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# DECOMPOSITION THEOREM AND LACUNARY CONVERGENCE OF RIESZ-BOCHNER MEANS OF FOURIER TRANSFORMS OF TWO VARIABLES

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**Introduction.** This paper is concerned with some inequalities related to Fourier transforms of functions of two variables. Our starting points are Fefferman's divergence theorem [4] of spherical means of Fourier transforms and Carleson-Sjölin theorem [1] on the norm convergence of the Riesz-Bochner means $s_R^\sigma(f)$.

In the previous paper [8] the author showed that the lacunary subsequence of the Riesz-Bochner means $s_R^\sigma(f)$ with positive order of a function $f$ in $L^p(\boldsymbol{R}^2)$, $4/3 \leqq p \leqq 2$, converges almost everywhere. In this note we shall apply a technique in [8] to prove Carleson-Sjölin theorem for $l^2$-valued functions. It gives a partial answer of a problem in Stein [9] and also implies lacunary convergence theorem in the previous paper.

In the last section we shall prove a decomposition theorem of Littlewood-Paley type for "weak" spherical truncations.

**1. Carleson-Sjölin theorem for $l^2$-valued functions.** For an integrable function $f$ on the two dimensional euclidean space $\boldsymbol{R}^2$ let $\hat{f}$ be the Fourier transform:

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{\boldsymbol{R}^2} f(x)e^{-i\xi x}dx\ , \quad \xi \in \boldsymbol{R}^2\ .$$

The Riesz-Bochner kernel $s_R^\sigma$ of order $\sigma \geqq 0$ is defined by $\hat{s}_R^\sigma(\xi) = (1-|\xi|^2/R^2)^\sigma$ for $|\xi| < R$ and $=0$ otherwise, and the Riesz-Bochner mean of $f$ by $s_R^\sigma(f) = s_R^\sigma * f$, the convolution of $f$ and $s_R^\sigma$.

THEOREM 1. *Let $\{R_n\}$ be a sequence of positive numbers with Hadamard's gap, i.e., $R_{n+1}/R_n > q$ $(n = 0, \pm 1, \pm 2, \cdots)$ for some $q > 1$. Let $4/3 \leqq p \leqq 4$ and $\sigma > 0$. Then*

$$(1.1) \qquad \|(\sum_n |s_{R_n}^\sigma(f_n)|^2)^{1/2}\|_p \leqq c\|(\sum_n |f_n|^2)^{1/2}\|_p$$

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*for* $\{f_n\} \in L^p \cap L^1(\boldsymbol{R}^2; l^2)$, *where* $\|\cdot\|_p$ *denotes the* $L^p$*-norm and* $c$ *a constant depending only on* $q$, $p$ *and* $\sigma$.

In the following we fix $q > 1$ and $\sigma > 0$ and denote by $c$ a positive constant depending only on $q$ and $\sigma$ which will be different in each occasion. In Theorem 1 the case $p = 4$ is most essential and other cases will follow from this case by duality and interpolation arguments. Our proof procceeds along the line of the previous paper [8] but for convenience' sake we give a complete proof.

Let $\phi$ be a $C^\infty$-function on $(-\infty, \infty)$ such that the support of $\phi \subset (1, 3)$ and $1 = \sum_{n=-\infty}^{\infty} \phi(2^{-n}\rho)$ for $\rho > 0$. For $\delta > 0$ put $\phi_{R,\delta}(\xi) = \phi(R^{-1}\delta^{-1}(R - |\xi|))$, $\xi \in \boldsymbol{R}^2$ and $s_R(f) = s^\sigma_{R,\delta}(f) = s^\sigma_R * \hat{\phi}_{R,\delta} * f$. Then (1.1) follows from the inequality

$$\left\|\left(\sum_n |s_{R_n}(f_n)|^2\right)^{1/2}\right\|_4 \leqq c\delta^\varepsilon \left\|\left(\sum_n |f_n|^2\right)^{1/2}\right\|_4 \tag{1.2}$$

where $\varepsilon$ is a positive constant depending only on $\sigma$ and $q$.

