Tôhoku Math. Journ.
33 (1981), 289-295.

GEOMETRIC QUANTIZATION FOR THE MECHANICS ON SPHERES

KIYOTAKA II

(Received April 20, 1979, revised October 30, 1980)

1. Introduction. The classical Hamiltonian (kinetic energy) H for the mechanical system consisting of a non-relativistic free particle of unit mass moving on a Riemannian manifold is given by one-half the Riemannian metric. Several authors say that the quantum Hamiltonian $\hat{H}$ corresponding to H is $2^{-1}\hbar^{1}\Delta$, where Δ is the Laplacian acting on functions and $\hbar = (2\pi)^{-1}h$, h = Planck's constant. See Blattner [4] and Simms and Woodhouse [20]. However, DeWitt [7] and Cheng [5] used the method of "Feynman's path integrals" to derive a different operator for the quantum Hamiltonian. See also Ben-Abraham and Lonke [1]. Elhadad [9] applied the method of "Maslov pairing" to the geodesic flow on the unit n-sphere S^n to obtain a quantum Hamiltonian. Also, Weinstein [21] has shown, in the case of S^n, that the N-th quasi-classical eigenvalue for H is $\lambda_N = 2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$, which is not equal to the N-th eigenvalue $\mu_N = 2^{-1}\hbar^2 N(N+n-1)$ of the operator $2^{-1}\hbar^2\Delta$ on S^n. Note that the multiplicity of λ_N is equal to that of μ_N. See Ii [11]. These results show that $2^{-1}\hbar^2\Delta$ may not necessarily be the quantum Hamiltonian corresponding to H. The correct quantum Hamiltonian $\hat{H}$ has to be determined from appropriate general principles. In the present paper, we apply the quantization procedure of Kostant to quantize the mechanical system consisting of a non-relativistic, positive energy free particle of unit mass moving on the unit n-sphere S^n ($n \geqq 3$). We construct a polarization which enables us to quantize H. The resulting quantum Hamiltonian $\hat{H}$ has λ_N as the N-th eigenvalue. Moreover using the same polarization, we define an operator $\tilde{L}^2$ which has $2\mu_N$ as the N-th eigenvalue and has the same eigenspaces as that of $\hat{H}$. From the construction $\tilde{L}^2$ may be identified with $\hbar^2\Delta$. Under this identification, the correct quantum Hamiltonian $\hat{H}$ on S^n is given by $2^{-1}\hbar^2(\Delta + 4^{-1}(n-1)^2)$. The referee pointed out that the idea of adding $4^{-1}(n-1)^2$ to the Laplacian was also found by Y. Akyildiz in

Partly supported by the Grant-in-Aid for Encouragement of Young Scientists, the Ministry of Education, Science and Culture, Japan.

his Berkeley thesis in connection with the representation of $SO(n+1, 2)$ on $L^2(S^n)$.

