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QUANTITATIVE THEOREMS ON LINEAR APPROXIMATION PROCESSES OF CONVOLUTION OPERATORS IN BANACH SPACES

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1. Introduction. Let X be an arbitrary (real or complex) Banach space with norm $\|\cdot\|_X$ and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Here we are concerned with linear approximation processes on X defined as follows:

DEFINITION 1. Let $\{L_{n,\lambda};\, n \in N,\ \lambda \in \Lambda\}$ be a family of operators in $B[X]$, where N denotes the set of all natural numbers and Λ is an arbitrary index set. The family $\{L_{n,\lambda}\}$ is said to be a linear approximation process on X if for every $f \in X$,

$$\lim_{n\to\infty} \|L_{n,\lambda}(f) - f\|_X = 0 \quad \text{uniformly in} \quad \lambda \in \Lambda .$$

In this note we would like to study the direct problem of approximation by particular linear approximation processes $\{L_{n,\lambda};\, n \in N,\ \lambda \in \Lambda\}$, $L_{n,\lambda}$ being of convolution type in connection with families of strongly continuous operators in $B[X]$. That is, we estimate the degree of convergence of $L_{n,\lambda}(f)$ to f (in the sense of our Definition 1) by a modulus of continuity of f, which can be defined in a natural way (cf. [9; p. 204]). We also study the multiplier type operators in connection with Fourier series expansions corresponding to a total, fundamental sequence of mutually orthogonal projections in $B[X]$ (cf. [3]).

The results obtained in this paper yield applications to the almost convergence of sequences of operators in $B[X]$. For the basic properties of almost convergent sequences of real numbers, see [7] (cf. [6], [8], [10]). We also give applications to the approximation problem in homogeneous Banach spaces due to Katznelson [5] (cf. [9. p. 206]), which are more than the Banach spaces $L^p_{2\pi}$, $1 \leqq p < \infty$, and $C_{2\pi}$ of all 2π-periodic, pth power Lebesgue integrable functions f with the norm

$$\|f\|_p = \left\{ (1/2\pi) \int_{-\pi}^{\pi} |f(t)|^p dt \right\}^{1/p} ,$$

and of all 2π-periodic continuous functions f with the norm

$$\|f\|_\infty = \max\{|f(t)|;\ |t| \leqq \pi\} ,$$

respectively. An excellent source for references and a systematic treatment on approximation by convolution integral operators in $L^p_{2\pi}$, $1 \leqq p < \infty$, and $C_{2\pi}$ can be found in the books of Butzer and Nessel [2] and DeVore [4].

2. Approximate identities for convolution operators. Let R denote the set of all real numbers and let $\mathscr{T} = \{T_t;\ t \in R\}$ be a family of operators in $B[X]$ with $T_0 = I$, the identity operator, such that for each $f \in X$ the map $t \to T_t(f)$ is strongly continuous. Therefore, the uniform boundedness principle implies that

$$M_{\mathscr{T}} = \sup\{\|T_t\|_{B[X]};\ |t| \leqq \pi\}$$

is finite. The convolution of f in X with k in $L^1_{2\pi}$ is the element $k*f$ in X given by

$$k*f = (1/2\pi)\int_{-\pi}^{\pi} k(t)T_t(f)dt ,$$

which exists as a Bochner integral (cf. [9; pp. 201-202]).

Let $k \in L^1_{2\pi}$ and $P \in B[X]$. Then the relation

$$(k*P)(f) = k*(P(f)) \quad (f \in X)$$

determines a convolution operator $k*P$ on X. Note that $k*P$ belongs to $B[X]$ and

$$\|k*P\|_{B[X]} \leqq M_{\mathscr{T}}\|k\|_1\|P\|_{B[X]} ,$$

and moreover, if $PT_t = T_tP$ for each $t \in R$, then $P(k*I) = (k*I)P$. In many cases we deal with linear approximation processes on X which can be generated via convolution operators of the form $k*I$, k being a non-negative or even function in $L^1_{2\pi}$.

In connection with convergence theorems we give the following standard definition.

DEFINITION 2. An approximate identity (for convolution) is a family $\{k_{n,\lambda};\ n \in N,\ \lambda \in \Lambda\}$ of elements in $L^1_{2\pi}$ such that

(1) $$\sup\{\|k_{n,\lambda}\|_1;\ n \in N,\ \lambda \in \Lambda\} < \infty ,$$

(2) $$\lim_{n\to\infty} (1/2\pi)\int_{-\pi}^{\pi} k_{n,\lambda}(t)dt = 1 \quad \text{uniformly in} \quad \lambda \in \Lambda$$

and

(3) $$\lim_{n\to\infty} \int_{\delta \leqq |t| \leqq \pi} |k_{n,\lambda}(t)|dt = 0 \quad \text{uniformly in} \quad \lambda \in \Lambda$$

for any fixed δ satisfying $0 < \delta < \pi$.

LEMMA 1. *Let* $\{k_{n,\lambda};\, n \in N,\ \lambda \in \Lambda\}$ *be an approximate identity and let* Φ *be a continuous X-valued function on* $[-\pi, \pi]$. *Then*

$$\lim_{n\to\infty}(1/2\pi)\int_{-\pi}^{\pi} k_{n,\lambda}(t)\Phi(t)dt = \Phi(0) \quad \textit{uniformly in} \quad \lambda \in \Lambda\,.$$

We omit the proof, which is elementary ([cf. [5; Chapter I, Lemma 2.2]). As an immediate consequence of Lemma 1, we have the following.

PROPOSITION 1. *Let* $\{k_{n,\lambda};\, n \in N,\ \lambda \in \Lambda\}$ *be an approximate identity. Then the family* $\{k_{n,\lambda} * I;\, n \in N,\ \lambda \in \Lambda\}$ *is a linear approximation process on* X.

PROOF. Let f be an arbitrary element in X and take $\Phi(t) = T_t(f)$ in Lemma 1.

COROLLARY 1. *Let* $\{k_{n,\lambda};\, n \in N,\ \lambda \in \Lambda\}$ *be a family of non-negative functions in* $L^1_{2\pi}$ *satisfying* (2) *with Fourier series expansions*

$$k_{n,\lambda}(t) \sim \sum_{j=-\infty}^{\infty} \hat{k}_{n,\lambda}(j)e^{ijt} \qquad (n \in N,\ \lambda \in \Lambda)\,,$$

where

$$\hat{k}_{n,\lambda}(j) = (1/2\pi)\int_{-\pi}^{\pi} k_{n,\lambda}(t)e^{-ijt}dt\,.$$

Suppose that $\lim_{n\to\infty}\{\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1))\} = 0$ *uniformly in* $\lambda \in \Lambda$, *where* $\mathrm{Re}(\hat{k}_{n,\lambda}(1))$ *denotes the real part of* $\hat{k}_{n,\lambda}(1)$. *Then the family* $\{k_{n,\lambda} * I;\, n \in N,\ \lambda \in \Lambda\}$ *is a linear approximation process on* X.