Put

$$I_{m,n} = \int |s_{R_m}(f_m) s_{R_n}(f_n)|^2 dx .$$

THEOREM (1.3) (Fefferman [4], cf. Córdoba [2]). *We have*

$$I_{n,n} \leqq c\delta^\varepsilon \int |f_n|^4 dx .$$

For a locally integrable function $g$ in $\boldsymbol{R}^2$ let $g^*$ be the Hardy-Littlewood maximal function, i.e.,

$$g^*(x) = \sup_{r>0} \frac{1}{\pi r^2} \int_{|x-y|<r} |g(y)|\, dy .$$

LEMMA (1.4.) *There exist* $\varepsilon > 0$ *and* $2 > \gamma > 1$ *such that*

$$I_{m,n} \leqq c\delta^\varepsilon \int (|f_m|^{2/\gamma})^{*\gamma} |s_{R_n}(f_n)|^2 dx \tag{1.5}$$

*for all* $\{f_n\} \in L^p \cap L^1(\boldsymbol{R}^2)$ *and for* $m$, $n$ *satisfying* $R_n/R_m < \delta^2$.

PROOF. Let $\{\psi^j;\ j = 0, 1, \cdots, [2\pi\delta^{-1}] - 1\}$ be a partition of unity on the unit circle such that $\psi^j(\omega) = \psi(\delta^{-1}(\omega - j\delta))$, $0 \leqq j < [2\pi\delta^{-1}] - 1$ where $\psi$ is a $C^\infty$-function on $(-\infty, \infty)$ with support contained in $(-3/4, 3/4)$. Define $s^j_R = s^{\sigma,j}_{R,\delta}$ by $\hat{s}^j_R(\xi) = \hat{s}_R(\xi)\psi^j(\omega)$, where $\xi = |\xi|(\cos\omega, \sin\omega)$. Then $s^j_{R_m}(f_m) = s^j_{R_m} * f_m$ satisfies

$$s_{R_m}(f_m) = \sum_j s^j_{R_m}(f_m) .$$

Since $\hat{s}^j_{R_m}(f_m) * \hat{s}_{R_n}(f_n) \cdot \hat{s}^k_{R_m}(f_m) * \hat{s}_{R_n}(f_n) \equiv 0$ if $|j - k| > 1$, we have

$$I_{m,n} \leqq 3 \sum_j \int |s^j_{R_m}(f_m) s_{R_n}(f_n)|^2 dx .$$

Define $\eta^j = \eta^j_{R_m,\delta}$ and $\eta = \eta_{R_m,\delta}$ by

$$\hat{\eta}^0(\xi_1, \xi_2) = \psi([(\xi_1 - R_m)^2 + \xi_2^2]/100\delta^2 R_m^2) ,$$

$\eta^j(\xi) = \eta^0(M_j \xi)$, where $M_j$ is the rotation of angle $\delta j$, and

$$\hat{\eta}(\xi) = \psi(|\xi|/100R_m) .$$

Then

$$s^j_{R_m}(f_m) = s^j_{R_m} * \eta^j * \eta * f_m .$$

Suppose that the support of $f_m$ is contained in the square $Q$ of side length $R_m^{-1}\delta^{-\gamma}$ with center at $O$, where $\gamma > 1$ is a number close to 1 but determined later. By Schwarz's inequality

$$|s^j_{R_m}(f_m)(x)|^2 \leqq \int |s^j_{R_m}|^2 dx \int |\eta^j * \eta * f_m|^2 dx .$$

Since $\int |s^j_{R_m}|^2 dx \leqq cR_m^2 \delta^{2\sigma+2}$,

$$\sum_j |s^j_{R_m}(f_m)(x)|^2 \leqq cR_m^2 \delta^{2\sigma+2} \sum_j \int |\eta^j * \eta * f_m|^2 dx . \tag{1.6}$$

By the Parseval relation $\sum_j \int |\eta^j * \eta * f_m|^2 dx \leqq c \int |\eta * f_m|^2 dx$. Furthermore by Young's inequality $\|\eta * f_m\|_2 \leqq \|\eta\|_{2/(3-\gamma)} \|f_m\|_{2/\gamma} \leqq cR_m^{\gamma-1} \|f_m\|_{2/\gamma}$. Thus the right hand side of (1.6) is bounded by $c\delta^\varepsilon \left((1/|Q|) \int_Q |f_m|^{2/\gamma} dx\right)^\gamma$, where $\varepsilon = 2(\sigma + 1 - \gamma^2)$ which is positive for $\gamma$ close to 1. Thus

$$\sum_j |s^j_{R_m}(f_m)(x)|^2 \leqq c\delta^\varepsilon (|f_m|^{2/\gamma})^{*\gamma}(x) \tag{1.7}$$

for $x \in 3Q$.