2. **Preliminaries.** Let $\boldsymbol{R}^{n+1}$ and $T^*\boldsymbol{R}^{n+1}$ be the $(n+1)$-space and its cotangent bundle with coordinates $x = (x_1, \cdots, x_{n+1})$ and $(x, y) = (x_1, \cdots, x_{n+1}, y_1, \cdots, y_{n+1})$, respectively. Let $|x|^2 = \sum x_j^2$, $|y|^2 = \sum y_j^2$ and $x \cdot y = \sum x_j y_j$, so that the cotangent bundle on the unit n-sphere S^n is represented by $T^*S^n = \{(x, y) \in T^*\boldsymbol{R}^{n+1} \mid |x| = 1, x \cdot y = 0\}$. The Hamiltonian of a free particle of unit mass on S^n is given by $H(x, y) = 2^{-1}|y|^2$. The phase space of the positive energy free particle of unit mass on S^n is given by $M = T^*S^n - \{0\text{-section}\} = \{(x, y) \in T^*S^n \mid |y| > 0\}$ with the action form $\omega = \sum y_j dx_j$ and the symplectic form $\Omega = -d\omega = \sum dx_j \wedge dy_j$. Let $C^\infty(M; \boldsymbol{R})$ be the space of real-valued smooth functions on M. Since Ω is real and non-degenerate, we can define for each $f \in C^\infty(M; \boldsymbol{R})$ a real vector field X_f on M by $X_f \rfloor \Omega = df$. X_f is called the Hamiltonian vector field generated by f. Let us define L_{jk} and $L^2 \in C^\infty(M; \boldsymbol{R})$ by $L_{jk}(x, y) = x_j y_k - x_k y_j$ $(1 \leqq j, k \leqq n+1)$ and $L^2 = \sum_{j<k} L_{jk}^2$, which are called angular momenta and square of angular momenta, respectively. Note that $H = 2^{-1}L^2$. Let us denote $X_j = \partial/\partial x_j$, $Y_j = \partial/\partial y_j$, $X = (X_1, \cdots, X_{n+1})$ and $Y = (Y_1, \cdots, Y_{n+1})$. For the sake of simplicity, we write $u \cdot X$ instead of $\sum u_j X_j$ for $u = (u_1, \cdots, u_{n+1})$. Then we have $X_H = \sum (y_j X_j - |y|^2 x_j Y_j) = y \cdot X - |y|^2 x \cdot Y$ and $X_{jk} \equiv X_{L_{jk}} = \sum_{i=1}^{n+1} \{(-\delta_{ij} x_k + \delta_{ik} x_j) X_i + (-\delta_{ij} y_k + \delta_{ik} y_j) Y_i$, which are complete vector fields on M. X_H is the geodesic flow vector field on M. See Moser [13].

3. **Quantum bundle $\boldsymbol{L}$.** Since the symplectic form Ω is exact, and M is simply connected $(n \geqq 3)$, there exists, up to isomorphism, unique quantum bundle $\boldsymbol{L}$ over M. For quantum bundles, see Kostant [12]. $\boldsymbol{L}$ is a trivial bundle; $\boldsymbol{L} = (M \times \boldsymbol{C}, p, M)$. Let $\Gamma(\boldsymbol{L})$ denote the space of smooth cross-sections of $\boldsymbol{L}$. Then $\Gamma(\boldsymbol{L})$ is identified with the space $C^\infty(M)$ of complex-valued smooth functions on M in a natural manner. The connection form θ on $\boldsymbol{L}$ is given by $\theta|_{(x,y,z)} = -\hbar^{-1} p^*\omega + i^{-1} z^{-1} dz$ $(z \in \boldsymbol{C} - \{0\})$, where $i = \sqrt{-1}$. The covariant differentiation ∇ corresponding to θ is given by $\nabla_Z \varphi = Z\varphi - i\hbar^{-1}\omega(Z)\varphi$ for any tangent vector Z to M and for any $\varphi \in C^\infty(M)$.

4. **Polarization P.** In this section, we construct a polarization of our symplectic manifold (M, Ω), which is invariant under the flows of X_H and X_{jk}. By means of this polarization, we quantize the "classical observables" H and L_{jk}. See Elhadad [8], Gawedzki [10], Kostant [12], Onofri [14], [15], [16], Simms [18], [19] and Simms and Woodhouse [20].

Let F_j $(1 \leqq j \leqq n+1)$ be vector fields on $T^*\boldsymbol{R}^{n+1}$ defined by

$F_j|_{(x,y)} = X_j - i|y|Y_j$. For any $(x, y) \in M$ and for any $v = (v_1, \cdots, v_{n+1}) \in \boldsymbol{R}^{n+1}$, such that $v \cdot x = v \cdot y = 0$, the vector $v \cdot F|_{(x,y)} = \sum v_j F_j|_{(x,y)}$ is tangent to M. Let $P_{(x,y)}$ be the complexified tangent space spanned by the vectors $X_H|_{(x,y)}$ and $\{v \cdot F|_{(x,y)} \mid v \in \boldsymbol{R}^{n+1}, v \cdot x = v \cdot y = 0\}$. Then $P: (x, y) \mapsto P_{(x,y)}$ defines a distribution on M, and we have:

LEMMA 1. *P is a polarization of (M, Ω), which is invariant under the flows of X_H and X_{jk}.*

PROOF. It is easy to see that P is an n-dimensional smooth, complex distribution on M. $\Omega(P, P) = 0$ is straightforward. Let $v = (v_1, \cdots, v_{n+1})$ and $w = (w_1, \cdots, w_{n+1})$ be $\boldsymbol{R}^{n+1}$-valued smooth functions on M such that $v \cdot x = v \cdot y = w \cdot x = w \cdot y = 0$. Then we have $[X_H, v \cdot F] = \sum (X_H v_j + i|y|v_j) F_j$, $[v \cdot F, w \cdot F] = \sum (a_j - i|y|b_j) F_j$, $[X_{jk}, X_H] = 0$ and $[X_{jk}, v \cdot F] = \sum c_i F_i$, where $a_j = (v \cdot X) w_j - (w \cdot X) v_j$, $b_j = (v \cdot Y) w_j - (w \cdot Y) v_j$ and $c_i = X_{jk} v_i + \delta_{ij} v_k - \delta_{ik} v_j$. a_j, b_j and c_i are real-valued and satisfy $\sum (X_H v_j) x_j = \sum (X_H v_j) y_j = a \cdot x = a \cdot y = b \cdot x = b \cdot y = 0$, $c \cdot x = X_{jk}(v \cdot x) = 0$ and $c \cdot y = X_{jk}(v \cdot y) = 0$. It follows that P is involutive and invariant under the flows of X_H and X_{jk}. $P \cap \bar{P}$ is a one-dimensional complex distribution spanned by X_H. $P + \bar{P}$ is a $(2n-1)$-dimensional involutive complex distribution spanned by X_H and the vectors of the form $v \cdot X$ and $v \cdot Y$. Thus we are done.

In the terminology of Gawedzki [10], P is a strongly admissible, positive polarization.

5. Half-P-forms. Let $\alpha = \sum y_j dy_j$ and $\beta_j = dx_j - i|y|^{-1} dy_j$ $(1 \leqq j \leqq n+1)$ be one-forms on M. Choose $\boldsymbol{R}^{n+1}$-valued (not necessarily continuous) functions $u_a = (u_a^1, \cdots, u_a^{n+1})$, $(1 \leqq a \leqq n-1)$, on M, such that the matrix ${}^t(x, |y|^{-1} y, u_1, \cdots, u_{n-1}) \in SO(n+1)$ at any point $(x, y) \in M$. Define $\beta = \bigwedge_{a=1}^{n-1} \sum_{1 \leqq j \leqq n+1} u_a^j \beta_j$. Then β is a smooth $(n-1)$-form on M, which is independent of the choice of $\{u_a\}$. Let $\mathscr{L}_Z$ denote the Lie derivation with respect to a vector field Z on M.

LEMMA 2. *$\mu = \alpha \wedge \beta$ is a nowhere-vanishing smooth n-form on M, which satisfies: (1) $Z \lrcorner \mu = 0$ for any vector Z from P, (2) $\mathscr{L}_{X_H} \mu = i(n-1)|y|\mu$, (3) $\mathscr{L}_{v \cdot F} \mu = 0$ for any $\boldsymbol{R}^{n+1}$-valued smooth function v on M with $v \cdot x = v \cdot y = 0$ and (4) $\mathscr{L}_{X_{jk}} \mu = 0$.*

