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EXTENSIONS OF DERIVATIONS AND AUTOMORPHISMS FROM C^*-ALGEBRAS TO THEIR INJECTIVE ENVELOPES

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1. Introduction and preliminaries. Quite recently, the first author showed that any unital C^*-algebra A has a unique injective envelope $I(A)$ which indeed is an AW^*-algebra and contains the regular monotone completion $\bar{A}$ of A as an AW^*-subalgebra. The injective envelope $I(A)$ (resp. the regular monotone completion $\bar{A}$) reflects closely the structure of A; e.g., any *-automorphism of A is extended to a unique *-automorphism of $I(A)$ (resp. $\bar{A}$) ([6]).

AW^*-algebras are more tractable than the general C^*-algebras. They have sufficiently many projections and are decomposed uniquely according to type. Moreover it is known that their derivations are inner ([10]).

On the other hand, $I(A)$ is an AW^*-factor if and only if A is prime, and in most cases $I(A)$ becomes a non-W^*, AW^*-algebra. To such an algebra the spatial theory of W^*-algebras cannot be applicable and to study it seems to be very interesting.

In this paper we shall consider the following questions: Whether can each derivation on a C^*-algebra be extended to a unique derivation on its injective envelope and whether can each automorphism (not necessarily *-preserving) of a C^*-algebra be extended to a unique automorphism of its injective envelope? The answers should be given *affirmatively* to both questions for a general C^*-algebra. As an application of the observation on derivations, we shall be able to introduce, for the general C^*-algebra A, the C^*-algebra $D(A)$, as a C^*-subalgebra of the regular monotone completion $\bar{A}$ of A (note that, if A is separable then $\bar{A}$ coincides with the regular σ-competion $\hat{A}$ of A [18] and hence $D(A)$ is a C^*-subalgebra of $\hat{A}$). This C^*-algebra $D(A)$ must coincide with Sakai's derived algebra $\mathscr{D}(A)$ ([14]) if A is factorial (see also Tomiyama [17]).

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We assume familiarity with basics of C^*-algebras, their derivations and automorphisms (e.g., [2] and [13]). Before going into discussions, however, we shall give the definitions, and the constructions of the injective envelope and the regular monotone completion of a C^*-algebra.

DEFINITION 1.1. An *extension* of a unital C^*-algebra A is a pair (B, κ) of a unital C^*-algebra B and a unital $*$-monomorphism κ of A into B. An extension (B, κ) is *injective* if B is injective, and *essential* if for any unital completely positive linear mapping φ of B into a unital C^*-algebra C, φ is completely isometric whenever $\varphi \circ \kappa$ is. An extension (B, κ) is an *injective envelope* of A if it is an injective extension of A such that the identity mapping id_B on B is a unique completely positive linear mapping of B into itself which fixes each element of $\kappa(A)$.

Let A be a unital C^*-algebra. Then there exists an injective C^*-algebra C containing A as a C^*-subalgebra (we may take C as the algebra $B(H)$ of all bounded linear operators on the universal Hilbert space H of A) and a minimal A-projection on C in the sense that φ is unital, completely positive, idempotent, satisfies $\varphi(a) = a$ for all $a \in A$ and has a minimal image among the images of all A-projections. One can check that the image $\mathrm{Im}\,\varphi$ of φ is a uniformly closed $*$-subspace of C (not necessarily closed with respect to the multiplication of C). Introducing a new product in $\mathrm{Im}\,\varphi$ via $a \circ b = \varphi(ab)$ $(a, b \in \mathrm{Im}\,\varphi)$, we get that, with respect to this multiplication, $\mathrm{Im}\,\varphi$ is a C^*-algebra and it contains A as a C^*-subalgebra. The injectivity of C and the minimality of φ imply that $(\mathrm{Im}\,\varphi, \mathrm{id}_A)$ is an injective envelope of A. It can be shown that $(\mathrm{Im}\,\varphi, \mathrm{id}_A)$ is unique in the following sense: If another injective envelope (B, κ) is given, then there exists a unique $*$-isomorphism γ of $\mathrm{Im}\,\varphi$ onto B such that $\gamma \circ \mathrm{id}_A = \kappa$. Moreover, it can be shown that $(\mathrm{Im}\,\varphi, \mathrm{id}_A)$ is the largest essential extension of A in the following sense: An extension (B, κ) of A is essential if and only if there exists a unital $*$-monomorphism λ of B into $\mathrm{Im}\,\varphi$ such that $\lambda \circ \kappa = \mathrm{id}_A$ ([5; Lemma 4.6]).