Next remark that for every $M \geqq 0$ there exists a constant $c$ such that

$$|s^j_{R_m}(x)| \leqq cR_m^2 \delta^{\sigma+2} (R_m \delta |x|)^{-M} \tag{1.8}$$

for $x \neq O$. Thus

$$|s^j_{R_m}(f_m)(x)| \leqq c\delta^\sigma (R_m \delta |x|)^{-M+2} (|f_m|^{2/\gamma})^{*\gamma/2}(x) \tag{1.9}$$

for $x \notin 3Q$.

To get an estimate for a general function $f_m$ divide $\boldsymbol{R}^2$ into non-overlapping squares $\{Q(\alpha)\}$ similar to $Q$ and with center at $R_m^{-1}\delta^{-\gamma}\alpha$ where $\alpha = (\alpha_1, \alpha_2)$ is a lattice point. Then

$$(1.0)\qquad \sum_j |s^j_{R_m}(f_m)(x)|^2 \leqq c\sum_j \sum_{|\alpha|\leqq 1} |s^j_{R_m}(f_m\chi_\alpha)(x)|^2 + c\sum_j \Big|\sum_{|\alpha|>1} s^j_{R_m}(f_m\chi_\alpha)(x)\Big|^2$$

where $\chi_\alpha$ is the characteristic function of $Q(\alpha)$. Suppose $x \in Q = Q(0)$. We apply (1.7) for the first term on the right hand side of (1.10) and (1.9) for the second term. Then we get

$$(1.11)\qquad \sum_j |s^j_{R_m}(f_m)(x)|^2$$
$$\leqq c\delta^\varepsilon(|f_m|^{2/\gamma})^{*\gamma}(x) + c\delta^{-1}[\delta^\sigma \sum_{|\alpha|>1} (\delta^{1-\gamma}|\alpha|)^{-M+2}(|f_m|^{2/\gamma})^{*\gamma/2}(x)]^2$$
$$\leqq c\delta^\varepsilon(|f_m|^{2/\gamma})^{*\gamma}(x)\,,$$

if $M$ is sufficiently large. Thus we get (1.11) for all $x$ in $\boldsymbol{R}^2$. Thus we get (1.5).

We shall use the following:

THEOREM (1.12) (Fefferman and Stein [4]). *Let $1 < r, p < \infty$ and $\{f_m\}$ be a sequence of $L^p(\boldsymbol{R}^d)$. Then*

$$\|(\sum_m f_m^{*r})^{1/r}\|_p \leqq c_{p,r}\|(\sum_m |f_m|^r)^{1/r}\|_p\,,$$

*where $c_{p,r}$ is a constant depending only on $p$ and $r$.*

PROOF OF THEOREM 1. Let $\sum^1_{m,n}$ be summation over $(m, n)$ such that $\delta^2 < R_m/R_n < \delta^{-2}$ and $\sum^2_{m,n}$ summation over $(m, n)$ such that $R_m/R_n > \delta^{-2}$ or $R_m/R_n < \delta^2$.

By Schwarz's inequality $I_{m,n} \leqq I^{1/2}_{m,m}I^{1/2}_{n,n}$. Thus $\sum^1_{m,n} I_{m,n} \leqq \sum^1_{m,n} I_{m,m}$. For every $m$ the number of $n$'s satisfying $\delta^2 < R_m/R_n < \delta^{-2}$ is less than $4\log\delta^{-1}/\log q$. Thus, by Theorem (1.3)

$$(1.13)\qquad \sum^1_{m,n} I_{m,n} \leqq c\log\delta^{-1}\sum_m I_{m,m} \leqq c\delta^\varepsilon\log\delta^{-1}\sum_m\int|f_m|^4dx\,.$$