By the above lemma, it follows that the bundle $\boldsymbol{D}$ of complex n-forms on M, vanishing after contraction with any vector from P is a trivial bundle. Let $\boldsymbol{D}^{1/2} = (M \times \boldsymbol{C}, p, M)$ be another complex line bundle (trivial bundle) over M, and ν denote the cross-section $(x, y) \mapsto (x, y, 1)$ of $\boldsymbol{D}^{1/2}$. Let $\iota: \boldsymbol{D}^{1/2} \otimes \boldsymbol{D}^{1/2} \to \boldsymbol{D}$ be the vector bundle isomorphism defined by $\iota(\nu \otimes \nu) = \mu$. We call the pair $(\boldsymbol{D}^{1/2}, \iota)$ the square root structure for P and sections of $\boldsymbol{D}^{1/2}$ half-P-forms on M. See Gawedzki [10] and Simms and Woodhouse [20]. Since $H^1(M, \boldsymbol{Z}_2) = 0$ $(n \geqq 3)$, the square root structure is unique. For any smooth vector field Z on M, let us define a Z-derivation $\mathscr{L}_Z^{1/2}$ acting on the space $\Gamma(\boldsymbol{D}^{1/2})$ of smooth cross-sections of $\boldsymbol{D}^{1/2}$ by the following: $\iota(2(\mathscr{L}_Z^{1/2}\sigma) \otimes \sigma) = \mathscr{L}_Z(\iota(\sigma \otimes \sigma))$, for any $\sigma \in \Gamma(\boldsymbol{D}^{1/2})$. See Gawedzki [10].

Let $\mathscr{D}'(M)$ be the space of generalized functions (distributions or 0-currents) on M. See de Rham [6] and Schwartz [17]. We call the tensor product $\mathscr{D}'(M) \otimes \Gamma(\boldsymbol{D}^{1/2})$, taken over the ring $C^\infty(M)$, the space of generalized half-P-forms on M. See Simms [19]. Finally, we have the space of generalized $\boldsymbol{L}$-valued half-P-forms on M, $\Gamma = \Gamma(\boldsymbol{L}) \otimes D'(M) \otimes \Gamma(\boldsymbol{D}^{1/2})$. Note that Γ is naturally identified with $\mathscr{D}'(M)$ by the correspondence $1 \otimes T \otimes \nu \leftrightarrow T$.

6. Quantum phase space $\mathscr{H}^P$. Let $\Gamma(P)$ denote the space of smooth, complex vector fields on M which belong to P at each point of M. A complex vector field Z on M is said to preserve the polarization P if $[Z, X] \in \Gamma(P)$ for any $X \in \Gamma(P)$. For each vector field Z, which preserves P, we define a linear operator δ_Z on Γ by $\delta_Z(\varphi \otimes T \otimes \sigma) = (\nabla_Z \varphi) \otimes T \otimes \sigma + \varphi \otimes ZT \otimes \sigma + \varphi \otimes T \otimes \mathscr{L}_Z^{1/2}\sigma$, where ZT is defined by $(ZT)(A) = -T(\mathscr{L}_Z A)$ for any smooth $2n$-form A on M of compact support. See Gawedzki [10] and Simms [19]. A cross-section $\gamma \in \Gamma$ is called P-horizontal if $\delta_Z(\gamma) = 0$ for all $Z \in \Gamma(P)$. Then by Lemma 2, a cross-section $1 \otimes T \otimes \nu \in \Gamma$ is P-horizontal if and only if $X_H T - i|y|(\hbar^{-1}|y| - 2^{-1}(n-1))T = 0$ and $(v \cdot F)T = 0$, for any v as in Lemma 2.

