty$, $0 \leqq h \leqq \infty$, and the semi-norms are given by

$$(1) \qquad \|\phi\|_{C_h^\gamma} = \sup_{-h\leqq s\leqq 0} e^{\gamma s}\|\phi(s)\|\,, \quad \|\phi\|_{M_h^\gamma} = \|\phi(0)\| + \int_{-h}^{0} e^{\gamma s}\|\phi(s)\|\,ds\,,$$

(refer to [9]). The following lemma is trivial.

LEMMA 1. (i) *Every admissible space contains every continuous function with compact support.*

(ii) *$C_h^\gamma$ and $M_h^\gamma$ have fading memories if $\gamma > 0$ or $h < \infty$.*

(iii) *If $h < \infty$, then $C_h^\gamma < C_h^\beta$ and $M_h^\gamma < M_h^\beta$ for any $\gamma$ and $\beta$.*

(iv) *If $\gamma > \beta$, then $C_h^\beta < M_h^\gamma$.*

(v) *If $0 \leqq k \leqq h \leqq \infty$, then $C_h^\gamma < C_k^\gamma$, $M_h^\gamma < M_k^\gamma$.*

By this lemma, $C_h^\gamma$, $M_h^\gamma$ will be written as $C_h$, $M_h$ if $h < \infty$, while $C^\gamma$, $M^\gamma$ are used for $C_h^\gamma$, $M_h^\gamma$ when $h = \infty$.

Throughout this paper, $I = [0, \infty)$ and $\mathscr{K}(I)$ denotes the class of real-valued continuous, non-decreasing functions on $I$, while $a \in \mathscr{K}^+(I)$ means $a \in \mathscr{K}(I)$ and $a(r) > 0$ for $r > 0$.

LEMMA 2. *Let $X$ be admissible, and let $\delta \in \mathscr{K}^+(I)$ be given. Then, for any $\tau > 0$ there exists a constant $\theta > 0$ and a $\rho \in \mathscr{K}^+(I)$ such that for any $\phi \in X_\tau$ we have*

$$(2) \qquad m[\{t \in [-\tau, 0]: \|\phi_t\|_X \geqq \theta\|\phi\|_X\}] \geqq \rho(\|\phi\|_X)\,,$$

*where $m[A]$ is the Lebesgue measure of $A \subset R$, whenever*

$$(3) \qquad \|\phi(s) - \phi(t)\| \leqq \varepsilon \text{ if } |s - t| \leqq \delta(\varepsilon)\,, \quad s, t \in [-\tau, 0]\,.$$

PROOF. Put $K = \sup_{0\leqq s\leqq\tau}K(s)$, $M = \sup_{0\leqq s\leqq\tau}M(s)$ for $K(s)$ and $M(s)$ given in (a$_3$) for the admissible space $X$, and let $\mu$ be the one in (a$_3$).

Suppose that $\phi \in X_\tau$. Then, clearly we must have $\|\phi_t\|_X \geqq \varepsilon/(2M)$ for all $t \in [-\tau, 0]$ if $\|\phi\|_X \geqq \varepsilon$ and if $\|\phi(t)\| \leqq \varepsilon/(2K)$ for all $t \in [-\tau, 0]$. On the other hand, if $\|\phi(s)\| \geqq \varepsilon/(2K)$ for an $s \in [-\tau, 0]$, then by (3) we have $\|\phi(t)\| \geqq \varepsilon/(4K)$ for $t \in [s - \delta(\varepsilon/(4K)), s + \delta(\varepsilon/(4K))] \cap [-\tau, 0]$ $(=J)$, and hence $\|\phi_t\|_X \geqq \mu\varepsilon/(4K)$ for $t \in J$. Thus, we can conclude (2) by setting $\theta = \min\{1/(2M), \mu/(4K)\}$, $\rho(\varepsilon) = \min\{\tau, \delta(\varepsilon/(4K))\}$ $(\leqq m[J])$. q.e.d.

A special case of the following lemma is stated in [3].

LEMMA 3. *Let $c \in \mathscr{K}^+(I)$, and let $\gamma$, $h$, $\tau$, $\tau \leqq h$, and $B$ be given positive constants ($h$ may be $\infty$). Then, there exist $d \in \mathscr{K}^+(I)$ and $-q \in \mathscr{K}(I)$ (namely $q(r)$ is non-increasing), $h \geqq q(r) > 0$, such that*

$$\inf_{-\tau \leqq t \leqq 0} \|\phi_t\|_{M_h^\gamma} \geqq \varepsilon \text{ implies } \int_{-q(\varepsilon)}^0 c(\|\phi(s)\|)ds \geqq d(\varepsilon) \tag{4}$$

*for any $\varepsilon > 0$ and $\phi \in M_{2h}^\gamma (\supset (M_h^\gamma)_h)$ satisfying $\|\phi(s)\| \leqq B$, $\|\phi_s\|_{M_h^\gamma} \leqq B$ for all $s \in [-q(\varepsilon), 0]$.*