PROOF. Let $0 < \delta < \pi$. Then we have

$$\int_{\delta \le |t| \le \pi} k_{n,\lambda}(t)dt \leqq 2\pi\{\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1))\}/(1 - \cos\delta)$$

for all $n \in N$ and for all $\lambda \in \Lambda$. Therefore, the desired assertion follows from Proposition 1.

REMARK 1. Let $\{L_p;\, p \in N\}$ be a sequence of operators in $B[X]$ and let $f,\ g \in X$. In view of the concept of almost convergence of sequences of real numbers due to Lorentz [7], we say that the sequence $\{L_p(f);\, p \in N\}$ is almost convergent to g in X if

$$\lim_{n\to\infty}\left\|(1/n)\sum_{p=m}^{m+n-1} L_p(f) - g\right\|_X = 0 \quad \text{uniformly in} \quad m = 1, 2, \cdots$$

(cf. [6], [8], [10]). Let

$$L_{n,m} = (1/n)\sum_{p=m}^{m+n-1} L_p \qquad (m,\ n = 1, 2, \cdots)\,. \tag{4}$$

Then the family $\{L_{n,m}; n \in N, m \in N\}$ is a linear approximation process on X if and only if for each $f \in X$, the sequence $\{L_p(f); p \in N\}$ is almost convergent to f in X.

REMARK 2. Let $\{k_p; p \in N\}$ be a sequence of functions in $L^1_{2\pi}$ having Fourier series expansions

$$k_p(t) \sim \sum_{j=-\infty}^{\infty} \hat{k}_p(j) e^{ijt} \qquad (p \in N) ,$$

and let

$$k_{n,m} = (1/n) \sum_{p=m}^{m+n-1} k_p \qquad (m, n = 1, 2, \cdots) . \tag{5}$$

Applying Proposition 1 and Corollary 1, we have the following:

(i) If the family $\{k_{n,m}; n \in N, m \in N\}$ is an approximate identity, then for each $f \in X$, the sequence $\{k_p * f; p \in N\}$ is almost convergent to f in X.

(ii) If each function k_p is non-negative and if $\{\hat{k}_p(0); p \in N\}$ and $\{\hat{k}_p(0) - \mathrm{Re}(\hat{k}_p(1)); p \in N\}$ are almost convergent respectively to one and zero, then $\{k_p * f; p \in N\}$ is almost convergent to f for each $f \in X$.

3. A quantitative theorem. In order to recast Corollary 1 in a quantitative form we shall need the following additional assumption upon the family $\mathscr{T} = \{T_t; t \in R\}$:

($\mathscr{T}$-1) There exists a constant $C_{\mathscr{T}} \geqq 1$, independent of f, s and t, such that

$$\| T_s(f) - T_t(f) \|_X \leqq C_{\mathscr{T}} \| T_{s-t}(f) - f \|_X \tag{6}$$

for all s, $t \in R$ and for all $f \in X$.

REMARK 3. If $\mathscr{T} = \{T_t; t \in R\}$ is a uniformly bounded strongly continuous group of operators in $B[X]$ (for the fundamentals of semi-group theory, see [1]), then (6) holds with $C_{\mathscr{T}} = \sup\{\|T_t\|_{B[X]}; t \in R\}$. If in addition each T_t is isometric, then $\|T_s(f) - T_t(f)\|_X = \|T_{s-t}(f) - f\|_X$.

We now introduce a modulus of continuity of elements in X associated with the family $\mathscr{T}$ (cf. [9; p. 204]).

DEFINITION 3. Suppose $\mathscr{T}$ satisfies ($\mathscr{T}$-1) and let $f \in X$. For $\delta \geqq 0$ we define the modulus of continuity of f associated with $\mathscr{T}$ by

$$\omega_{\mathscr{T}}(X; f, \delta) = \sup\{\|T_t(f) - f\|_X; |t| \leqq \delta\} .$$

The modulus of continuity has the following fundamental properties:

LEMMA 2. *Suppose $\mathscr{T}$ satisfies ($\mathscr{T}$-1) and let $f \in X$.*

(i) $\omega_{\mathscr{T}}(X; f, \delta)$ *is a non-decreasing function of* δ *on* $[0, \infty)$ *and* $\omega_{\mathscr{T}}(X; f, 0) = 0$.

(ii) $\omega_{\mathscr{T}}(X; f, \eta\delta) \leqq (1 + \eta C_{\mathscr{T}})\omega_{\mathscr{T}}(X; f, \delta)$ *for each* $\eta, \delta \geqq 0$.

(iii) $\lim_{\delta \to 0+}\omega_{\mathscr{T}}(X; f, \delta) = 0$.

PROOF. The parts (i) and (iii) are obvious by the definition and the strong continuity of the map $t \to T_t(f)$ at $t = 0$. Condition ($\mathscr{T}$-1) yields that

$$\omega_{\mathscr{T}}(X; f, \delta + \eta) \leqq C_{\mathscr{T}}\omega_{\mathscr{T}}(X; f, \delta) + \omega_{\mathscr{T}}(X; f, \eta) .$$

Hence by induction on n we have

$$\omega_{\mathscr{T}}(X; f, n\delta) \leqq \{1 + (n - 1)C_{\mathscr{T}}\}\omega_{\mathscr{T}}(X; f, \delta) \quad (n \in N) ,$$

from which the part (ii) follows.

LEMMA 3. *Let k be a non-negative function in $L^1_{2\pi}$ with its Fourier series expansion*

$$k(t) \sim \sum_{j=-\infty}^{\infty} \hat{k}(j)e^{ijt} .$$

Then we have

$$(1/2\pi)\int_{-\pi}^{\pi} |t| k(t)dt \leqq \pi\{(1/2)(\hat{k}(0) - \mathrm{Re}(\hat{k}(1)))\}^{1/2}\{\hat{k}(0)\}^{1/2}$$

and

$$(1/2\pi)\int_{-\pi}^{\pi} t^2 k(t)dt \leqq (\pi^2/2)\{\hat{k}(0) - \mathrm{Re}(\hat{k}(1))\} .$$

The proof follows by elementary computations using the inequality $2x/\pi \leqq \sin x \quad (0 \leqq x \leqq \pi/2)$ and Hölder's inequality (cf. [2; Lemma 1.5.7]).