In what follows, we regard an extension (B, κ) of A as a C^*-algebra B which contains A as a C^*-subalgebra by identifying κ with the inclusion, and the injective envelope of A is denoted by $I(A)$. Note that $I(A)$ is a monotone complete AW^*-algebra ([16]).

DEFINITION 1.2. A *regular monotone completion* of a unital C^*-algebra A is a monotone complete C^*-algebra $\bar{A}$ which contains A as a C^*-subalgebra and satisfies the following properties:

(i) $\bar{A}_h$ itself is the smallest monotone closed subspace of $\bar{A}_h$ which contains A_h; and

(ii) Each x in $\bar{A}_h$ is the supremum in $\bar{A}_h$ of the set $\{a \in A_h: a \leqq x\}$.

Let us introduce the following notations: For $x \in A_h$ and $\mathscr{F} \subset A_h$, $\mathscr{F} \leqq x$ means that $y \leqq x$ for all $y \in \mathscr{F}$, and $\mathrm{Sup}_A \mathscr{F} = x$ means that the supremum of $\mathscr{F}$ in A exists and it is equal to x. For an extension B of A and $x \in B_h$, we denote the set $\{a \in A_h: a \leqq x\}$ by $(-\infty, x]_A$.

Keeping these notations in mind, we shall describe briefly a way to construct $\bar{A}$ ([6]). Let $\tilde{A}_h = \{x \in I(A)_h: x = \mathrm{Sup}_{I(A)}(-\infty, x]_A\}$ and $\tilde{A} = \tilde{A}_h + i\tilde{A}_h$. Then, the first author showed that $\tilde{A}$ is a monotone complete C^*-subalgebra of $I(A)$ which contains A as a C^*-subalgebra (it becomes a maximal regular extension of A). Define $\bar{A}$ as the monotone closure of A in $\tilde{A}$. Then it can be shown to be the regular monotone completion of A.

In this connection, $\hat{A}$ which is defined as the monotone σ-closure of A in $\bar{A}$ is nothing but the regular σ-completion of A introduced by Wright [18] because it is uniquely determined by A.

We have the following inclusions

$$A \subset \hat{A} \subset \bar{A} \subset \tilde{A} \subset I(A)$$

(respective inclusions are *supremum preserving* [6]) and moreover

$$Z_A \subset Z_{\hat{A}} \subset Z_{\bar{A}} = Z_{\tilde{A}} = Z_{I(A)},$$

where Z_B means, in general, the center of a C^*-algebra B.

In what follows, we suppose that the C^*-algebra A in consideration acts on its universal Hilbert space H, to simplify the arguments without any loss of generality.

For a non unital C^*-algebra A, we can consider $I(A_1)$, $\tilde{A}_1$, $\bar{A}_1$ and $\hat{A}_1$, where $A_1 = C^*(A, 1_H)$, the C^*-algebra generated by A and the identity operator 1_H on H. In what follows, we employ the notations $I(A)$, $\bar{A}$ and $\hat{A}$ instead of $I(A_1)$, $\bar{A}_1$ and $\hat{A}_1$, respectively (see also [15]) when A is non unital.

2. Extension of derivations from A to $I(A)$. Let B be a C^*-algebra (not necessarily unital). We denote by $\mathrm{Der}(B)$ the Lie algebra of all derivations on B and by $\mathrm{Der}(B; C)$ the Lie subalgebra of $\mathrm{Der}(B)$, of all derivations on B which leave a C^*-subalgebra C of B invariant.

For $\delta \in \mathrm{Der}(B)$, δ^* means the derivation on B defined via $\delta^*(x) = \delta(x^*)^*$ $(x \in B)$. δ is said to be *skew-adjoint* if it satisfies that $\delta^* = -\delta$.

$(A_h)^m$ denotes the set of all elements in $I(A)_h$ each of which can be obtained as a supremum of an inceasing net from A_h (note that $(A_h)^m \subset \bar{A}$).

The following theorem plays a key rôle for the later discussions.

THEOREM 2.1. (1) *For any* $\delta \in \operatorname{Der}(I(A); A)$, $\|\delta\| = \|\delta|_A\|$ *and there is an element* g *in* $(A_h)^m + i(A_h)^m$ *such that* $\delta = \operatorname{ad} g$.