By Lemma (1.4)

$$\sum^2_{m,n} I_{m,n} \leqq c\delta^\varepsilon \sum_{m,n}\int(|f_m|^{2/\gamma})^{*\gamma}|s_{R_n}(f_n)|^2dx\,.$$

Put $S = \sum_{m,n} I_{m,n} = \|(\sum_m |s_{R_m}(f_m)|^2)^{1/2}\|_4^4$. Then by Schwarz's inequality and Theorem (1.12)

$$\sum^2_{m,n} I_{m,n} \leqq c\delta^\varepsilon\Big[\int(\sum_m (|f_m|^{2/\gamma})^{*\gamma})^2dx\Big]^{1/2} S^{1/2}$$
$$\leqq c\delta^\varepsilon\Big[\int(\sum_m |f_m|^2)^2dx\Big]^{1/2} S^{1/2}\,.$$

Combining the last inequality with (1.13) we get

$$S \leqq c\delta^{\varepsilon} \| (\sum_m |f_m|^2)^{1/2} \|_4^2 S^{1/2} + c\delta^{\varepsilon} \log \delta^{-1} \| (\sum_m |f_m|^2)^{1/2} \|_4^4 ,$$

which proves (1.2) with norm $c\delta^{\varepsilon/8}$.

2. **Lacunary convergence of $s_R^\sigma(f)$.** In this section we shall prove the almost everywhere convergence of $s_{R_m}^\sigma(f)$ where $f \in L^p(\boldsymbol{R}^2)$, $4/3 \leqq p \leqq 4$, $\sigma > 0$ and $\{R_m\}$ is a lacunary sequence with Hadamard's gap $q > 1$.

Put $\phi_m(\xi) = \phi(2R_m^{-1}|\xi|)$. Remark that $\phi_m(\xi) = 1$ if $3R_m/4 \leqq |\xi| \leqq R_m$.

By the lacunarity of $\{R_m\}$ we have (cf., e.g., [5; p. 120]):

LEMMA (2.1). *Let $1 < p < \infty$. Then*

$$\| (\sum_m |f * \hat{\phi}_m|^2)^{1/2} \|_p \leqq c_p \| f \|_p$$

*for $f \in L^p(\boldsymbol{R}^2)$, where $c_p$ is a constant not depending on $f$.*

Let $\psi_m(\xi) = 1 - \phi_m(\xi)$ for $|\xi| \leqq R_m$ and $=0$ otherwise.

LEMMA (2.2). *Suppose $\sigma \geqq 0$. Then*

$$\sup_n | s_{R_n}^\sigma * \hat{\psi}_n * f(x) | \leqq c f^*(x)$$

*for $f \in L^1(\boldsymbol{R}^2)$.*

PROOF. Since $\hat{s}_{R_n}^\sigma(\xi)\psi_n(\xi) = \eta(R_n^{-1}\xi)$ for some $C^\infty$-function $\eta$ with compact support, Lemma (2.2) follows from a routine work.

THEOREM 2. *Let $\{R_n\}$ be a sequence of positive numbers with Hadamard's gap $q > 1$. Let $4/3 \leqq p \leqq 4$ and $\sigma > 0$. Then $s_{R_n}^\sigma(f)$ converges almost everywhere to $f$ for all $f \in L^p(\boldsymbol{R}^2)$.*

PROOF. Suppose $f \in L^p$. Since $s_{R_n}^\sigma = s_{R_n}^\sigma * \hat{\psi}_n + s_{R_n}^\sigma * \hat{\phi}_n$,

$$\sup_n |s_{R_n}^\sigma(f)(x)| \leqq c f^*(x) + \sup_n |s_{R_n}^\sigma(\hat{\phi}_n * f)| .$$

By Lemma (2.1) and Theorem 1

$$\| (\sum | s_{R_n}^\sigma(\hat{\phi}_n * f) |^2)^{1/2} \|_p \leqq c \| f \|_p .$$

Thus

$$\| \sup_n | s_{R_n}^\sigma(\hat{\phi}_n * f) | \|_p \leqq c \| f \|_p .$$

Thus by the Hardy-Littlewood maximal theorem

$$\| \sup_n | s_{R_n}^\sigma(f) | \|_p \leqq c \| f \|_p$$

from which our theorem follows.