LEMMA 4. *Suppose that $\mathscr{T}$ satisfies ($\mathscr{T}$-1). Let k be a non-negative function in $L^1_{2\pi}$, and $f \in X$. Then we have*

$$\begin{aligned}(7)\quad & \|k*f - \hat{k}(0)f\|_X \\ & \leqq \{\hat{k}(0)\}^{1/2}\omega_{\mathscr{T}}(X; f, \delta)[\{\hat{k}(0)\}^{1/2} + (\pi C_{\mathscr{T}}/\delta)\{(1/2)(\hat{k}(0) - \mathrm{Re}(\hat{k}(1)))\}^{1/2}]\end{aligned}$$

for each $\delta > 0$.

PROOF. We have

$$k*f - \hat{k}(0)f = (1/2\pi)\int_{-\pi}^{\pi} k(t)\{T_t(f) - f\}dt ,$$

which implies

$$\|k*f - \hat{k}(0)f\|_X \leqq (1/2\pi)\int_{-\pi}^{\pi} k(t)\| T_t(f) - f\|_X dt$$

$$\leqq (1/2\pi)\int_{-\pi}^{\pi} k(t)\omega_{\mathscr{T}}(X; f, |t|)dt .$$

Therefore it follows from the part (ii) of Lemma 2 that

$$\|k*f - \hat{k}(0)f\|_X \leqq \omega_{\mathscr{T}}(X; f, \delta)(1/2\pi)\int_{-\pi}^{\pi}\{1 + (|t|C_{\mathscr{T}}/\delta)\}k(t)dt ,$$

which implies (7) by Lemma 3. The proof of the lemma is complete.

Given a family $\{L_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ of operators in $B[X]$, let

$$|||A_n(f) - f|||_X = \sup\{\|L_{n,\lambda}(f) - f\|_X; \lambda \in \Lambda\} \qquad (n \in N, f \in X) .$$

Note that $\{L_{n,\lambda}\}$ is a linear approximation process on X if and only if $\lim_{n\to\infty}|||A_n(f) - f|||_X = 0$ for every $f \in X$.

We are now in a position to recast Corollary 1 in a quantitative form as follows.

THEOREM 1. *Let $\{k_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ be a family of non-negative functions in $L^1_{2\pi}$ such that for each $n \in N$*

$$\alpha_n = \sup\{(\hat{k}_{n,\lambda}(0))^{1/2}; \lambda \in \Lambda\} \tag{8}$$

is finite. Suppose that $\mathscr{T} = \{T_t; t \in R\}$ satisfies ($\mathscr{T}$-1). Then for the family $\{k_{n,\lambda}*I; n \in N, \lambda \in \Lambda\}$ we have

$$|||A_n(f) - f|||_X \leqq \|f\|_X\gamma_n + \{\alpha_n + (1/2)^{1/2}\pi C_{\mathscr{T}}\}\alpha_n\omega_{\mathscr{T}}(X; f, \beta_n) \tag{9}$$

for all $n \in N$ and for all $f \in X$, where

$$\beta_n = \sup\{[\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1))]^{1/2}; \lambda \in \Lambda\} \tag{10}$$

and

$$\gamma_n = \sup\{|\hat{k}_{n,\lambda}(0) - 1|; \lambda \in \Lambda\} . \tag{11}$$

In particular, if $\hat{k}_{n,\lambda}(0) = 1$ for all $n \in N$ and for all $\lambda \in \Lambda$, then (9) *reduces to*

$$|||A_n(f) - f|||_X \leqq \{1 + (1/2)^{1/2}\pi C_{\mathscr{T}}\}\omega_{\mathscr{T}}(X; f, \beta_n) . \tag{12}$$

PROOF. Taking $k = k_{n,\lambda}$ in Lemma 4, we have

$$\|k_{n,\lambda}*f - \hat{k}_{n,\lambda}(0)f\|_X \leqq \alpha_n\{\alpha_n + (1/2)^{1/2}\pi C_{\mathscr{T}}(\beta_n/\delta)\}\omega_{\mathscr{T}}(X; f, \delta) . \tag{13}$$

If $\beta_n > 0$, take $\delta = \beta_n$ in (13). Then the inequality

$$\|k_{n,\lambda}*f - f\|_X \leqq \|k_{n,\lambda}*f - \hat{k}_{n,\lambda}(0)f\|_X + |\hat{k}_{n,\lambda}(0) - 1|\,\|f\|_X \tag{14}$$

implies (9). If $\beta_n = 0$, then (13) reduces to

$$\|k_{n,\lambda}*f - \hat{k}_{n,\lambda}(0)f\|_X \leqq \alpha_n^2\omega_{\mathscr{T}}(X; f, \delta) .$$

Letting $\delta \to 0+$, we have $k_{n,\lambda}*f = \hat{k}_{n,\lambda}(0)f$ by (iii) of Lemma 2. Thus

(14) reduces to

$$\|k_{n,\lambda}*f - f\|_X = |\hat{k}_{n,\lambda}(0) - 1|\,\|f\|_X\,,$$

which implies (9), and the proof of the theorem is complete.

In connection with even functions we shall need the following condition ($\mathscr{T}$-2) instead of ($\mathscr{T}$-1):

($\mathscr{T}$-2) For every $s,\ t,\ u \in R$ and for every $f \in X$,

$$\|T_s(f) + T_t(f) - 2T_u(f)\|_X = \|T_{s-u}(f) + T_{t-u}(f) - 2f\|_X\,. \tag{15}$$

REMARK 4. ($\mathscr{T}$-2) already implies ($\mathscr{T}$-1) with $C_{\mathscr{T}} = 1$;

$$\|T_s(f) - T_t(f)\|_X = \|T_{s-t}(f) - f\|_X \qquad (f \in X,\ \ s,\, t \in R)\,.$$

If $\mathscr{T} = \{T_t;\, t \in R\}$ is a strongly continuous group of isometric operators in $B[X]$, then ($\mathscr{T}$-2) always holds.

DEFINITION 4. Suppose $\mathscr{T}$ satisfies ($\mathscr{T}$-2) and let $f \in X$. For $\delta \geqq 0$ we define the generalized modulus of continuity of f associated with $\mathscr{T}$ by

$$\omega^*_{\mathscr{T}}(X; f,\, \delta) = \sup\{\|T_t(f) + T_{-t}(f) - 2f\|_X;\ 0 \leqq t \leqq \delta\}\,.$$

LEMMA 5. *Suppose* $\mathscr{T}$ *satisfies* ($\mathscr{T}$-2) *and let* $f \in X$.