(2) *For any* $\delta \in \operatorname{Der}(A)$, *we can get a unique* $I(\delta) \in \operatorname{Der}(I(A); A)$ *such that* $I(\delta)|_A = \delta$; *and the mapping* $\operatorname{Der}(A) \ni \delta \to I(\delta) \in \operatorname{Der}(I(A); A)$ *is an isometric Lie isomorphism of* $\operatorname{Der}(A)$ *onto* $\operatorname{Der}(I(A); A)$.

(3) $I(\delta)$ *is skew-adjoint if and only if so is* δ.

LEMMA 2.1 ([1]). *Let A be a unital C^*-algebra acting on its universal Hilbert space H. Let φ be any A-projection on $B(H)$. Then φ is an A-module homomorphism, that is, $\varphi(ax) = a\varphi(x)$ and $\varphi(xa) = \varphi(x)a$ hold for all $x \in B(H)$ and $a \in A$.*

If a C^*-algebra A is non unital, then by Kadison [7], each derivation δ on A can be extended to a unique derivation δ_1 on A_1 and in fact $\delta_1(a + \lambda 1_H) = \delta(a)$ for all $a \in A$ and $\lambda \in \boldsymbol{C}$ (the complex numbers). The mapping $\delta \to \delta_1$ is an isometric Lie isomorphism of $\operatorname{Der}(A)$ onto $\operatorname{Der}(A_1)$, and δ_1 is skew-adjoint if and only if so is δ. Hence, to prove Theorem 2.1, we may assume that A is unital. As was mentioned before, we may suppose that $I(A) = \operatorname{Im}\varphi$ for some minimal A-projection on $B(H)$ and the multiplication in it is: $a \circ b = \varphi(ab)$ $(a, b \in \operatorname{Im}\varphi)$.

LEMMA 2.2. *Keeping the notations as above, for any x_1, x_2 in $B(H)_h$, $x_1 \leqq x_2$ in $B(H)_h$ implies $\varphi(x_1) \leqq \varphi(x_2)$ in $I(A)_h$. If $x_1, x_2 \in I(A)_h$, then the converse implication also holds.*

PROOF. Since, for any $x \in B(H)$, $\operatorname{Sp}_{B(H)}\varphi(x) \supset \operatorname{Sp}_{I(A)}\varphi(x)$, where $\operatorname{Sp}_A(x)$ is the spectrum of an element x of a C^*-algebra A, $x_1 \leqq x_2$ in $B(H)_h$ implies $\varphi(x_1) \leqq \varphi(x_2)$ in $I(A)_h$. Conversely if $x_1, x_2 \in I(A)_h$ and $x_1 \leqq x_2$ in $I(A)_h$, then $x_2 - x_1 = y^* \circ y = \varphi(y^*y)$ for some $y \in I(A)$, thus $x_2 - x_1 \geqq 0$ in $B(H)_h$. This completes the proof.

PROOF OF THEOREM 2.1. For any $\delta \in \operatorname{Der}(A)$ it can be shown by [7] that the bitranspose δ'' of δ is a σ-weakly continuous derivation on the bidual A'' of A, which is the σ-weak closure of A in $B(H)$, and it is an extension of δ with $\|\delta''\| = \|\delta\|$. Then by [13] (see also [7]), there is a generator g_0 for δ'' in A'' such that $\|\delta''\| = 2\|g_0\|$. Let $g = \varphi(g_0)$ and $\delta^{(1)} = \operatorname{ad} g$ (where $\operatorname{ad} g(x) = g \circ x - x \circ g$ for $x \in I(A)$). Then, for any a in A,

$$\begin{aligned}\delta^{(1)}(a) &= \varphi(g_0) \circ a - a \circ \varphi(g_0) \\ &= \varphi(\varphi(g_0)a - a\varphi(g_0)) = \varphi^2(g_0 a - a g_0) \quad \text{(by Lemma 2.1)} \\ &= \varphi(\delta''(a)) = \varphi(\delta(a)) = \delta(a) \quad \text{(because } \varphi|_A = \mathrm{id}_A) .\end{aligned}$$

Thus $\delta^{(1)}$ is an extension of δ to $I(A)$ and $\delta^{(1)} \in \mathrm{Der}\,(I(A); A)$. Moreover, since $\|\delta^{(1)}\| \leqq 2\|g\| \leqq 2\|g_0\| \leqq \|\delta''\| = \|\delta\| \leqq \|\delta^{(1)}\|$, it follows that $\|\delta^{(1)}\| = \|\delta\|$.