**3. Decomposition theorem.** Define $k_n$ by the Fourier transform $\hat{k}_n(\xi) = \phi(2^{-n}|\xi|)$. Let $1 < p < \infty$ and $f \in L^p(\boldsymbol{R}^2)$. Then we have

$$\|f\|_p \approx \left\|\left(\sum_{n=-\infty}^{\infty} |k_n * f|^2\right)^{1/2}\right\|_p, \tag{3.1}$$

that is, two norms of $f$ are equivalent (cf., e.g., [5; p. 120]).

Remark that $\hat{k}_n \in C^\infty$, the support of $\hat{k}_n \subset \{2^n \leqq |\xi| \leqq 3\cdot 2^n\}$ and $\hat{k}_n = 1$ on $\{3\cdot 2^{n-1} \leqq |\xi| \leqq 2^{n+1}\}$, and that (3.1) is valid with $\hat{k}_n$ replaced by the characteristic function of $\{\xi;\ 2^n \leqq |\xi| < 2^{n+1}\}$ if and only if $p=2$ (Fefferman [4]).

Let $\sigma > 0$ and $1 > \tau > 0$. Define $D_n$ ($n = 0, \pm 1, \pm 2, \cdots$) as follows. Put $\hat{D}_0(\xi) = 1$ for $|\xi| < 2$, $[(2 + \tau - |\xi|)/\tau]^\sigma$ for $2 \leqq |\xi| < 2 + \tau$ and $0$ for $2 + \tau \leqq |\xi|$, and $\hat{D}_n(\xi) = \hat{D}_0(2^{-n}\xi)$. Let $\Delta_n = D_n - D_{n-1}$.

THEOREM 3. *Let* $4/3 \leqq p \leqq 4$. *Then*

$$\|f\|_p \approx \left\|\left(\sum_{n=-\infty}^{\infty} |\Delta_n * f|^2\right)^{1/2}\right\|_p \tag{3.2}$$

*for* $f \in L^p(\boldsymbol{R}^2)$.

PROOF. Suppose $\{f_n\} \in L^4(l^2)$. First we remark that (1.1) holds if $\{s^\sigma_{R_n}(f_n)\}$ is replaced by $\{D_n * f_n\}$. In fact, (1.3) with $\{D_n * f_n\}$ in place of $\{s_{R_n}(f_n)\}$ is valid by an elementary reduction process. On the other hand we have an estimate similar to (1.8) with $D^j_m$ and $2^m$ in place of $s^j_{R_m}$ and $R_m$ respectively where a definition of $D^j_m$ will be obvious. Thus we get Lemma (1.4) for the kernels $D_n$.

Thus we have

$$\left\|\left(\sum_{n=-\infty}^{\infty} |D_n * f_n|^2\right)^{1/2}\right\|_4 \leqq c\left\|\left(\sum_{n=-\infty}^{\infty} |f_n|^2\right)^{1/2}\right\|_4. \tag{3.3}$$

Thus

$$\|(\sum |\Delta_n * f_n|^2)^{1/2}\|_4 \leqq c\|(\sum |f_n|^2)^{1/2}\|_4.$$

By the duality and interpolation arguments we have

$$\|(\sum |\Delta_n * f_n|^2)^{1/2}\|_p \leqq c\|(\sum |f_n|^2)^{1/2}\|_p \tag{3.4}$$

for $\{f_n\} \in L^p(l^2)$, $4/3 \leqq p \leqq 4$.

Let $f \in L^p(\boldsymbol{R}^2)$. Then $\{(k_{n-1} + k_n + k_{n+1}) * f\} \in L^p(l^2)$ and by (3.1) and (3.4) we have

$$\|(\sum |\Delta_n * f|^2)^{1/2}\|_p \leqq c\|f\|_p. \tag{3.5}$$

On the other hand, since

$$\int fg\,dx = \sum_n \int (\Delta_n * f)(k_{n-1} + k_n + k_{n+1}) * g\,dx$$

for smooth functions $f$ and $g$ with compact support, we have an opposite inequality.

REMARK. The author discussed with H. Dappa to organize this note. His result ([3]) on radial multipliers will be related to Theorem 3.

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