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THE HAUSDORFF DIMENSION OF LIMIT SETS OF SOME FUCHSIAN GROUPS

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1. Preliminaries. Let Γ and Λ be a non-elementary finitely generated Fuchsian group of the second kind and its limit set, respectively. Put $M_t(\delta, \Lambda) = \inf \sum_i |I_i|^t$, where the infimum is taken over all coverings of Λ by sequences $\{I_i\}$ of sets I_i with the spherical diameter $|I_i|$ less than a given number $\delta > 0$. Further, put $M_t(\Lambda) = \sup M_t(\delta, \Lambda)$, which is called the t-dimensional Hausdorff measure of Λ. It is shown in [2] that if $\infty \notin \Lambda$, $M_t(\Lambda) = \sup_\delta \inf \sum_i \mathrm{dia}^t (I_i)$, where the infimum is taken over all coverings of Λ by sequences $\{I_i\}$ of sets I_i with the Euclidean diameter $\mathrm{dia}(I_i)$. We call $d(\Lambda) = \inf\{t > 0;\ M_t(\Lambda) = 0\}$ the Hausdorff dimension of Λ. In [3] Beardon proved that $d(\Lambda) < 1$ for the limit set $\Lambda(\not\ni \infty)$ of any finitely generated Fuchsian group of the second kind.

The purpose of this note is to show the continuity of $d(\Lambda)$ with respect to quasiconformal deformations of Γ.

Let w be a K-quasiconformal mapping of the unit disc D onto itself and $w(0) = 0$. The following distortion theorem is due to Mori [5].

PROPOSITION 1. *Let w be a K-quasiconformal mapping of D onto itself and $w(0) = 0$. Then for every pair of points z_1, z_2 with $|z_1| \leqq 1$, $|z_2| \leqq 1$,*

$$|w(z_1) - w(z_2)| < 16|z_1 - z_2|^{1/K}, \quad (z_1 \neq z_2).$$

Let Γ be a finitely generated Fuchsian group acting on D. We say that Γ has a type $(g; n; m)$ if $S = D/\Gamma$ is obtained from a compact surface of genus g by removing j ($\geqq 0$) points, m ($\geqq 0$) conformal discs and if there are finitely many, say k ($\geqq 0$), ramification points on S, where $n = j + k$. Suppose that to each ramification point a_i $(i = 1, 2, \cdots, k)$ on S, there is assigned an integer ν_i, $1 < \nu_1 \leqq \nu_2 \leqq \cdots \leqq \nu_k < +\infty$. Then we say that Γ has the signature $(g; \nu_1, \nu_2, \cdots, \nu_k, \nu_{k+1}, \cdots, \nu_n; m)$, where $\nu_{k+1} = \cdots = \nu_{n-1} = \nu_n = \infty$. We call an isomorphism χ of a Fuchsian group Γ_0 onto Γ_1 quasiconformal if there exists a quasiconformal mapping w which maps D onto itself and $w(0) = 0$ such that $\chi(A) = wAw^{-1}$ for all $A \in \Gamma_0$. The following proposition was proved by Bers [4].

PROPOSITION 2. *Assume that Γ_0, Γ_1 have the same signature $(g; \nu_1, \nu_2, \cdots, \nu_n; m)$. Then Γ_0 is quasiconformally isomorphic to Γ_1.*

2. Statement of the theorem. Let $B(D)$ denote the set of all bounded measurable functions $\mu(z)$ $(|z| < \infty)$ with $\operatorname{ess\,sup}_{|z|<\infty} |\mu(z)| < 1$, which satisfy the condition $\overline{\mu(z)} = \mu(1/\bar{z})\bar{z}^2/z^2$. The Beltrami equation $f_{\bar{z}} = \mu f_z$ has one and only one normalized solution $w^\mu(z)$ with $w^\mu(0) = 0$, $w^\mu(1) = 1$ which maps D quasiconformally onto itself. Set $B(D, \Gamma) = \{\mu \in B(D) \mid \mu(A)\bar{A}'/A' = \mu(z)$ for all $A \in \Gamma\}$. Let Γ_0, Γ_1 be finitely generated Fuchsian groups of the second kind with the same signature. By Proposition 2, Γ_0 is quasiconformally isomorphic to Γ_1. For any real number s $(0 \leqq s \leqq 1)$, $s\mu \in B(D, \Gamma_0)$ if $\mu \in B(D, \Gamma_0)$. Hence $\Gamma_s = w^{s\mu}\Gamma_0(w^{s\mu})^{-1}$ is also a Fuchsian group leaving the unit disc D.