PROOF. For convenience, set $|||\phi||| = \int_{-h}^0 e^{\gamma s}\|\phi(s)\|ds$ for $\phi \in M_{2h}^\gamma$. First of all, we prove that there are $d^* \in \mathscr{K}^+(I)$ and $-q^* \in \mathscr{K}(I)$ such that $h \geqq q^*(r) > 0$ for $r \geqq 0$ and

$$|||\phi||| \geqq \varepsilon \text{ implies } \int_{-q^*(\varepsilon)}^0 c(\|\phi(s)\|)ds \geqq d^*(\varepsilon) \tag{5}$$

if $\|\phi(s)\| \leqq B$ and $|||\phi_s||| \leqq B$ for $s \in [-q^*(\varepsilon), 0]$. Let $J = \{s \in [-k, 0]: \|\phi(s)\| \geqq \delta\}$ for $k > 0$ and $\delta > 0$. Then,

$$|||\phi||| \leqq e^{-\gamma k}|||\phi_{-k}||| + \delta\int_{-k}^0 e^{\gamma s}ds + \int_J e^{\gamma s}\|\phi(s)\|ds \leqq e^{-\gamma k}B + \delta/\gamma + Bm[J]$$

if $\|\phi(s)\| \leqq B$ on $[-k, 0]$ and $|||\phi_{-k}||| \leqq B$ (the first terms in the center and right hand sides will be deleted if $h \leqq k$). Hence, by setting $k = q^*(\varepsilon) = \min\{h, [\log(3B/\varepsilon)]/\gamma\}$ and $\delta = \delta(\varepsilon) = \gamma\varepsilon/3$, we have $m[J] \geqq \varepsilon/(3B)$ if $|||\phi||| \geqq \varepsilon$, which implies (5) with $d^*(r) = c(\gamma r/3)r/(3B)$. Since $|||\phi||| \geqq \int_{-\tau}^0 e^{\gamma s}\|\phi(s)\|ds$, we can find a $t \in [-\tau, 0]$ so that $\|\phi(t)\| \leqq \{\gamma/(1 - e^{\gamma\tau})\}|||\phi|||$, while $|||\phi_t||| \leqq e^{\gamma\tau}|||\phi|||$ for $t \in [-\tau, 0]$. Therefore, $\|\phi_t\|_{M_h^\gamma} = \|\phi(t)\| + |||\phi_t||| \leqq \nu|||\phi|||$ with $\nu = \gamma/(1 - e^{-\gamma\tau}) + e^{\gamma\tau}$, namely, $\inf_{-\tau \leqq t \leqq 0}\|\phi_t\|_{M_h^\gamma} \geqq \varepsilon$ implies $|||\phi||| \geqq \varepsilon/\nu$. This together with (5) implies (4), where $q(r) = q^*(r/\nu)$, $d(r) = d^*(r/\nu)$. q.e.d.

**3. Functional differential equation.** Let $X$ be an admissible space. Consider the functional differential equation

$$\dot{x}(t) = f(t, x_t) \tag{E}$$

and assume that $f(t, \phi)$ is completely continuous on $I\times X$. For the fundamental properties of the solutions of (E), see [9]. Here, we reproduce only the following lemma (cf. [9; Theorem 2.5]).

LEMMA 4. *Suppose that the zero solution of* (E) *is unique for the initial value problem. Then, there exist continuous functions* $L(t, s, r)$ *on* $I^3$, *non-decreasing in* $r$, $L(t, s, 0)=0$, *and* $\delta_0(t, s)$ *on* $I^2$, $\delta_0(t, s)>0$, *such that any solution* $x(t)$ *of* (E) *satisfies*

$$\text{(L)} \qquad \|x_t\|_X \leqq L(t, s, \|x_s\|_X) \quad \textit{if} \quad \|x_s\|_X \leqq \delta_0(t, s)\,, \quad t\geqq s\,.$$

REMARK 1. If $L(t, s, r)$ and $\delta_0(t, s)$ can be chosen so that $L(t, s, r)=L(t-s, 0, r)$ and $\delta_0(t, s)=\delta_0(t-s, 0)$, then the relation (L) will be referred to as (UL). A sufficient condition for (E) to have the property (UL) is $\|f(t, \phi)\|\leqq Bg(\|\phi\|_X)$ on $I\times X$, where $B$ is a constant, $g\in\mathscr{K}^+(I)$ and $\lim_{r\to 0+}\int_r^1(1/g(s))ds=\infty$.

**4. Basic inequalities.** Hereafter, we assume that:

($h_1$) $v(t)$ and $w(t)$ are nonnegative continuous functions on $[\tau, \infty)$ for a $\tau\geqq 0$ and $\dot v(t)$ denotes the upper right Dini derivative.

($h_2$) $F(r)\in\mathscr{K}(I)$ and $F(r)>r$ for $r>0$.