(i) $\omega^*_{\mathscr{T}}(X; f,\, \delta)$ *is a non-decreasing function of* δ *on* $[0,\, \infty)$ *and* $\omega^*_{\mathscr{T}}(X; f,\, 0) = 0$.

(ii) $\omega^*_{\mathscr{T}}(X; f,\, \eta\delta) \leqq (1 + \eta)^2\omega^*_{\mathscr{T}}(X; f,\, \delta)$ *for each* $\eta,\ \delta \geqq 0$.

(iii) $\omega^*_{\mathscr{T}}(X; f,\, \delta) \leqq 2\omega_{\mathscr{T}}(X; f,\, \delta)$ *for each* $\delta \geqq 0$. *Thus in particular,* $\lim_{\delta\to 0+}\omega^*_{\mathscr{T}}(X; f,\, \delta) = 0$.

PROOF. The parts (i) and (iii) are obvious by definition and the part (iii) of Lemma 2. Condition ($\mathscr{T}$-2) yields that $\omega^*_{\mathscr{T}}(X; f,\, n\delta) \leqq n^2\omega^*_{\mathscr{T}}(X; f,\, \delta)$ $(n \in N)$, from which the part (ii) follows.

LEMMA 6. *Suppose that* $\mathscr{T}$ *satisfies* ($\mathscr{T}$-2). *Let* k *be a non-negative, even function in* $L^1_{2\pi}$, *and* $f \in X$. *Then we have*

$$\begin{aligned}\|k*f - \hat{k}(0)f\|_X \leqq \omega^*_{\mathscr{T}}(X; f,\, \delta)[(\pi^2/4)(\hat{k}(0) - \hat{k}(1))(1/\delta^2)\\ + (1/2)(\hat{k}(0))^{1/2}\{(\hat{k}(0))^{1/2} + 2^{1/2}\pi(\hat{k}(0) - \hat{k}(1))^{1/2}/\delta\}]\end{aligned} \tag{16}$$

for each $\delta > 0$.

PROOF. Since k is even and positive, we have

$$k*f - \hat{k}(0)f = (1/2\pi)\int_0^\pi k(t)\{T_t(f) + T_{-t}(f) - 2f\}dt\,,$$

and so

$$\|k*f - \hat{k}(0)f\|_X \leqq (1/2\pi)\int_0^\pi k(t)\|T_t(f) + T_{-t}(f) - 2f\|_X dt$$
$$\leqq (1/2\pi)\int_0^\pi k(t)\omega_{\mathscr{T}}^*(X; f, t)dt .$$

Therefore it follows from the part (ii) of Lemma 5 that

$$\|k*f - \hat{k}(0)f\|_X \leqq (1/2\pi)\omega_{\mathscr{T}}^*(X; f, \delta)\int_0^\pi (1 + t/\delta)^2 k(t)dt ,$$

which implies (16) by Lemma 3.

THEOREM 2. *Let $\{k_{n,\lambda}; n \in N, \lambda \in \Lambda\}$ be a family of non-negative, even functions in $L_{2\pi}^1$ such that for each $n \in N$, α_n defined by* (8) *is finite. Suppose that $\mathscr{T} = \{T_t; t \in R\}$ satisfies ($\mathscr{T}$-2). Then for the family $\{k_{n,\lambda}*I; n \in N, \lambda \in \Lambda\}$ we have*

(17) $$|||A_n(f) - f|||_X \leqq \|f\|_X\gamma_n + (1/2)\{\pi^2/2 + \alpha_n(2^{1/2}\pi + \alpha_n)\}\omega_{\mathscr{T}}^*(X; f, \beta_n)$$

for all $n \in N$ and for all $f \in X$, where α_n, β_n and γ_n are numbers defined by (8), (10) *and* (11), *respectively. In particular, if $\hat{k}_{n,\lambda}(0) = 1$ for all $n \in N$ and for all $\lambda \in \Lambda$, then* (17) *reduces to*

(18) $$|||A_n(f) - f|||_X \leqq (1/2)(1 + 2^{-1/2}\pi)^2\omega_{\mathscr{T}}^*(X; f, \beta_n) .$$

PROOF. In view of Lemma 6 the proof is essentially similar to that of Theorem 1, and so we omit the details.

COROLLARY 2. *Let $\{k_p; p \in N\}$ be a sequence of functions in $L_{2\pi}^1$, and let $\{k_{n,m}; n \in N, m \in N\}$ be the family of functions defined by* (5) *such that for each $n \in N$, α_n defined by* (8) *is finite. Then the following statements hold:*

(i) *Suppose that $\mathscr{T} = \{T_t; t \in R\}$ satisfies ($\mathscr{T}$-1) and each k_p is non-negative. Then for the family $\{k_{n,m}*I; n \in N, m \in N\}$,* (9) *holds for all $n \in N$ and for all $f \in X$. If in addition $\hat{k}_p(0) = 1$ for each $p \in N$, then* (12) *holds for all $n \in N$ and for all $f \in X$.*

(ii) *Suppose that $\mathscr{T} = \{T_t; t \in R\}$ satisfies ($\mathscr{T}$-2) and each k_p is non-negative and even. Then for the family $\{k_{n,m}*I; n \in N, m \in N\}$,* (17) *holds for all $n \in N$ and for all $f \in X$. If in addition $\hat{k}_p(0) = 1$ for each $p \in N$, then* (18) *holds for all $n \in N$ and for all $f \in X$.*

REMARK 5. Let $\{k_p;\ p \in N\}$ be a sequence of non-negative, even functions in $L_{2\pi}^1$ with Fourier series expansions

$$k_p(t) \sim 1 + 2\sum_{j=1}^\infty a_p(j)\cos jt \qquad (p \in N) .$$

Suppose that $\mathscr{T} = \{T_t;\ t \in R\}$ satisfies ($\mathscr{T}$-2) and there exists a constant

$C > 0$ such that $1 - a_p(1) \leqq C/p$ for all $p \in N$. Then the latter statement of the part (ii) of Corollary 2 implies that

(19) $|||A_n(f) - f|||_X \leqq (1/2)(1 + 2^{-1/2}\pi)^2 \omega^*_{\mathscr{T}}(X; f, \{C(\gamma + \log(n + 1))/n\}^{1/2})$

for all $n \in N$ and for all $f \in X$, where γ is Euler's constant:

$$\gamma = \lim_{p\to\infty}\left(\sum_{j=1}^{p}(1/j) - \log p\right) = 0.5772156649015328\cdots .$$

We mention some concrete examples of non-negative, even functions k_p, $p \in N$.