If δ is skew-adjoint, then Olesen and Pedersen's theorem ([11]) tells us that the above g_0 can be taken to be the minimal positive generator h for δ, which is a strong limit in $B(H)$ of an increasing net $\{h_\alpha\}$ from A_h. We can show that $\varphi(h)$ is the supremum of $\{h_\alpha\}$ in $(I(A))_h$. In fact, $h_\alpha \leqq h$ in $B(H)_h$ for any α implies that $h_\alpha = \varphi(h_\alpha) \leqq \varphi(h)$ in $I(A)_h$ for any α. On the other hand, if $h_\alpha \leqq x$ for any α in $I(A)_h$, then by Lemma 2.2, $h_\alpha \leqq x$ in $B(H)_h$ for any α and so $h \leqq x$ in $B(H)_h$. Hence, again by Lemma 2.2, $\varphi(h) \leqq x$ in $I(A)_h$. Thus we have $\varphi(h) = \mathrm{Sup}_{I(A)} h_\alpha$ ([16; Th. 7.1]). Therefore $\varphi(h)$ is in $(A_h)^m$.

Let $\delta^{(2)}$ be an extension of δ to $I(A)$. Then, to show that $\delta^{(2)} = \delta^{(1)}$, we may assume that δ, $\delta^{(1)}$ and $\delta^{(2)}$ are skew-adjoint, by considering the respective Cartesian decompositions. Let us consider the uniformly continuous one-parameter groups $\alpha_t^{(1)} = \exp\,(it\,\delta^{(1)})$ and $\alpha_t^{(2)} = \exp\,(it\,\delta^{(2)})$ $(-\infty < t < \infty)$ of *-automorphisms of $I(A)$. Since $\beta_t = \alpha_t^{(1)-1}\alpha_t^{(2)}$ is a completely isometric mapping of $I(A)$ into $I(A)$ such that $\beta_t|_A = \mathrm{id}_A$, we have $\beta_t = \mathrm{id}_{I(A)}$ by Definition 1.1. Thus, $\alpha_t^{(2)} = \alpha_t^{(1)}$ for any t; and this implies that $\delta^{(2)} = \delta^{(1)}$. Let us define $I(\delta)$ to be the unique extension of δ to $I(A)$. It is obvious that the mapping $\delta \to I(\delta)$ is a Lie isomorphism of $\mathrm{Der}\,(A)$ onto $\mathrm{Der}\,(I(A); A)$. What was proved above shows that this mapping is isometric and that $I(\delta)$ is skew-adjoint if and only if so is δ. Therefore all the statements in Theorem 2.1 are proved.

REMARK 2.1. Since $Z_{I(A)} = Z_{\bar{A}}$, we can easily show by Theorem 2.1 (1) that *any* generator of $I(\delta)$ is found in $\bar{A}$. Therefore, $\bar{\delta} = I(\delta)|_{\bar{A}}$ is a derivation on $\bar{A}$, which is a unique extension of δ to $\bar{A}$, In fact, the above argument which shows that the extension of δ to $I(A)$ is unique can be applied if we observe that $I(\bar{A}) = I(A)$ and replace A by $\bar{A}$.

3. Derived algebra $D(A)$ of A.

Given a derivation on an AW^*-algebra, Olesen proved that it is inner, and Halpern proved that it has a unique *minimal generator* ([10], [4]):

THEOREM 3.1 ([10], [4]). *Let B be an AW^*-algebra with the center Z_B and δ a derivation on B. Then there is a unique generator $h(\delta)$ (called the minimal generator) for δ in B such that*

$$\|\delta|_{pB}\|/2 = \|h(\delta)p\|$$

for each projection p in Z_B, where $\delta|_{pB}$ is the derivation restricted to pB.