($h_3$) $p(t, r)$ is continuous on $I\times(0, \infty)$, nondecreasing in $r$ and satisfies $p(t, r)\leqq t$, $p(t, r)\to\infty$ as $t\to\infty$.

($h_4$) $c(t, r)$ is nonnegative, continuous on $I^2$, nondecreasing in $r$, $c(t, r)\leqq c(r)$ for a $c\in\mathscr{K}^+(I)$ and for all $t\geqq 0$.

The following lemma is trivial.

LEMMA 5. *Under the assumption* ($h_3$), $\sigma(t, r)=\sup\{s: p(s, r)\leqq t\}$ *is continuous on* $I\times(0, \infty)$, *nonincreasing in* $r$, $\sigma(t, r)\geqq t$, *and* $p(t, r)\geqq\tau$ *if* $t\geqq\sigma(\tau, r)$ *and* $r>0$. *Moreover, we can find a* $\beta(\tau; v)>0$ *for which* $v(t)\leqq\beta(\tau; v)$ *on* $[\tau, \sigma(\tau, \beta(\tau; v))]$ *if* $v(t)$ *is continuous there.*

Now, we shall state the following theorems.

THEOREM 1. *Let* $\tau\geqq 0$ *be given, and suppose that* $\dot v(t)\leqq -w(t)$ *whenever* $v(t)>0$, $p(t, v(t))\geqq\tau$ *and* $v(s)\leqq F(v(t))$ *for all* $s\in[p(t, v(t)), t]$. *Then, we have*

(i) $v(t)\leqq\beta(\tau; v)$ *for all* $t\geqq\tau$,

(ii) $\limsup_{t\to\infty}v(t)=\beta>0$ *implies* $\lim_{t\to\infty}v(t)=\beta$,

(iii) $\int^\infty w(t)dt<\infty$ *if* $\lim_{t\to\infty}v(t)>0$.

PROOF. (i) Set $\beta(\tau; v)=\beta>0$, and suppose that $v(t_1)>\beta$ for a $t_1>\tau$. Clearly $t_1>\sigma(\tau, \beta)$ by Lemma 5. Then, we can find a $t_2\in[\sigma(\tau, \beta), t_1]$ so that $v(t_2)>\beta$, $\dot v(t_2)>0$ and $v(t)\leqq v(t_2)$ for all $t\in[\tau, t_2]$.

Since $p(t_2, v(t_2)) \geqq p(t_2, \beta) \geqq \tau$ and $v(t) \leqq v(t_2) \leqq F(v(t_2))$ for $t \in [p(t_2, v(t_2)), t_2]$, we have $\dot{v}(t_2) \leqq 0$, a contradiction.

(ii) For given $\beta > 0$ we can find an $\varepsilon > 0$ for which

$$\beta + \varepsilon \leqq F(\beta - \varepsilon)\ , \quad \beta - \varepsilon > 0\ . \tag{6}$$

If $\liminf_{t\to\infty} v(t) = \gamma < \beta$, then letting $\varepsilon < \beta - \gamma$ we can choose $t_1$ and $t_2$ so that $v(t) \leqq \beta + \varepsilon$ for all $t \geqq t_1$, $t_2 \geqq \sigma(t_1, \beta - \varepsilon)$, $v(t_2) \geqq \beta - \varepsilon$ and $\dot{v}(t_2) > 0$. By the same argument as in (i) there arises a contradiction, since $v(t) \leqq \beta + \varepsilon \leqq F(\beta - \varepsilon) \leqq F(v(t_2))$ $(t \in [t_1, t_2])$ and $p(t_2, v(t_2)) \geqq p(t_2, \beta - \varepsilon) \geqq t_1$.

(iii) Suppose that $\lim_{t\to\infty} v(t) = \beta > 0$. Choose $t_1$ and $t_2$ so that $|v(t) - \beta| < \varepsilon$ for $t \geqq t_1$ and $t_2 \geqq \sigma(t_1, \beta - \varepsilon)$, where $\varepsilon > 0$ satisfies (6). Hence, as in (ii), we have

$$\dot{v}(t) \leqq -w(t) \quad \text{for all} \quad t \geqq t_2\ , \tag{7}$$

which implies $\int^{\infty} w(t)dt < \infty$ since $v(t) \geqq 0$, $w(t) \geqq 0$. q.e.d.

COROLLARY. *Suppose that $v(t)$ and $w(t)$ satisfy the conditions in Theorem 1. Then, the following assertions hold:*

(i) *If for any $\sigma \geqq \tau$, $\Gamma > \gamma > 0$, $G > 0$ there is a $T$ such that*

$$\int_{\sigma}^{\sigma+T} w(t)dt \geqq G \ \text{as long as}\ \Gamma \geqq v(t) \geqq \gamma \quad \text{on} \quad [\sigma, \sigma + T]\ , \tag{8}$$

*then* $\lim_{t\to\infty} v(t) = 0$.