Now we shall prove the following theorem.

THEOREM 1. *Let Γ_0, Γ_1 be finitely generated Fuchsian groups of the second kind with the same signature. Let Γ_s be a Fuchsian group constructed above for any real number s $(0 \leqq s \leqq 1)$ and let Λ_s be the limit set of Γ_s. Then $d(\Lambda_s)$ is continuous in s $(0 \leqq s \leqq 1)$.*

Before going into the proof of Theorem 1, we shall show the following lemma.

LEMMA 1. *Let F be a compact set in $\{|z| \leqq 1\}$. Then*

$$K^{-1}d(F) \leqq d(w(F)) \leqq Kd(F) ,$$

where $w(z)$ is a K-quasiconformal mapping of the unit disc onto itself and $w(0) = 0$.

PROOF OF LEMMA 1. First, we shall prove the second inequality $d(w(F)) \leqq Kd(F)$. Assume that $Kd(F) < d(w(F))$ for some K-quasiconformal mapping w of the unit disc onto itslf with $w(0) = 0$. Take and fix $t > 0$ such that $Kd(F) < Kt < d(w(F))$. Then $M_t(F) = 0$. By the definition of the Hausdorff measure, for any $\varepsilon > 0$, there are a positive number δ and a covering $\{I_i\}$ of F with $\operatorname{dia}(I_i) < \delta$ such that $\sum_i \operatorname{dia}^t(I_i) < \varepsilon$. Let d_i be the diameter of $w(I_i \cap D)$. Then we have $d_i \leqq 16(\operatorname{dia}(I_i))^{1/K}$ by Proposition 1. We take a disc I_i' with radius d_i centered at some point $w_i \in w(I_i \cap F)$ $(i = 1, 2, \cdots)$. It is easily seen that $\{I_i'\}$ is a covering of $w(F)$. It is well known that $d(E) \leqq 2$ for any compact set E of $\hat{C} = C \cup \{\infty\}$. Therefore we have

$$\sum_i \operatorname{dia}^{Kt}(I_i') \leqq 32^{Kt} \sum_i \operatorname{dia}^t(I_i) < 32^2 \cdot \varepsilon .$$

As ε is arbitrary, we obtain $M_{Kt}(w(F)) = 0$. This contradicts the assumption $Kt < d(w(F))$. The first inequality is given similarly by considering the inverse K-quasiconformal mapping w^{-1}. Therefore we have our lemma.

PROOF OF THEOREM 1. By Proposition 2, Γ_0 is quasiconformally isomorphic to Γ_1, that is, there is a quasiconformal mapping w^μ such that $\Gamma_1 = w^\mu \Gamma_0 (w^\mu)^{-1}$. Denote by $w^{s\mu}$, $w^{t\mu}$ the normalized quasiconformal mappings for $s\mu, t\mu \in B(D, \Gamma_0)$, $0 \leqq s, t \leqq 1$, respectively. Set $w^{s\mu} = w^\eta \circ w^{t\mu}$. Then we have $\eta \circ w^{t\mu} = (s - t)\cdot\mu\cdot(1 - st|\mu|^2)^{-1}\cdot w_z^{t\mu}\cdot(\overline{w_z^{t\mu}})^{-1}$ (see [1, p. 9]). Set $K = \operatorname{ess\,sup}(1 + |\eta|)/(1 - |\eta|)$. Then w^η is a K-quasiconformal mapping such that $w^\eta(\Lambda_t) = \Lambda_s$ and $w^\eta(0) = 0$. We have from Lemma 1

$$|\log d(\Lambda_s) - \log d(\Lambda_t)| \leqq \operatorname{ess\,sup}[s\mu, t\mu] ,$$

where $[a, b]$ denotes the non-Euclidean distance between two points a and b in D measured by the metric $ds = 2|dw|(1 - |w|^2)^{-1}$ in D. Thus we have Theorem 1.