Given a $\delta \in \mathrm{Der}\,(A)$, since $I(\delta)$ is a derivation on the AW^*-algebra

$I(A)$, Theorem 2.1 and Remark 2.1 tell us that the minimal generator $h(\delta)$ for $I(\delta)$ is in $\bar{A}$. We shall show that $h(\delta)$ is also the minimal generator of $\bar{\delta}$. Since, for each projection p in $Z_{\bar{A}}$ $(=Z_{I(A)})$ the injective envelope of pA coincides with $pI(A)$ by [6], putting $\delta_p = \delta|_{pA}$, we have $I(\delta_p) = I(\delta)|_{pI(A)}$ and $\|\delta_p\| = \|I(\delta_p)\| = \|I(\delta)|_{pI(A)}\|$ by Theorem 2.1. Moreover, $pI(A) \supset p\bar{A} \supset \overline{pA} \supset pA$ implies that $\|\delta_p\| = \|I(\delta)|_{pI(A)}\| \geqq \|\bar{\delta}|_{p\bar{A}}\| \geqq \|\delta_p\|$ and hence $\|I(\delta)|_{pI(A)}\| = \|\bar{\delta}|_{p\bar{A}}\|$. Therefore we have

$$\|\bar{\delta}|_{p\bar{A}}\| = \|I(\delta)|_{pI(A)}\| = 2\|h(\delta)p\| \geqq \|(\mathrm{ad}\, h(\delta)|_{\bar{A}})|_{p\bar{A}}\| \geqq \|\bar{\delta}|_{p\bar{A}}\| ,$$

and hence $\|\bar{\delta}|_{p\bar{A}}\| = 2\|h(\delta)p\|$.

Summing this consideration up, we get:

LEMMA 3.1. *For any $\delta \in \mathrm{Der}\,(A)$, the minimal generator $h(\delta)$ for $I(\delta)$ is contained in $\bar{A}$ and it is also the minimal generator for $\bar{\delta}$.*

DEFINITION 3.1. Keeping the notations as above, $D(A)$, with A unital, denotes the C^*-algebra generated by the system $\{A, h(\delta)\colon \delta \in \mathrm{Der}\,(A)\}$. When A is non unital, the notation $D(A)$ means $D(A_1)$.

THEOREM 3.2. *$D(A)$ has the following properties:*

(1) *$D(A)$ is a C^*-subalgebra of $\bar{A}$.*

(2) *For each $\delta \in \mathrm{Der}\,(A)$, there is an $h(\delta)$ in $D(A)$ such that $\delta = \mathrm{ad}\, h(\delta)|_A$ and $\|\delta\|/2 = \|h(\delta)\|$.*

(3) *For any closed ideal J of $D(A)$, $J \cap A = \{0\}$ implies that $J = \{0\}$.*

(4) *If A is factorial, then $D(A)$ is the derived algebra for A in the sense of Sakai* [14] (*see also* [17]).

To prove this theorem, we need the following lemma.

LEMMA 3.2. *Let B be a C^*-subalgebra of $I(A)$ which contains A_1 as a C^*-subalgebra. Then, for any closed two-sided ideal I of B, $A \cap I = \{0\}$ implies that $I = \{0\}$.*

PROOF. If A is non unital, then $A \cap I = \{0\}$ if and only if $A_1 \cap I = \{0\}$. Thus, to prove the lemma, we may assume that A is unital. Let I be any closed two-sided ideal of B with $A \cap I = \{0\}$ and ϕ_I the canonical quotient mapping from B onto B/I. Since $A \subset B \subset I(A)$, B is an essential extension of A. Thus ϕ_I is completely isometric because $\phi_I|_A$ is completely isometric (in fact, it is a $*$-monomorphism). It follows that ϕ_I is one-to-one and so $I = \{0\}$. Thus the lemma follows.

PROOF OF THEOREM 3.2. (1) and (2) Obvious.

(3) Immediate from Lemma 3.2.

(4) We can claim actually that if π is a faithful pseudo-factorial

*-representation of a C^*-algebra A (thus A is pseudo-factorial and hence prime), then it can be extended to a unique *-isomorphism $\hat{\pi}$ of $D(A)$ onto the C^*-algebra D_π generated by $\{\pi(A), 1_{H_\pi}, h(\pi\delta\pi^{-1}); \delta \in \mathrm{Der}(A)\}$, where 1_{H_π} is the identity operator on the representation space H_π of π. In fact, since A is pseudo-factorial and $\bar{A}$ is an AW^*-factor, it follows that $h(\bar{\delta})$ and $h(\pi\delta\pi^{-1})$ can be determined uniquely by the equality

$$\|h(\bar{\delta})\| = \|\delta\|/2 = \|\pi\delta\pi^{-1}\|/2 = \|h(\pi\delta\pi^{-1})\| \quad \text{(Theorem 3.1).}$$

Thus, in the same way as in [17] we get, by Lemma 3.2, the desired extension $\hat{\pi}$ which is determined uniquely by π. The details may be omitted.