(ii) *If for any $\varepsilon > 0$ there exist positive numbers $\eta(\varepsilon)$, $\xi(\varepsilon)$ and $\zeta(\varepsilon)$ such that*

$$m[\{s \in [t - \eta(\varepsilon), t]\colon w(s) \geqq \xi(\varepsilon)\}] \geqq \zeta(\varepsilon) \tag{9}$$

*whenever $w(t) \geqq \varepsilon$ and $v(t) \geqq \varepsilon$, then $\lim_{t\to\infty} v(t) = 0$ or $w(t) \to 0$ as $t \to \infty$.*

PROOF. Suppose that $\lim_{t\to\infty} v(t) = \beta > 0$ by noting Theorem 1 (ii) and, hence, that we have (7) together with $|v(t) - \beta| < \varepsilon$ for $t \geqq t_2$. Then, under the assumption in (i) immediately we have a contradiction, since $\beta - \varepsilon < v(t_2 + T) \leqq v(t_2) - \int_{t_2}^{t_2+T} w(s)ds \leqq \beta + \varepsilon - \int_{t_2}^{t_2+T} w(s)ds < \beta - \varepsilon$ by choosing $T$ for $\sigma = t_2$, $\Gamma = \beta + \varepsilon$, $\gamma = \beta - \varepsilon$, $G = 2\varepsilon$.

(ii) Suppose that $w(t)$ does not converge to 0 as $t \to \infty$, that is, $w(s_k) \geqq \varepsilon$ for an $\varepsilon > 0$ and a divergent sequence $\{s_k\}$, where we may assume that $\beta \geqq 2\varepsilon$, $s_{k+1} - s_k \geqq \eta(\varepsilon)$, $s_1 \geqq t_2 + \eta(\varepsilon)$. By the assumptions we have $\int_{s_1-\eta(\varepsilon)}^{s_k} w(s)ds \geqq k\xi(\varepsilon)\zeta(\varepsilon)$, which diverges as $k \to \infty$. This contradicts (7), and hence $w(t)$ should converge to 0 as $t \to \infty$. q.e.d.

REMARK 2. If $w(t) = c(t, v(t))$ for a $c(t, r)$ in $(\mathrm{h}_4)$, then the condition

(8) is equivalent to

$$\int_{\sigma}^{\sigma+T} c(t, \gamma)dt \geqq G \ . \tag{10}$$

THEOREM 2. *Let* $\tau \geqq 0$ *be given, and suppose that* $\dot{v}(t) \leqq -w(t)$ *whenever* $p(t, v(t)) \geqq \tau$ *and* $v(t) > 0$. *Then, we have*

(i) *if* $v(t_1) \leqq \varepsilon$ *for an* $\varepsilon > 0$ *and for a* $t_1 \geqq \sigma(\tau, \varepsilon)$, *then* $v(t) \leqq \varepsilon$ *for all* $t \geqq t_1$,

(ii) *under the condition* (8), *there is a* $T_1$, *which depends on* $\tau$, $\beta(\tau; v)$ *and given* $\varepsilon > 0$, *such that* $v(t) \leqq \varepsilon$ *for all* $t \geqq \tau + T_1$.

PROOF. (i) Suppose that $v(t_2) > \varepsilon$ for a $t_2 > t_1$. Then, we may assume that $\dot{v}(t_2) > 0$, which yields a contradiction, since $p(t_2, v(t_2)) \geqq p(t_2, \varepsilon) \geqq \tau$.

(ii) Let $T = T(\tau, \varepsilon)$ be chosen so as to satisfy (8) with $\sigma = \sigma(\tau, \varepsilon)$, $\Gamma = \beta(\tau; v)$, $\gamma = \varepsilon$ and $G = \beta(\tau; v)$. Then, $T_1(\tau, \varepsilon) = T(\tau, \varepsilon) + \sigma(\tau, \varepsilon) - \tau$ is a desirable one. In fact, suppose that $v(t) \geqq \varepsilon$ for all $t \in [\sigma(\tau, \varepsilon), \tau + T_1]$. Then, $p(t, v(t)) \geqq p(t, \varepsilon) \geqq \tau$ and, hence, $\dot{v}(t) \leqq -w(t)$ there. Therefore, since $\varepsilon \leqq v(t) \leqq \beta(\tau; v)$ by Theorem 1 (i), we have $\int_{\sigma(\tau,\varepsilon)}^{\tau+T_1} w(t)dt \leqq v(\sigma(\tau, \varepsilon)) - v(\tau + T_1) \leqq \beta(\tau; v) - \varepsilon$, which contradicts (8), where we note $\tau + T_1 = \sigma(\tau, \varepsilon) + T$. Thus, there is a $t_1 \in [\sigma(\tau, \varepsilon), \tau + T_1]$ for which $v(t_1) \leqq \varepsilon$, and hence $v(t) \leqq \varepsilon$ for all $t \geqq t_1$ by (i). q.e.d.