3. Application. Let G_α be the Hecke group generated by $P_\alpha: z \mapsto z + 2(1 + \alpha)$ and $E: z \mapsto -z^{-1}$ $(0 \leqq \alpha < \infty)$. Then G_α is a Fuchsian group of the second kind except when $\alpha = 0$. Let Λ_α be the limit set of G_α. The following inequality was proved by Beardon [3]:

$$d(\Lambda_\alpha) \geqq 1 - 8(3\alpha + 18\alpha^{1/2}) \tag{1}$$

for a sufficiently small number $\alpha > 0$. On the other hand, there is a positive number α_0 depending only on any given small number ε such that

$$1/2 < d(\Lambda_\alpha) < 1/2 + \varepsilon , \quad (\alpha \geqq \alpha_0) , \tag{2}$$

(see [3], [6]).

Now we shall prove the following.

THEOREM 2. *Assume that* $1/2 < s < 1$. *Then there is a Hecke group* G_α *with* $d(\Lambda_\alpha) = s$.

PROOF. Take and fix a number s $(1/2 < s < 1)$. Then there is an arbitrarily small number $\varepsilon > 0$ such that $1/2 + \varepsilon \leqq s \leqq 1 - \varepsilon$. Let ε be fixed. Then we can find Hecke groups with real parameters p and q such that $d(\Lambda_p) > 1 - \varepsilon$ for $0 < p \leqq p_1$ and $1/2 < d(\Lambda_q) < 1/2 + \varepsilon$ for $0 < q_1 \leqq q$ by (1) and (2). The mapping $T(z) = (z - \sqrt{-1})(z + \sqrt{-1})^{-1}$ sends the upper half-plane H onto the unit disc D. Let $G'_\alpha = TG_\alpha T^{-1}$. Then G'_α is a Fuchsian group of the second kind generated by

$$P'_\alpha = \begin{pmatrix} 1 + (1 + \alpha)\sqrt{-1} & -(1 + \alpha)\sqrt{-1} \\ (1 + \alpha)\sqrt{-1} & 1 - (1 + \alpha)\sqrt{-1} \end{pmatrix} \quad \text{and} \quad E' = \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix} .$$

Denote by $I_\alpha = \{z;\ |z-(1+(1+\alpha)^{-1}\sqrt{-1})| = (1+\alpha)^{-1}\}$ and $I_\alpha^{-1} = \{z;\ |z-(1-(1+\alpha)^{-1}\sqrt{-1}| = (1+\alpha)^{-1}\}$ the isometric circles of P'_α and $(P'_\alpha)^{-1}$, respectively. Let R'_α be the fundamental region of G'_α whose boundary consists of I_α, I_α^{-1}, the imaginary axis and two arcs lying on $\{|z|=1\}$. By Proposition 2, there is a quasiconformal mapping W^μ such that $W^\mu(0)=0$, $W^\mu(1)=1$ and $W^\mu G'_{q_1}(W^\mu)^{-1} = G'_{p_1}$. By Theorem 1, there is a Fuchsian group $G_t^* = W^{t\mu}G'_{q_1}(W^{t\mu})^{-1}$ with the property $d(\Lambda(G_t^*)) = s$. It is easily shown that G_t^* is freely generated by

$$P_t^* = \begin{pmatrix} 1+\lambda & -\lambda \\ \lambda & 1-\lambda \end{pmatrix} \quad \text{and} \quad E' = \begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & \sqrt{-1} \end{pmatrix}.$$

It is easy to verify that if $|\lambda| \leqq 1$, G_t^* is a Fuchsian group of the first kind. As G_t^* is a Fuchsian group of the second kind acting on the unit disc D, we have $1+\lambda = \overline{1-\lambda}$. Thus λ is pure imaginary and further $|\lambda| > 1$. Replacing $|\lambda|$ by $(1+\alpha)$ $(\alpha > 0)$, we have the Hecke group $T^{-1}G_t^*T = G_\alpha$ with $d(\Lambda_\alpha) = s$. Thus we have the desired result.

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