(1°) $k_p(t) = 1 + 2\sum_{j=1}^{p}\{1 - j/(p+1)\}\cos jt$, the Fejér kernel (in this case, $C = 1$).

(2°) $k_p(t) = 1 + 2\sum_{j=1}^{p}\{(p!)^2/((p-j)!\,(p+j)!)\}\cos jt$, the de La Vallée Poussin kernel (in this case, $C = 1$).

(3°) $k_p(t) = \{3/(p(2p^2+1))\}\{(\sin(pt/2))/\sin(t/2)\}^4$, the Jackson kernel (in this case, $C = 3/2$).

(4°) $k_p(t) = 1 + 2\sum_{j=1}^{p} a_p(j)\cos jt$ with

$$\begin{aligned} a_p(j) = \{&(p - j + 3)\sin((j+1)\pi/(p+2)) \\ &- (p - j + 1)\sin((j-1)\pi/(p+2))\}\{2(p+2)\sin(\pi/(p+2))\}^{-1}, \end{aligned}$$

the Fejér-Korovkin kernel (in this case, $C = \pi$, cf. [2; pp. 79–80]).

Definition 5. Suppose $\mathscr{T} = \{T_t;\, t \in R\}$ satisfies ($\mathscr{T}$-1). An element $f \in X$ is said to satisfy the Lipschitz condition with constant M and exponent α, or to belong to the class $\mathrm{Lip}_{\mathscr{T}}(X; \alpha)_M$, $M > 0$, $\alpha > 0$, if $\omega_{\mathscr{T}}(X; f, \delta) \leqq M\delta^\alpha$ for all $\delta \geqq 0$. Further, we let $\mathrm{Lip}_{\mathscr{T}}(X; \alpha) = \bigcup\{\mathrm{Lip}_{\mathscr{T}}(X; \alpha)_M;\, M > 0\}$.

Definition 6. Suppose $\mathscr{T} = \{T_t;\, t \in R\}$ satisfies ($\mathscr{T}$-2). An element $f \in X$ is said to satisfy the generalized Lipschitz condition with constant M and exponent α, or to belong to the class $\mathrm{Lip}^*_{\mathscr{T}}(X; \alpha)_M$, $M > 0$, $\alpha > 0$, if $\omega^*_{\mathscr{T}}(X; f, \delta) \leqq M\delta^\alpha$ for all $\delta \geqq 0$. Further, we let $\mathrm{Lip}^*_{\mathscr{T}}(X; \alpha) = \bigcup\{\mathrm{Lip}^*_{\mathscr{T}}(X; \alpha)_M;\, M > 0\}$.

Remark 6. Under the hypotheses of Theorem 1, if f belongs to $\mathrm{Lip}_{\mathscr{T}}(X; \alpha)_M$, then (12) implies that

$$|||A_n(f) - f|||_X \leqq M\{1 + (1/2)^{1/2}\pi C_{\mathscr{T}}\}\beta_n^\alpha$$

for all $n \in N$. Under the hypotheses of Theorem 2, if f belongs to $\mathrm{Lip}^*_{\mathscr{T}}(X; \alpha)_M$, then (18) implies that

$$|||A_n(f) - f|||_X \leqq (M/2)(1 + 2^{-1/2}\pi)^2\beta_n^\alpha$$

for all $n \in N$. In particular under the hypotheses of Remark 5, if f belongs to $\mathrm{Lip}^*_{\mathscr{T}}(X; \alpha)_M$, then (19) implies that

$$|||A_n(f) - f|||_X \leqq (M/2)(1 + 2^{-1/2}\pi)^2\{C(\gamma + \log(n+1))/n\}^{\alpha/2}$$

for all $n \in N$.

THEOREM 3. *Let $\mathscr{T} = \{T_t;\ t \in R\}$ be a uniformly bounded strongly continuous group of operators in $B[X]$, having G as its infinitesimal generator with domain $D(G)$ and M as its bound. Let $\{k_{n,\lambda};\ n \in N, \lambda \in \Lambda\}$ be a family of non-negative, even functions in $L^1_{2\pi}$ such that for each $n \in N$, α_n defined by* (8) *is finite. Then for the family $\{k_{n,\lambda}*I;\ n \in N, \lambda \in \Lambda\}$ we have*

$$\begin{aligned}(20)\quad |||A_n(f) - f|||_X \leqq \|f\|_X\gamma_n &+ 2^{-1/2}\pi\|G(f)\|_X\alpha_n\beta_n \\ &+ 2^{-1/2}\pi\{1 + M(2^{-3/2}\pi)\}\beta_n\omega_{\mathscr{T}}(X; G(f), \beta_n)\end{aligned}$$

for all $n \in N$ and for all $f \in D(G)$, where α_n, β_n and γ_n are numbers defined by (8), (10) *and* (11), *respectively. In particular, if $\hat{k}_{n,\lambda}(0) = 1$ for all $n \in N$ and for all $\lambda \in \Lambda$, then* (20) *reduces to*

$$\begin{aligned}(21)\quad |||A_n(f) - f|||_X \leqq 2^{-1/2}\pi\|G(f)\|_X\beta_n & \\ + 2^{-1/2}\pi\{1 + M(2^{-3/2}\pi)\}&\beta_n\omega_{\mathscr{T}}(X; G(f), \beta_n)\,.\end{aligned}$$

PROOF. Let k be a non-negative, even function in $L^1_{2\pi}$, and $f \in D(G)$. Then we have

$$\begin{aligned}(22)\quad \|k*f - \hat{k}(0)f\|_X \leqq\ & 2^{-1/2}\pi\{\hat{k}(0)(\hat{k}(0) - \hat{k}(1))\}^{1/2}\|G(f)\|_X \\ &+ 2^{-1/2}\pi\{\hat{k}(0) - \hat{k}(1)\}^{1/2}[1 + M(2^{-3/2}\pi/\delta)\{\hat{k}(0) \\ &- \hat{k}(1)\}^{1/2}]\omega_{\mathscr{T}}(X; G(f), \delta)\end{aligned}$$

for each $\delta > 0$. Indeed,

$$\begin{aligned}k*f - \hat{k}(0)f &= (1/2\pi)\left\{\int_0^\pi k(t)(T_t(f) - f)dt + \int_0^\pi k(t)(T_{-t}(f) - f)dt\right\} \\ &= g + h\,,\end{aligned}$$

say. Since

$$T_t(f) - f = \int_0^t T_u(G(f))du \qquad (t > 0)\,,$$

we have

$$\begin{aligned}\|g\|_X \leqq & \left\{(1/2\pi)\int_0^\pi tk(t)dt\right\}\|G(f)\|_X \\ &+ (1/2\pi)\int_0^\pi k(t)\left\{\int_0^t \|T_u(G(f)) - G(f)\|du\right\}dt\,.\end{aligned}$$