Clearly, the assumption on $\dot{v}(t)$ in Theorem 2 is stronger than that in Theorem 1. However, the following proposition shows that under some subsistent conditions Theorem 1 can be converted to Theorem 2. The proposition can be proved in the same way as in [11; Theorem 4].

PROPOSITION 1. *Let* $\dot{v}(t) \leqq -w(t)$ *as long as* $p(t, v(t)) \geqq \tau$, $v(t) > 0$ *and* $v(s) \leqq F(v(t))$ *for* $s \in [p(t, v(t)), t]$, *and suppose that*

(11) $\qquad \sigma(t, r) - t$ *is positive, nondecreasing in* $t$

*and* $F(r)/r$ *is nondecreasing. Put* $u(t) = \sup_{\tau \leqq s \leqq t} v(s)\exp[\alpha(s, v(s))(s - t)]$ *for* $\alpha(t, r) = \{\sigma(t, F^{-1}(r)/2) - t\}^{-1}\log[r/F^{-1}(r)]$. *Then,* $u(t)$ *satisfies the condition in Theorem* 2, *or more precisely,* $\dot{u}(t) \leqq -d(t)$ *if* $p(t, u(t)) \geqq \tau$ *and* $u(t) > 0$, *where* $d(t) = \min\{w(t), \alpha(t, u(t))u(t)\}$. *Moreover, if* $w(t)$ *and* $\alpha(t, r)$ *satisfy* (8) *and* (10), *respectively, and if* $\alpha(t, r)$ *is independent of* $t$ *or* $w(t) = c(v(t))$ *for a* $c \in \mathscr{K}(I)$, *then* $d(t)$ *satisfies* (8).

**5. Definition of stability.** Let $x(t)$ be an arbitrary solution of (E). The definitions of stability in $(X, Y)$ can be given in the usual way (refer to [18]) associated to the relation

(12) $$\|x_\tau\|_X \leqq \delta \quad \text{and} \quad t \geqq \tau + T \quad \text{imply} \quad \|x_t\|_Y \leqq \varepsilon ,$$

where $(Y, \|\cdot\|_Y)$ is an admissible space satisfying $X < Y$.

For example, the zero solution of (E) is said to be *stable in* $(X, Y)$ if for any $\varepsilon > 0$ and $\tau \geqq 0$ we can find a $\delta > 0$ so that (12) holds with $T = 0$ and *uniformly asymptotically stable in* $(X, Y)$ if it is stable in $(X, Y)$ with $\delta$ independent of $\tau$ and if there are a constant $\delta > 0$ and a function $T$ of $\varepsilon > 0$ for which (12) holds whatever $\tau \geqq 0$ is. The following proposition is obvious.

PROPOSITION 2. *If $X_2 < X_1 < Y_1 < Y_2$, then the stability in $(X_1, Y_1)$ implies the stability in $(X_2, Y_2)$ of the same type. Especially, the stability in $(X, R^n)$ follows from the stability in $(X, X)$, while the reverse also holds when $X$ has a fading memory.*

REMARK 3. By the condition $(a_3)$ on the admissible space, if $\|x_t\|_Y$ is bounded on a finite interval, so is $\|x(t)\|$, and hence $\|x_t\|_X$ is bounded on any finite interval by $(a_3)$. Since we assume that $f(t, \phi)$ is completely continuous, this fact guarantees that any solution $x(t)$ of (E) is continuable up to infinity as long as $\|x_t\|_Y$ remains bounded.

**6. Liapunov function.** A Liapunov function is a collection $\{V(t, \phi; \tau) \colon \tau \geqq 0\}$ of real-valued, continuous functions $V(t, \phi; \tau)$, defined on $\{(t, \phi) \colon \phi \in X_{t-\tau},\ t \geqq \tau\}$, satisfying

(A) $$a(\|\phi\|_Y) \leqq V(t, \phi; \tau)$$

and

(B) $$V(t, \phi; \tau) \leqq b(t, \tau, \|\phi\|_{X_{t-\tau}})$$

for an $a \in \mathscr{K}^+(I)$ and for a function $b(t, \tau, r)$, continuous on $I^3$, nondecreasing in $r$, $b(t, \tau, 0) = 0$. Define $\dot{V}_{(E)}(t, \phi; \tau) = \sup[\limsup_{s \to t+0} \{V(s, x_s; \tau) - V(t, \phi; \tau)\}/(s - t)]$ for a solution $x(s)$ of (E) satisfying $x_t = \phi$, where "sup" runs over such solutions. Clearly, for any solution $x(t)$ of (E) starting at $t = \tau$ we have $x_t \in X_{t-\tau}$ and $v(t) = V(t, x_t; \tau)$ satisfies $\dot{v}(t) \leqq \dot{V}_{(E)}(t, x_t; \tau)$.