REMARK 3.1. For an arbitrary C^*-algebra A, the C^*-algebra $D(A)$ considered to be its derived algebra and, by the above theorem, Sakai's derived algebra $\mathscr{D}(A)$ with A factorial, can be realized as a C^*-subalgebra $D(A)$ of $\bar{A}$.

If A is separable then $\hat{A} = \bar{A}$, and hence $D(A)$ is a C^*-subalgebra of $\hat{A}$. However, in general, $D(A) \not\subset \hat{A}$. In fact, let H be a non separable Hilbert space and A the algebra $C(H)$ of all compact operators on H. Then one can easily check that $\hat{A} = S(H) + C1_H$, where $S(H)$ is the algebra of operators on H with separable ranges, and $\bar{A} = I(A) = B(H)$. Let p be a projection on H such that p and $1 - p \notin S(H)$. Then the minimal generator of the derivation $\mathrm{ad}\, p|_A$ is not in $\hat{A}$. Therefore $D(A) \not\subset \hat{A}$.

Since A is prime if and only if $\bar{A}$ is a factor ([6]), we see easily that A is prime if and only if so is $D(A)$.

If A is separable, then A is primitive if and only if $D(A)$ is primitive and A is NGCR if and only if $D(A)$ is NGCR ([15]).

4. Extensions of automorphisms of A to $I(A)$. Let η be a positive automorphism of a C^*-algebra A in the following sense:

DEFINITION 4.1 ([9]). An automorphism η (not necessarily *-preserving) of a C^*-algebra B is said to be positive if $\eta = \eta'$ (where η' is the automorphism of B defined via $\eta'(x)^* = \eta^{-1}(x^*)$ $(x \in B)$) and the spectrum $\mathrm{Sp}(\eta)$ of η is on the positive half axis $[0, +\infty)$.

Then, according to [9; Theorems 8.1 and 8.3], there is an invertible positive element h in A'' such that $\eta = \mathrm{Ad}\, h|_A$, where $\mathrm{Ad}\, h$ is the automorphism implemented by h. Since both $\mathrm{Sp}(\eta)$ and $\mathrm{Sp}(\mathrm{Ad}\, h)$ are on $(0, +\infty)$, the principal branch Log of the logarithm can be applied to η and to $\mathrm{Ad}\, h$, and $\mathrm{Log}\,\eta = \mathrm{Log}\,\mathrm{Ad}\, h|_A$ holds. Thus, the formula $\mathrm{Log}\,\mathrm{Ad}\, h =$

ad Log h ([9; Lemma 5.3 (b)]) implies that

$$\mathrm{Log}\,\eta = \mathrm{ad}\,\mathrm{Log}\,h|_A .$$

Let us put $\delta = \mathrm{Log}\,\eta$, extend it via Theorem 2.1 to a derivation $I(\delta)$ on $I(A)$ and put $I(\eta) = \exp I(\delta)$. Then $I(\eta)$ is an automorphism of $I(A)$ which is an extension of η. Moreover, it turns out to be implemented by an invertible positive element in $I(A)$ and hence to be positive. In fact, by the proof of Theorem 2.1,

$$I(\delta) = \mathrm{ad}\,\varphi(\mathrm{Log}\,h) ,$$

where φ is, as in the preceding sections, a minimal A-projection; so that

$$I(\eta) = \exp\mathrm{ad}\,\varphi(\mathrm{Log}\,h) = \mathrm{Ad}\exp\varphi(\mathrm{Log}\,h) .$$

(e.g., [9; Lemma 5.3 (a)]).

Let $\eta^{(1)}$ be a positive automorphism of $I(A)$, which is an extension of η. Since $\mathrm{Log}\,\eta^{(1)}$ is a derivation on $I(A)$ which is an extension of δ, we have $\mathrm{Log}\,\eta^{(1)} = I(\delta)$ by Theorem 2.1. Thus we know that $\eta^{(1)} = \exp I(\delta) = I(\eta)$.

Now we proved the following:

LEMMA 4.1. *Any positive automorphism η of a C^*-algebra A can be extended to a unique positive automorphism $I(\eta)$ of $I(A)$.*

We denote by Aut(A) the group of all automorphisms (not necessarily *-preserving) of a C^*-algebra A and by Aut$(A; B)$ the subgroup of Aut(A), of all automorphisms of A of which restrictions to a C^*-subalgebra B of A become automorphisms of B.