The second integral is, by virtue of the part (ii) of Lemma 2, majorized by

$$(1/2\pi)\int_0^\pi k(t)\left\{\int_0^t (1 + (M/\delta)u)\omega_{\mathscr{T}}(X; G(f), \delta)du\right\}dt$$
$$= \omega_{\mathscr{T}}(X; G(f), \delta)(1/2\pi)\int_0^\pi tk(t)\{1 + (M/2\delta)t\}dt .$$

In a similar manner, we obtain the same estimate for $\|h\|_X$, and consequently

$$\|k*f - \hat{k}(0)f\|_X \leqq \left\{(1/\pi)\int_0^\pi tk(t)dt\right\}\|G(f)\|_X$$
$$+ \left[(1/\pi)\int_0^\pi tk(t)\{1 + (M/2\delta)t\}dt\right]\omega_{\mathscr{T}}(X; G(f), \delta) .$$

This, in virtue of Lemma 3, proves the desired estimate (22).

Taking $k = k_{n,\lambda}$ in (22), we finish the proof exactly as that of Theorem 1.

REMARK 7. Let $\{k_p; p \in N\}$ and $\mathscr{T} = \{T_t;\ t \in R\}$ be as in Remark 5 and Theorem 3, respectively. Then (21) reduces to

$$(23)\quad \||A_n(f) - f\||_X \leqq 2^{-1/2}\pi\|G(f)\|_X\{C(\gamma + \log(n + 1))/n\}^{1/2}$$
$$+ 2^{-1/2}\pi\{1 + M(2^{-3/2}\pi)\}\{C(\gamma + \log(n + 1))/n\}^{1/2}$$
$$\times \omega_{\mathscr{T}}(X; f, \{C(\gamma + \log(n + 1))/n\}^{1/2})$$

for all $n \in N$ and for all $f \in D(G)$. In particular, if $f \in D(G)$ belongs to $\mathrm{Lip}_{\mathscr{T}}(X; \alpha)_K$, then (23) implies that $\||A_n(f) - f\||_X \leqq 2^{-1/2}\pi\|G(f)\|_X\{C(\gamma + \log(n + 1))/n\}^{1/2} + 2^{-1/2}\pi K\{1 + M(2^{-3/2}\pi)\}\{C(\gamma + \log(n + 1))/n\}^{(1+\alpha)/2}$ for all $n \in N$.

4. **Multiplier operators.** In this section we would like to discuss certain families $\{T_t;\ t \in R\}$ of multiplier operators. Let Z denote the set of all integers, and let $\{P_j\}_{j \in Z}$ be a sequence of projections in $B[X]$ satisfying the following properties:

(i) The projections P_j are mutually orthogonal, i.e., for all $j, m \in Z$ there holds $P_j P_m = \delta_{j,m} P_m$, $\delta_{j,m}$ being Kronecker's symbol.

(ii) The sequence $\{P_j\}$ is total, i.e., $P_j(f) = 0$ for all $j \in Z$ implies $f = 0$.

(iii) The sequence $\{P_j\}$ is fundamental, i.e., the linear subspace of X spanned by the ranges $P_j(X)$, $j \in Z$, is dense in X. Then for each $f \in X$ the series $\sum_{j=-\infty}^{\infty} P_j(f)$ is called the (formal) Fourier series expansion of f (with respect to $\{P_j\}$), and the following notation is used (cf. [3]):

(24) $$f \sim \sum_{j=-\infty}^{\infty} P_j(f) .$$

Let $\mathscr{S}$ denote the set of all sequences $a = \{a_j\}_{j \in Z}$ of scalars. An element $a \in \mathscr{S}$ is called a multiplier sequence for X (corresponding to $\{P_j\}$) if for each $f \in X$ there exists an element $f_a \in X$ such that $a_j P_j(f) = P_j(f_a)$ for all $j \in Z$, thus

(25) $$f_a \sim \sum_{j=-\infty}^{\infty} a_j P_j(f) .$$

Note that f_a is uniquely determined by f, since $\{P_j\}$ is total and so the map $f \to f_a$ defines a bounded linear operator of X into itself by the closed graph theorem. An element $T \in B[X]$ is called a multiplier operator on X if it permits an expansion of type (25).

REMARK 8. The expansion (24) represents a slight generalization of the concept of Fourier series in a Banach space X associated with a fundamental, total, biorthogonal system $\{f_j, f_j^*\}_{j \in Z}$. Here $\{f_j\}_{j \in Z}$ and $\{f_j^*\}_{j \in Z}$ are sequences of elements in X and X^* (the dual space of X), respectively, such that the linear subspace of X spanned by $\{f_j\}$ is dense in X (fundamental), $f_j^*(f) = 0$ for all $j \in Z$ implies $f = 0$ (total), and $f_j^*(f_m) = \delta_{j,m}$ for all $j, m \in Z$ (biorthogonal). Then (24) and (25) read

$$f \sim \sum_{j=-\infty}^{\infty} f_j^*(f) f_j , \qquad T(f) \sim \sum_{j=-\infty}^{\infty} a_j f_j^*(f) f_j ,$$

respectively.

The following proposition shows that if $\mathscr{T} = \{T_t; t \in R\}$ is generated via multiplier operators on X with associated multiplier sequences of exponential type, then every convolution operator $k * I$ with $k \in L_{2\pi}^1$ is a multiplier operator on X.

PROPOSITION 2. *Let $\mathscr{T} = \{T_t;\ t \in R\}$ be a family of operators in $B[X]$ such that $\sup\{\|T_t\|_{B[X]};\ t \in R\}$ is finite and*

(26) $$T_t(f) \sim \sum_{j=-\infty}^{\infty} \exp(a_j t) P_j(f) \qquad (t \in R, \quad f \in X) ,$$

where $a = \{a_j\}$ is a sequence in $\mathscr{S}$. Then $\mathscr{T}$ is a strongly continuous group of operators in $B[X]$, and with each $k \in L_{2\pi}^1$ the convolution operator $k*I$ on X is a multiplier operator on X with associated multiplier sequence $c = \{c_j\}_{j \in Z} \in \mathscr{S}$ defined by

$$c_j = (1/2\pi)\int_{-\pi}^{\pi} k(t)\exp(a_j t)dt\ , \quad j \in Z\ ,$$

thus

(27) $$k*f \sim \sum_{j=-\infty}^{\infty} c_j P_j(f)$$

for every $f \in X$. *Furthermore, the infinitesimal generator* G *of* $\mathscr{T}$ *with domain* $D(G)$ *satisfies*