Now, we shall state the following theorems concerning the stability in $(X, Y)$. The conditions (L), (UL) are those given in Lemma 4 and Remark 1. It is clear that we can replace $\|x_t\|_Y \leqq \varepsilon$ for an arbitrary small $\varepsilon > 0$ in (12) by $V(t, x_t; \tau) \leqq \varepsilon$ under the condition (A).

THEOREM 3. *Assume that* (L) *holds and that there is a Liapunov function $\{V(t, \phi; \tau) \colon \tau \geqq 0\}$ which satisfies*

(C) $$\dot{V}_{(E)}(t, \phi; \tau) \leqq - W(t, \phi; \tau)$$

*for a nonnegative continuous function* $W(t, \phi; \tau)$ *whenever*

($c_1$) $V(t, \phi; \tau) > 0$,

($c_2$) $p(t, V(t, \phi; \tau)) \geqq \tau$,

($c_3$) $V(s, \phi_{s-t}; \tau) \leqq F(V(t, \phi; \tau))$ *for* $s \in [p(t, V(t, \phi; \tau)), t]$, *where* $p(t, r)$ *and* $F(r)$ *are those in* ($h_3$) *and* ($h_2$), *respectively* (*see Section* 4). *Then, the zero solution of* (E) *is stable in* $(X, Y)$. *Moreover,*

(i) *it is asymptotically stable in* $(X, Y)$ *under the conditions*

($H_1$) $v(t) = V(t, x_t; \tau)$, $w(t) = W(t, x_t; \tau)$ *satisfy* (8) *for any solution* $x(t)$ *of* (E), *or*

($H_2$) $\|f(t, \phi)\| \leqq N$ *for a constant* $N$, *and* $W(t, \phi; \tau) = c(\|\phi\|_Y)$ *for a* $c \in \mathscr{K}^+(I)$;

(ii) *it is equiasymptotically stable in* $(X, Y)$ *under the condition* ($H_1$) *and one of the following conditions*:

($H_3$) $p(t, r) - t$ $(=p(r))$ *is positive and independent of* $t$,

($H_4$) $W(t, \phi; \tau) = c(V(t, \phi; \tau))$ *for a* $c \in \mathscr{K}^+(I)$, *and the condition* (11) *holds with* $\int^{\infty}\{\sigma(t, r) - t\}^{-1}dt = \infty$ *for the* $\sigma(t, r)$ *in Lemma* 5,

($H_5$) *the condition* ($c_3$) *is dropped, that is,* (C) *holds whenever* ($c_1$) *and* ($c_2$) *are satisfied.*

PROOF. Let $x(t)$ be a solution of (E) starting at $t = \tau$ for a $\tau \geqq 0$, and set $v(t) = V(t, x_t; \tau)$ and $w(t) = W(t, x_t; \tau)$. Clearly, we can find a $\delta > 0$ for any given $\varepsilon > 0$ so that

$$\sup_{\tau \leqq s \leqq \sigma(\tau, \varepsilon)} b(s, \tau, L(s, \tau, \delta)) \leqq \varepsilon\,, \quad \delta \leqq \inf_{\tau \leqq s \leqq \sigma(\tau, \varepsilon)} \delta_0(s, \tau)\,. \tag{13}$$

Therefore, $\|x_\tau\|_X \leqq \delta$ implies $v(t) \leqq \varepsilon$ on $[\tau, \sigma(\tau, \varepsilon)]$ by Lemma 4, and hence, $v(t) \leqq \varepsilon$ for all $t \geqq \tau$ by Theorem 1 (i) because we can put $\beta(\tau; v) = \varepsilon$ there under the choice of $\delta$. This shows that the zero solution of (E) is stable in $(X, Y)$, and we can prove (i) under ($H_1$). Now, assume the condition ($H_2$). Since $\|f(t, \phi)\| \leqq N$ guarantees that $x(t)$ satisfies (3) with $\delta(r) = r/N$ in Lemma 2, we have $m[\{s \in [t-1, t]: \|x_s\|_Y \geqq \theta\|x_t\|_Y\}] \geqq \rho(\|x_t\|_Y)$ for a $\theta > 0$ and $\rho \in \mathscr{K}^+(I)$, that is, $m[\{s \in [t-1, t]: w(s) \geqq c(\theta\|x_t\|_Y)\}] \geqq \rho(\|x_t\|_Y)$. Hence, the condition (9) in Corollary (ii) to Theorem 1 is satisfied with $\eta(\varepsilon) = 1$, $\xi(\varepsilon) = c(\theta c^{-1}(\varepsilon))$, $\zeta(\varepsilon) = \rho(c^{-1}(\varepsilon))$, where $c^{-1}(r) = \inf\{s\colon c(s) \geqq r\}$, and the desired conclusion follows from the corollary. (ii) is proved by Theorem 2 (ii) under the condition ($H_5$), while the condition ($H_5$) is verified under ($H_3$) or ($H_4$) by Proposition 1, where we note that $F(r)/r$ is nondecreasing as in Proposition 1 on $[0, r_0]$ for an $r_0 > 0$, which is sufficient to prove the stability, if $F(r)$ is chosen suitably. q.e.d.