Any automorphism ρ of a C^*-algebra A has *the polar decomposition*:

$$\rho = \pi\eta$$

with a unique pair of a *-automorphism π and a positive automorphism η of A ([9; Theorem 7.1], cf. [8]). We obtain the following:

THEOREM 4.1. (1) *For any $\rho \in \mathrm{Aut}(A)$, we get a unique $I(\rho) \in \mathrm{Aut}(I(A); A)$ such that $I(\rho)|_A = \rho$; and the mapping $\rho \to I(\rho)$ is a uniformly bicontinuous group isomorphism of* Aut(A) *onto* Aut$(I(A); A)$.

(2) *$I(\rho)$ is *-preserving (resp. positive) if and only if ρ is *-preserving (resp. positive).*

(3) *$I(\rho_t)$ $(-\infty < t < \infty)$ is a uniformly continuous one-parameter group if and only if ρ_t $(-\infty < t < \infty)$ is a uniformly continuous one-parameter group; in this case $I(\rho_t)$ has the form*

$$I(\rho_t) = \mathrm{Ad}\exp tg \qquad \textit{for all } t \quad (-\infty < t < \infty) ,$$

with $g \in \bar{A}$.

PROOF. (1) Let ρ have the polar decomposition $\rho = \pi\eta$. By [5; Corollary 4.2], π can be extended to a unique *-automorphism $I(\pi)$ of $I(A)$. Let us put $I(\rho) = I(\pi)I(\eta)$. Then this is an automorphism of $I(A)$ which is an extension of ρ. Let $\rho^{(1)}$ be an automorphism of $I(A)$ which is an extension of ρ, and have the polar decomposition $\rho^{(1)} = \pi^{(1)}\eta^{(1)}$. Since $\rho^{(1)\prime}|_A = \rho'$, we have $\rho^{(1)\prime}\rho^{(1)}|_A = \rho'\rho$ and hence

$$\eta^{(1)}|_A = (\rho^{(1)\prime}\rho^{(1)})^{1/2}|_A = (\rho'\rho)^{1/2} = \eta ,$$

because, in general, the polar decomposition $\rho = \pi\eta$ of an automorphism ρ of a C^*-algebra implies that $\eta = (\rho'\rho)^{1/2}$, the square root of $\rho'\rho$ with its spectrum on $(0, +\infty)$ ([9]). Therefore, by Lemma 4.1, we have $\eta^{(1)} = I(\eta)$. Moreover, it follows that

$$\pi^{(1)}|_A = (\rho^{(1)}\eta^{(1)-1})|_A = \rho\eta^{-1} = \pi$$

and hence $\pi^{(1)} = I(\pi)$. Thus we have

$$\rho^{(1)} = \pi^{(1)}\eta^{(1)} = I(\pi)I(\eta) = I(\rho) .$$

Next, suppose that a sequence $\{\rho_n\}$ of automorphisms of A converges uniformly to an automorphism ρ of A. Then

$$\|\rho_n\rho^{-1} - \mathrm{id}_A\| = \|(\rho_n - \rho)\rho^{-1}\| \leqq \|\rho_n - \rho\|\|\rho^{-1}\| \to 0 \ (n \to \infty) .$$

Therefore, for all n sufficiently large, $\mathrm{Sp}(\rho_n\rho^{-1})$ lies in the open half plane $\Omega = \{\lambda \in \boldsymbol{C}: \mathrm{Re}\,\lambda > 0\}$. This implies that $\delta_n = \mathrm{Log}(\rho_n\rho^{-1})$ is a derivation of A (e.g., [13; 4.1.18]). Thus, for all n sufficiently large, we have

$$I(\rho_n)I(\rho)^{-1} = I(\rho_n\rho^{-1}) = \exp I(\delta_n) = \mathrm{id}_{I(A)} + \sum_{k=1}^{\infty} I(\delta_n)^k/k!$$

to obtain

$$\begin{aligned}\|I(\rho_n) - I(\rho)\| &= \|(I(\rho_n)I(\rho)^{-1} - \mathrm{id}_{I(A)})I(\rho)\| \leqq \left\|\sum_{k=1}^{\infty} I(\delta_n)^k/k!\right\|\|I(\rho)\| \\ &\leqq \left(\sum_{k=1}^{\infty}\|\delta_n\|^k/k!\right)\|I(\rho)\| \quad \text{(by Theorem 2.1)} \\ &= (\exp\|\delta_n\| - 1)\|I(\rho)\| \to 0 \quad (n \to \infty) .\end{aligned}$$

Then we conclude that $\|I(\rho_n) - I(\rho)\| \to 0$ $(n \to \infty)$, as required.