(28) $$G(f) \sim \sum_{j=-\infty}^{\infty} a_j P_j(f)$$

for all $f \in D(G)$. *If, furthermore, with the nth Cesâro mean operator* σ_n *defined by*

$$\sigma_n = \sum_{j=-n}^{n} \{1 - |j|/(n+1)\}P_j\ ,$$

the sequence $\{\sigma_n\}$ *is uniformly bounded, i.e.,*

(29) $$\sup\{\|\sigma_n\|_{B[X]};\ n = 0, 1, 2, \cdots\} < \infty\ ,$$

then

(30) $$D(G) = \left\{f \in X;\ g \sim \sum_{j=-\infty}^{\infty} a_j P_j(f) \text{ for some } g \in X\right\}.$$

PROOF. Since $\{P_j\}$ is total, the expansion (26) implies that $\mathscr{T}$ forms a group of operators in $B[X]$. We have

$$\lim_{t\to s} \|T_t(h) - T_s(h)\|_X = \lim_{t\to s} |\exp(a_j t) - \exp(a_j s)|\,\|h\|_X = 0$$

for every $h \in P_j(X)$, $j \in Z$, and so the map $t \to T_t(f)$ is strongly continuous for each $f \in X$, since $\{P_j\}$ is fundamental and $\mathscr{T}$ is uniformly bounded. Let $k \in L^1_{2\pi}$ and $f \in X$. Then we have, for all $j \in Z$,

$$P_j(k*f) = (1/2\pi)\int_{-\pi}^{\pi} k(t)P_j(T_t(f))dt = (1/2\pi)\int_{-\pi}^{\pi} k(t)\exp(a_j t)P_j(f)dt = c_j P_j(f)\ ,$$

which implies (27), and so the first assertion of the proposition is proved.

Suppose now that f belongs to $D(G)$. Then for each $j \in Z$ we have

$$P_j(G(f)) = \lim_{t\to 0}(1/t)P_j(T_t(f) - f) = \lim_{t\to 0}(1/t)\{\exp(a_j t) - 1\}P_j(f) = a_j P_j(f)\ ,$$

which implies (28), and therefore $D(G)$ is contained in the set on the right-hand side of (30). Suppose next that (29) is satisfied. Let f be an element of X such that

$$g \sim \sum_{j=-\infty}^{\infty} a_j P_j(f)$$

for some $g \in X$. Then $\sigma_n(g) = G(\sigma_n(f))$ for all $n \in N$. Since $\{P_j\}$ is fundamental, (29) implies that $\lim_{n\to\infty} \|\sigma_n(h) - h\|_X = 0$ whenever h belongs to X. Thus we have

$$\lim_{n\to\infty} \|\sigma_n(f) - f\|_X = 0 \quad \text{and} \quad \lim_{n\to\infty} \|G(\sigma_n(f)) - g\|_X = 0 ,$$

which imply that $f \in D(G)$ and $G(f) = g$, since G is a closed operator. This proves (30), and the proof of the proposition is complete.

REMARK 9. Condition (29) is a standard one in the study of multiplier sequences and summation processes of Fourier series expansions in Banach spaces (cf. [3]). For the particular sequence $\{a_j\}_{j\in Z} = \{-ij\}_{j\in Z}$, (27) reduces to

$$k*f \sim \sum_{j=-\infty}^{\infty} \hat{k}(j) P_j(f) .$$

In view of Remarks 3 and 4 and Proposition 2, we have the following theorem in which the convolution operators in question have Fourier series expansions of the form (27).

THEOREM 4. *Let* $\mathscr{T} = \{T_t;\ t \in R\}$ *be a family of multiplier operators on* X *with Fourier series expansions* (26). *Then the following statements hold.*

(i) *Let* $\{k_{n,\lambda};\ n \in N,\ \lambda \in \Lambda\}$ *be as in Theorem* 1. *Suppose that* $M = \sup\{\|T_t\|_{B[X]};\ t \in R\}$ *is finite. Then for all* $n \in N$, $\lambda \in \Lambda$ *and for all* $f \in X$

$$k_{n,\lambda}*f \sim \sum_{j=-\infty}^{\infty} c_{n,\lambda}(j) P_j(f) , \tag{31}$$

where

$$c_{n,\lambda}(j) = (1/2\pi)\int_{-\pi}^{\pi} k_{n,\lambda}(t)\exp(a_j t)dt$$

and furthermore, (9) *holds with* $C_{\mathscr{T}} = M$.

(ii) *Let* $\{k_{n,\lambda};\ n \in N,\ \lambda \in \Lambda\}$ *be as in Theorem* 2. *Suppose that* $\|T_t(f)\|_X = \|f\|_X$ *for all* $t \in R$ *and for all* $f \in X$. *Then for all* $n \in N$, $\lambda \in \Lambda$ *and for all* $f \in X$, (31) *holds and furthermore,* (17) *holds.*

(iii) *Let* $\{k_{n,\lambda};\ n \in N,\ \lambda \in \Lambda\}$ *be as in Theorem* 3. *Suppose that* $M = \sup\{\|T_t\|_{B[X]};\ t \in R\}$ *is finite. Then for every* $n \in N$, $\lambda \in \Lambda$ *and for every* $f \in X$, (31) *holds and furthermore,* (20) *holds.*

5. Homogeneous Banach subspaces of $L^1_{2\pi}$. In this section we apply the results obtained in the preceding section to homogeneous Banach subspaces of $L^1_{2\pi}$. Let X be a linear subspace of $L^1_{2\pi}$. X is called a homogeneous Banach subspace of $L^1_{2\pi}$ if it is a Banach space with norm $\|\cdot\|_X$ which satisfies the following properties (cf. [5; p. 14], [9; p. 206]):

(H-1) There exists a constant $C > 0$ such that $\|f\|_1 \leqq C\|f\|_X$ for all $f \in X$.

(H-2) For each $f \in X$ and $t \in R$, $T_t(f)$ belongs to X and $\|T_t(f)\|_X = \|f\|_X$, where T_t is the translation operator, i.e.,

$$T_t(f)(u) = f(u - t)\ , \quad u \in R\ .$$

(H-3) For each $f \in X$, the map $t \to T_t(f)$ is a continuous X-valued function on R.

Examples of homogeneous Banach subspaces of $L^1_{2\pi}$ are the following:

(1°) $L^p_{2\pi}$, $1 \leqq p < \infty$ (note that (H-3) is not satisfied when $X = L^\infty_{2\pi}$).