THEOREM 4. *In Theorem* 3 *assume that* (L) *is replaced by* (UL),

(14) $$b(t, \tau, r) = b(t - \tau, 0, r) \quad in \quad (\mathrm{B})$$

*and* $(\mathrm{H}_3)$ *is satisfied. Then, the zero solution of* (E) *is uniformly stable in* $(X, Y)$, *and it is uniformly asymptotically stable in* $(X, Y)$ *under one of the following conditions*:

$(\mathrm{H}_6)$ *the condition* $(\mathrm{H}_1)$ *holds, and* $T$ *appearing in* (8) *in connection with* $(\mathrm{H}_1)$ *can be chosen independent of* $\sigma$,

$(\mathrm{H}_7)$ *in addition to* $(\mathrm{H}_2)$ *and* $(\mathrm{H}_5)$ *there is an* $l > 0$ *such that* $Y_l < X$.

PROOF. It is clear that $\delta$ can be chosen as a function of $\varepsilon$ alone so as to satisfy (13) under the assumptions, where we note $\sigma(t, r) = t + p(r)$ under $(\mathrm{H}_3)$. Therefore, the zero solution of (E) is uniformly stable in $(X, Y)$, and the sufficiency of $(\mathrm{H}_6)$ is immediate. Since the uniform stability is established, in order to prove the uniform asymptotic stability it is sufficient to show that for any $\varepsilon > 0$ we can find $T$ so that

(15) $$\inf\{\|x_t\|_X \colon \sigma(\tau, \varepsilon) \leqq t \leqq \tau + T\} \leqq \varepsilon$$

if $\|x(t)\| \leqq \beta$ $(t \geqq \tau)$ and $\|x_\tau\|_X \leqq \beta$ for a $\beta > 0$. On the other hand, by Theorem 2 (i) under $(\mathrm{H}_5)$ it is also sufficient to get $\inf\{v(t) \colon \sigma(\tau, \varepsilon) \leqq t \leqq \tau + T\} \leqq \varepsilon$ for a $T$ independent of $\tau$. Assume the condition $(\mathrm{H}_7)$, and suppose that $v(t) \geqq \varepsilon$ and $\|x_t\|_X \geqq \varepsilon$ on $[\sigma(\tau, \varepsilon), \tau + T]$, while $\|x_\tau\|_X \leqq \beta$ and $\|x_t\|_Y \leqq \beta$ for a $\beta > 0$ and all $t \geqq \tau$. Since $Y_l < X$, that is, $\|x_t\|_X \leqq \gamma \sup_{t-l \leqq s \leqq t} \|x_s\|_Y$ for a $\gamma > 0$, we can find a sequence $\{t_k\}$ so that $t_k \in [\sigma(\tau, \varepsilon) + (2k - 1)l,\ \sigma(\tau, \varepsilon) + 2kl]$ and $\|x_{t_k}\|_Y \geqq \varepsilon/\gamma$ as long as $\sigma(\tau, \varepsilon) + 2kl \leqq \tau + T$. Therefore from Lemma 2 it follows that $\int_{t_k - l}^{t_k} w(t)dt \geqq c(\varepsilon\theta/\gamma)\rho(\varepsilon/\gamma)$ $(=d(\varepsilon))$ for a $\theta > 0$ and a $\rho \in \mathscr{K}^+(I)$. Hence, $\int_{\sigma(\tau,\varepsilon)}^{\sigma(\tau,\varepsilon)+2kl} w(t)dt \geqq kd(\varepsilon)$, which yields

$$v(t) \leqq v(\sigma(\tau, \varepsilon)) - kd(\varepsilon) \leqq b(\sigma(\tau, \varepsilon) - \tau, 0, \gamma\beta) - kd(\varepsilon)$$

as long as $t \in [\sigma(\tau, \varepsilon) + 2kl, \tau + T]$ and, then, a contradiction if $T \geqq p(\varepsilon) + 2b(p(\varepsilon), 0, \gamma\beta)l/d(\varepsilon)$, where we note that $\|x_\sigma\|_X \leqq \gamma\|x_\sigma\|_{Y_l} \leqq \gamma\beta$ and $\sigma(\tau, \varepsilon) - \tau = p(\varepsilon)$. q.e.d.

REMARK 4. It can be easily seen that if $V(t, \phi; \tau)$ is independent of $\tau$ and satisfies (C), that is, $p(t, r) \equiv t$, then we can drop the conditions (L) and (UL) in Theorems 3 and 4, which correspond to a Liapunov type theorems (cf. [13], [18]). On the other hand, for a special phase space such as $C_h^\gamma$, if $V(t, \phi; \tau)$ satisfies $V(t, \phi; \tau) \leqq b(t, \|\phi\|_{C_l^\gamma})$ for an $l \leqq h$ and (C) under the condition $(\mathrm{c}_3)$ for a $p(t, r) \geqq t - h + l$, then again we can drop (L) in Theorem 3 by noting that the relation (13) can be replaced by

$\sup_{\tau-h+l\leqq s\leqq\tau}b(s, \delta)\leqq\varepsilon$. When $l=0$, this corresponds to a Razumikhin type theorem (cf. [5], [13]). The condition ($H_7$) in Theorem 4 is a generalization of [13; Theorem 31.4] in some sense.