For each integer N, $N > 2^{-1}(n-1)$, let us denote $r_N = \hbar(N + 2^{-1}(n-1))$. Define a submanifold: $M_N = \{(x, y) \in M \mid |y| = r_N\}$ of M with the inclusion $p_N \colon M_N \to M$. Let $\bigwedge^q(M)$ denote the space of smooth q-forms on M. Define $\eta = |y|^{-1}(y \cdot Y) \rfloor \Omega^n \in \bigwedge^{2n-1}(M)$. Then η satisfies $(y \cdot Y) \rfloor \eta = 0$, $d(|y|) \wedge \eta = \Omega^n$ and $p_N^*(\mathscr{L}_{X_H}\eta) = 0$. It follows that $\eta_N = p_N^*\eta \in \bigwedge^{2n-1}(M_N)$ is non-vanishing and invariant under the flow of X_H restricted to M_N. For any $A \in \bigwedge^{2n}(M)$, $A = a\Omega^n$ with $a \in C^\infty(M)$, let $A_N = p_N^*(a\eta) \in \bigwedge^{2n-1}(M_N)$. For any $T_N \in \mathscr{D}'(M_N)$, let us define $\tilde{T}_N \in \mathscr{D}'(M)$ by $\tilde{T}_N(A) = T_N(A_N)$, for any $A \in \bigwedge^{2n}(M)$ of compact support. In the following, we shall determine the subspace $\mathscr{H}^P$ of Γ composed of P-horizontal cross-sections of the form $1 \otimes \sum_N \tilde{T}_N \otimes \nu$. If we write $\mathscr{H}_N^P = \{1 \otimes \tilde{T}_N \otimes \nu \in \mathscr{H}^P \mid T_N \in \mathscr{D}'(M_N)\}$, then $\mathscr{H}^P = \bigoplus \mathscr{H}_N^P$.

LEMMA 3. *$\mathscr{H}_N^P$ is non-trivial if and only if N is non-negative. In this case, $\mathscr{H}_N^P$ is given by $\mathscr{H}_N^P = \{1 \otimes \tilde{T}_N \otimes \nu \mid T_N = \sum_{|K|=N} c_K z^K\}$, where*

$c_K \in \boldsymbol{C}$, $z = (z_1, \cdots, z_{n+1})$, $z_j = x_j - ir_N^{-1}y_j \in C^\infty(M_N)$, $K = (k_1, \cdots, k_{n+1})$ *and* $|K| = \sum k_j$.

Note that

$$\dim \mathscr{H}_N^P = \frac{2N+n-1}{N}\binom{N+n-2}{n-1},$$

which is equal to the multiplicity of the N-th eigenvalue of the Laplacian Δ acting on functions on S^n. See Berger-Gauduchon-Mazet [2].

7. Kostant quantization for H and L_{jk}. Following the Kostant quantization prescription, we shall assign for H and L_{jk} linear operators $\hat{H} = i^{-1}\hbar\delta_{X_H} + H$ and $\hat{L}_{jk} = i^{-1}\hbar\delta_{X_{jk}} + L_{jk}$ on $\mathscr{H}^P$. We call $\hat{L}_{jk}$ the angular momentum operators. Furthermore, we define $\tilde{L}^2 = \sum_{i<k}(\hat{L}_{jk})^2$, which we call the square of angular momentum operators.

LEMMA 4. (1) $\hat{H}|_{\mathscr{H}_N^P} = 2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$ *(multiplication operator).* (2) $\hat{L}_{jk}(1 \otimes z^A \otimes \nu) = 1 \otimes i^{-1}\hbar(a_k z^B - a_j z^C) \otimes \nu$, *where* $z = (z_1, \cdots, z_{n+1})$, $z_j = x_j - ir_N^{-1}y_k$, $A = (a_1, \cdots, a_{n+1})$, $\sum a_j = N$, $z^A = z_1^{a_1} \cdots z_{n+1}^{a_{n+1}}$, $B = (a_1, \cdots, a_j + 1, \cdots, a_k - 1, \cdots, a_{n+1})$ *and* $C = (a_1, \cdots, a_j - 1, \cdots, a_k + 1, \cdots, a_{n+1})$. (3) $\tilde{L}^2|_{\mathscr{H}_N^P} = \hbar^2 N(N+n-1)$ *(multiplication operator).*

PROOF. Since $\delta_{X_H} = 0$ on $\mathscr{H}^P$, we have $\hat{H}|_{\mathscr{H}_N^P} = H|_{\mathscr{H}_N^P} = 2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$. Thus (1) is proved. To prove (2), it is sufficient to note $\nabla_{X_{jk}}1 = -i\hbar^{-1}L_{jk}$ and $\mathscr{L}_{X_{jk}}^{1/2}\nu = 0$, which follow from Lemma 2. To prove (3), it is sufficient to note $z \cdot z = 0$ on M_N.