(2°) $C_{2\pi}$.

(3°) $C^{(n)}_{2\pi}$ = the linear subspace of $C_{2\pi}$ of all n-times continuously differentiable functions f with norm

$$\|f\|_{C^{(n)}_{2\pi}} = \sum_{j=0}^{n} (1/j!)\|f^{(j)}\|_\infty\ .$$

(4°) $AC_{2\pi}$ = the linear subspace of $L^1_{2\pi}$ of all 2π-periodic absolutely continuous functions f with norm

$$\|f\|_{AC_{2\pi}} = \|f\|_1 + \|f'\|_1\ .$$

(5°) $0 < \alpha < 1$, $\mathrm{lip}^\alpha_{2\pi}$ = the linear subspace of $C_{2\pi}$ consisting of all functions f for which

$$F(f) = \sup\{|f(t + h) - f(t)|/|h|^\alpha;\ h \neq 0,\ t \in R\} < \infty$$

and

$$\lim_{h \to 0}\,(\sup\{|f(t + h) - f(t)|/|h|^\alpha;\ t \in R\}) = 0\ ,$$

with norm

$$\|f\|_{\mathrm{lip}^\alpha_{2\pi}} = \|f\|_\infty + F(f)\ .$$

(6°) $D(L)$ = the domain in $L^1_{2\pi}$ of a closed operator L with range in $L^1_{2\pi}$ such that for each $t \in R$, T_t commutes with L, with norm

$$\|f\|_{D(L)} = \|f\|_1 + \|L(f)\|_1\ .$$

Now let X be a homogeneous Banach subspace of $L^1_{2\pi}$ with norm $\|\cdot\|_X$. Recall that $\mathscr{T} = \{T_t;\ t \in R\}$ is the family of translation operators.

Therefore, we have

$$\omega_{\mathscr{T}}(X; f, \delta) = \sup\{\|f(\cdot - t) - f(\cdot)\|_X; |t| \leqq \delta\}$$

and

$$\omega^*_{\mathscr{T}}(X; f, \delta) = \sup\{\|f(\cdot + t) + f(\cdot - t) - 2f(\cdot)\|_X; 0 \leqq t \leqq \delta\},$$

respectively.

Let $k \in L^1_{2\pi}$ and $f \in X$. Then

$$(k * f)(u) = (1/2\pi)\int_{-\pi}^{\pi} k(t)f(u - t)dt . \tag{32}$$

Defining the sequence $\{P_j\}_{j \in Z}$ by $P_j(f)(t) = \hat{f}(j)e^{ijt}$, it is obvious that $\{P_j\}$ is a total, fundamental sequence of mutually orthogonal projections in $B[X]$ since $\lim_{n\to\infty} \|\sigma_n(g) - g\|_X = 0$, whenever g belongs to X by virtue of Theorems 2.11 and 2.12 of [5; Chapter I]. Furthermore, we have

$$T_t(f) \sim \sum_{j=-\infty}^{\infty} e^{-ijt}P_j(f), \qquad t \in R$$

and

$$k * f \sim \sum_{j=-\infty}^{\infty} \hat{k}(j)P_j(f) .$$

Consequently, under the above setting all the results obtained in the preceding sections are applicable to homogeneous Banach subspaces X. In particular, the result corresponding to the part (ii) of Corollary 2 extends Theorem 7 of Mohapatra [8] for the real Banach space $C_{2\pi}$ to the more general homogeneous Banach subspaces of $L^1_{2\pi}$ and yields the better estimate of the degree of almost convergence.

Finally, for homogeneous Banach subspaces X of $L^1_{2\pi}$ we recast Corollary 1 in connection with the test function class $\{u_0, u_1, u_2\}$, where $u_0(t) = 1$, $u_1(t) = \cos t$ and $u_2(t) = \sin t$ for all $t \in R$.

THEOREM 5. *Let $\{k_{n,\lambda};\ n \in N,\ \lambda \in \Lambda\}$ be a family of non-negative functions in $L^1_{2\pi}$. Suppose that for $j = 1, 2$ and for each $f \in X$, $u_j f$ belongs to X and $\|u_j f\|_X \leqq \|f\|_X$, and $\|u_0\|_X = 1$. Then the following three statements are equivalent*:

(i) *For every $f \in X$,*

$$\lim_{n\to\infty} \|k_{n,\lambda} * f - f\|_X = 0 \quad \textit{uniformly in} \quad \lambda \in \Lambda ;$$

(ii) *For $j = 0, 1, 2$,*

$$\lim_{n\to\infty} \|k_{n,\lambda} * u_j - u_j\|_X = 0 \quad \textit{uniformly in} \quad \lambda \in \Lambda ; \tag{33}$$

(iii) $$\lim_{n\to\infty} \hat{k}_{n,\lambda}(0) = 1 \quad \textit{uniformly in} \ \lambda \in \Lambda$$

and

(34) $$\lim_{n\to\infty} \{\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1))\} = 0 \quad \textit{uniformly in } \lambda \in \Lambda \, .$$

PROOF. It is clear that (i) implies (ii) since u_0, u_1 and u_2 belong to X. Suppose that (ii) is valid. In view of the general formula (32) we have

$$\{\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1))\}u_0 = k_{n,\lambda} * u_0 - u_1 k_{n,\lambda} * u_1 - u_2 k_{n,\lambda} * u_2 \, ,$$

which implies

$$\hat{k}_{n,\lambda}(0) - \mathrm{Re}(\hat{k}_{n,\lambda}(1)) \leqq \| k_{n,\lambda} * u_0 - u_0 \|_X + \| k_{n,\lambda} * u_1 - u_1 \|_X + \| k_{n,\lambda} * u_2 - u_2 \|_X \, ,$$

since $\| u_1 f \|_X \leqq \| f \|_X$, $\| u_2 f \|_X \leqq \| f \|_X$ whenever f belongs to X, and $\| u_0 \|_X = 1$. Thus letting n tend to infinity in the above inequality, we have (34). For $j = 0$, (33) is equivalent to

$$\lim_{n\to\infty} \hat{k}_{n,\lambda}(0) = 1 \quad \text{uniformly in } \lambda \in \Lambda \, ,$$

and therefore (iii) holds. It follows from Corollary 1 that (iii) implies (i), and the theorem is proved.

We close with the following remark.

REMARK 10. The equivalence of (i) and (ii) in Theorem 5 extends Theorem 5 of King and Swetits [6] for sequences of positive convolution integral operators on $C_{2\pi}$ to the more general homogeneous Banach subspaces of $L^1_{2\pi}$.

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