THEOREM 5. *Let $X=M_h^\gamma$ for a $\gamma>0$ and $0\leqq h\leqq\infty$, and suppose that a Liapunov function $V(t, \phi)$ satisfies* (A), (B) *with* (14), *and* (C) *wtih $W(t, \phi)=c(\|\phi(0)\|)$ for a $c\in\mathscr{K}^+(I)$. Then, the zero solution of* (E) *is uniformly asymptotically stable in $(X, Y)$ and, hence, in $(C_h^\beta, R^n)$ for any $\beta<\gamma$.*

PROOF. By Theorem 4 and Remark 4, the zero solution of (E) is uniformly stable in $(X, Y)$. Hence, it is sufficient to get (15) for a $T$. Suppose that it is not the case, that is, $\|x_t\|_{M_h^\gamma}\geqq\varepsilon$ on $[\tau, \tau+T]$ for an $\varepsilon>0$. Then, we have $\int_{t_k}^{t_{k+1}}W(t, x_t)dt\geqq d(\varepsilon)$ with $t_k=\tau+2q(\varepsilon)k$ by Lemma 3, where $q(r)$ and $d(r)$ are those given there for $c(r)$. Thus, by setting $T\geqq 2q(\varepsilon)\beta/d(\varepsilon)$ we have a contradiction if $b(\tau, \tau, \|x_\tau\|_X)\leqq\beta$. The rest of the proof follows from Lemma 1 and Proposition 2. q.e.d.

REMARK 5. Clearly, the condition (C) in Theorem 5 suffices to hold under the conditions ($c_1$) and ($c_2$) if already we know the uniform stability. This theorem is a slight extension of [3; Theorem 1].

## REFERENCES

[1] B. I. BARNEA, A method and new results for stability and instability of autonomous functional equations, SIAM J. Appl. Math. 17 (1969), 681–697.

[2] S. R. BERNFELD AND J. R. HADDOCK, A variation of Razumikhin's method for retarded functional differential equations, Nonlinear Systems and Applications, Academic Press, New York-San Fransisco-London, 1977, 561–566.

[3] T. A. BURTON, Uniform asymptotic stability in functional differential equations, Proc. Amer. Math. Soc. 68 (1978), 195–199.

[4] R. DATKO, The uniform asymptotic stability of certain linear differential-difference equations, Proc. Roy. Soc. Edinburgh 74A (1974/75), 71–78.

[5] R. D. DRIVER, Existence and stability of solutions of a delay-differential system, Arch. Rational Mech. Anal. 10 (1962), 401–426.

[6] R. GRIMMER AND G. SEIFERT, Stability properties of Volterra integro-differential equations, J. Differential Equations 19 (1975), 142–166.

[7] S. E. GROSSMAN AND J. A. YORKE, Asymptotic behavior and exponential stability criteria for differential delay equations, J. Differential Equations, 12 (1972), 236–255.

[8] J. K. HALE, Theory of Functional Differential Equations, Appl. Math. Sci., Vol. 3, Springer-Verlag, New York-Heidelberg-Berlin, 1977.

[9] J. K. HALE AND J. KATO, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), 11–41.

[10] J. KATO, On Liapunov-Razumikhin type theorems for functional differential equations, Funkcial. Ekvac. 16 (1973), 225–239.

[11] J. KATO, Stability problem in functional differential equations with infinite delay,

Funkcial. Ekvac. 21 (1978), 63-80.

[12] J. KATO, Stability in functional differential equations, Lecture Notes in Math., Springer-Verlag, Berlin-Heidelberg-New York, 799 (1980), 252-262.

[13] N. N. KRASOVSKII, Stability of Motion, Stanford Univ. Press, Stanford, California, 1963.

[14] V. LAKSHMIKANTHAM AND S. LEELA, Differential and Integral Inequalities, Vol. 2, Academic Press, New York-London, 1969.

[15] B. S. RAZUMIKHIN, On the stability of systems with a delay, Prikl. Mat. Meh. 20 (1956), 500-512.

[16] K. SAWANO, Exponentially asymptotic stability for functional differential equations with infinite retardations, Tôhoku Math. J. 31 (1979), 363-382.

[17] G. SEIFERT, Liapunov-Razumikhin conditions for asymptotic stability in functional differential equations of Volterra type, J. Differential Equations 16 (1974), 289-297.

[18] T. YOSHIZAWA, Stability Theory by Liapunov's Second Method, Publ. Vol. 9, Japan Math. Soc. Tokyo, 1966.

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