Summing up, we have the following:

THEOREM. *There exists a polarization P on $M = T^*S^n - \{0\text{-section}\}$, which is invariant under the geodesic flow and under the natural $SO(n+1)$-action on M. By means of this polarization, the classical Hamiltonian H and the functions L_{jk}'s are geometrically quantized. For $n \geqq 3$, the corresponding quantum Hamiltonian $\hat{H}$ has $2^{-1}\hbar^2(N + 2^{-1}(n-1))^2$ as the N-th eigenvalue ($N \geqq 0$) with the eigenspace $\mathscr{H}_N^P$ of dimension*

$$\frac{2N+n-1}{N}\binom{N+n-2}{n-1}.$$

Moreover, an operator $\tilde{L}^2$, defined by $\sum_{j<k}(\hat{L}_{jk})^2$, has $\hbar^2 N(N+n-1)$ as the N-th eigenvalue with the eigenspace $\mathscr{H}_N^P$.

As "classical observables", energy H and one-half the square of angular momenta, $2^{-1}\sum L_{jk}^2$, are equal, but as "quantum observables", $\hat{H}$ and $2^{-1}\sum(\hat{L}_{jk})^2$ are different by an additive constant; $\hat{H} = 2^{-1}(\tilde{L}^2 + \hbar^2(2^{-1}(n-1))^2)$.

A similar observation may be possible for such manifolds as compact symmetric spaces of rank one. See Besse [3].

8. **Appendix.** Let Q be the restriction to M of the vertical polarization of $p: T^*S^n \to S^n$, and E the bundle of complex n-forms on M, vanishing after contraction with any vector from Q. In the following, we use the same letter p for the restriction of p to M. E is a trivial bundle and $p^*\mu$ is a nowhere-vanishing cross-section of E, where $\mu \in \bigwedge^n(S^n)$ is the volume form on S^n. Let $(E^{1/2}, \iota)$ be the square root structure for Q and ν the cross-section of the trivial bundle $E^{1/2}$ such that $\iota(\nu \otimes \nu) = p^*\mu$. For each vector field Z on M, which preserves the polarization Q, we define a linear operator δ_Z on $\Gamma(L) \otimes \Gamma(E)$ by $\delta_Z(\varphi \otimes \nu) = (\nabla_Z \varphi) \otimes \nu + \varphi \otimes \mathscr{L}_Z^{1/2}\nu$. Q-horizontal sections are similarily defined, which are of the form $(f \circ p) \otimes \nu$ for $f \in C^\infty(S^n)$. The space $\mathscr{H}^Q$ of Q-horizontal sections is naturally identified with $C^\infty(S^n)$ by the correspondence $(f \circ p) \otimes \nu \leftrightarrow f$. Since X_{jk} preserves Q, we can define a linear operator $\hat{L}_{jk}$ by $\hat{L}_{jk} = i^{-1}\hbar\delta_{X_{jk}} + L_{jk}$ on $\mathscr{H}^Q$. We also call $\hat{L}_{jk}$ the angular momentum operator. Furthermore, if we define $\tilde{L}^2 = \sum_{j<k}(\hat{L}_{jk})^2$, then we have $\tilde{L}^2((f \circ p) \otimes \nu) = ((\hbar^2 \Delta f) \circ p) \otimes \nu$. Thus, under the identification of $\mathscr{H}^Q$ with $C^\infty(S^n)$, $\tilde{L}^2$ is nothing but $\hbar^2$ times the Laplacian Δ acting on functions on S^n, (the Casimir operator). Since X_H does not preserve Q, we cannot quantize H in the same way as above as a linear operator on $\mathscr{H}^Q$. But, by Lemma 4 and the above calculation, it is reasonable to say that if we quantize the classical Hamiltonian H as an operator on $\mathscr{H}^Q$, then we should have the operator $\hat{H} = 2^{-1}(\tilde{L}^2 + \hbar^2(2^{-1}(n-1))^2)$ as the corresponding quantum Hamiltonian. If we identify $\mathscr{H}^Q$ with $C^\infty(S^n)$, then $\hat{H}$ is given by $2^{-1}\hbar^2(\Delta + (2^{-1}(n-1))^2)$.