(2) Obvious from what was mentioned above.

(3) It is obvious that if $I(\rho_t)$ $(-\infty < t < \infty)$ is a uniformly continuous one-parameter group then so is ρ_t $(-\infty < t < \infty)$.

Suppose that ρ_t $(-\infty < t < \infty)$ is a uniformly continuous one-parameter group of automorphisms of A. Then $\rho_t = \exp t\delta$ with a derivation δ on A. Thus by Theorem 2.1, there is an element g in $\bar{A}$ such that $I(\delta) = \mathrm{ad}\, g$. It follows immediately that for all t $(-\infty < t < \infty)$,

$$I(\rho_t) = \exp tI(\delta) = \mathrm{Ad} \exp tg \ .$$

Now we proved all the statements in Theorem 4.1.

5. **Concluding remarks.** In Sections 2 and 4 we considered how to extend derivations and automorphisms (not necessarily *-preserving) of a C^*-algebra A to its injective envelope $I(A)$. In this closing section, we discuss questions whether each derivation on (resp. automorphism of) A can be extended to a unique derivation on (resp. automorphism of) a C^*-subalgebra B of $I(A)$ which contains A.

(1) A derivation δ on A can be extended to a derivation on B if and only if $I(\delta)(B) \subset B$; in this case, $I(\delta)|_B$ is the unique extension of δ to B. Indeed, suppose that there is an extension δ_1 of δ to B. Then the extension $I(\delta_1)$ of δ_1 to $I(B)$ coincides with δ on A. Since $I(B) = I(A)$, we have $I(\delta_1) = I(\delta)$ by Theorem 2.1 (2). Therefore $\delta_1 = I(\delta_1)|_B = I(\delta)|_B$.

(2) We may apply a similar argument to show that an automorphism ρ of A can be extended to an automorphism of B if and only if $I(\rho)(B) = B$; in this case $I(\rho)|_B$ is the unique extension of ρ to B.

(3) If B contains $D(A)$, then the condition stated in (1) is satisfied by each derivation δ on A, and the condition stated in (2) is satisfied by each positive automorphism ρ of A (because ρ is of the form $\rho = \exp \delta = \mathrm{Log}\, \rho$). Therefore, when B contains $D(A)$, an automorphism ρ of A satisfies $I(\rho)(B) = B$ if and only if the *-preserving part $I(\pi)$ of $I(\rho)$ in its polar decomposition satisfies that $I(\pi)(B) = B$.

(4) Each automorphism ρ of A can be extended to a unique automorphism $\bar{\rho}$ of $\bar{A}$ and Theorem 4.1 holds when $I(A)$ and $I(\rho)$ are replaced by $\bar{A}$ and $\bar{\rho}$, respectively. To see this it is sufficient to observe that $\bar{A}$ contains $D(A)$ and $I(\pi)(\bar{A}) = \bar{A}$ and that $I(\rho)|\bar{A} = \bar{\rho}$.

(5) In general, the condition stated in (2) cannot hold for a C^*-algebra A, a C^*-subalgebra B of $I(A)$ which contains A and a *-automorphism α of A. We give here such an example. Let A be the C^*-algebra of all complex continuous functions on the one-dimensional torus $\boldsymbol{T}$. Then $I(A)$ turns out to be $B(\boldsymbol{T})/m(\boldsymbol{T})$ where $B(\boldsymbol{T})$ is the algebra of all bounded Baire functions on $\boldsymbol{T}$ and $m(\boldsymbol{T})$ is the ideal of all meager functions in $B(\boldsymbol{T})$. Let p be the projection in $I(A)$ defined by the characteristic function of $\{\exp 2\pi i\theta\colon 0 \leqq \theta < 1/4\}$ and, B the C^*-subalgebra of $I(A)$ generated by A and p. And, let α be the *-automorphism of A

defined by the rotation on T with angle π. Then $I(\alpha)$ is the *-automorphism of $I(A)$ canonically induced from α and $I(\alpha)(p) \notin B$.

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