References

[1] S. I. Ben-Abraham and A. Lonke, Quantization of a general dynamical system, J. Math. Phys. 14 (1973), 1935–1937.

[2] M. Berger, P. Gauduchon and E. Mazet, Le spectre d'une variété riemannienne, Lecture Notes in Math. 194, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[3] A. L. Besse, Manifolds all of whose geodesics are closed, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[4] R. J. Blattner, Quantization and representation theory, Proc. Symp. Pure Math. 26 (1974), 147–165.

[5] K. S. Cheng, Quantization of a general dynamical system by Feynman's path integration formulation, J. Math. Phys. 13 (1972), 1723–1726.

[6] G. de Rham, Variétés différentiables, Hermann, Paris, 1955.

[7] B. S. DeWitt, Dynamical theory in curved spaces. I, A review of the classical and quantum action principles, Rev. Mod. Phys. 26 (1957), 377–397.

[8] J. Elhadad, Sur l'interprétation en géométrie symplectique des états quantiques de

l'atome d'hydrogène, Symposia Math. 24 (1974), 259-291.
[9] J. ELHADAD, Quantification du flot géodésique de la sphère S^n, C. R. Acad. Sci. Paris 285 (1977), 961-964.
[10] K. GAWEDZKI, Fourier-like kernels in geometric quantization, Diss. Math. 128 (1976).
[11] K. II, On the multiplicities of the spectrum for quasi-classical mechanics on spheres, Tôhoku Math. J. 30 (1978), 517-524.
[12] B. KOSTANT, Quantization and unitary representations, Lecture Notes in Math. 170, Springer-Verlag, Berlin-Heidelberg-New York, 1970, 87-208.
[13] J. MOSER, Regularization of Kepler's problem and the averaging method on a manifold, Comm, Pure Appl. Math. 23 (1970), 609-636.
[14] E. ONOFRI, $SO(n, 2)$-singular orbits and their quantization, Colloques Internat. du CNRS 237 (1975), 155-161.
[15] E. ONOFRI, Dynamical quantization of the Kepler manifold, J. Math. Phys. 17 (1976), 401-408.
[16] E. ONOFRI, V. Fock, 40 years later, Lecture Notes in Math. 570, Springer-Verlag, Berlin-Heidelberg-New York, 1977, 72-75.
[17] L. SCHWARTZ, Théorie des distributions, Hermann, Paris, 1966.
[18] D. J. SIMMS, Geometric quantization of energy levels in the Kepler problem, Symposia Math. 24 (1974), 125-137.
[19] D. J. SIMMS, Metalinear structure and a geometric quantization of the harmonic oscillator, Colloques Internat. du CNRS 237 (1975), 163-174.
[20] D. J. SIMMS AND N. M. J. WOODHOUSE, Lectures on geometric quantization, Lecture Notes in Phys. 53, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
[21] A. WEINSTEIN, Quasi-classical mechanics on spheres, Symposia Math. 24 (1974), 25-32.

DEPARTMENT OF MATHEMATICS
YAMAGATA UNIVERSITY
YAMAGATA, 990
JAPAN