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EXISTENCE PROBLEM OF TRANSVERSE FOLIATIONS FOR SOME FOLIATED 3-MANIFOLDS

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1. Introduction. Recently several foliators began the study of an ordered set $\mathscr{W}=(\mathscr{F}_1,\cdots,\mathscr{F}_k)$ of codimension one foliations of a manifold

M^n *in general position*, that is,

$$\dim T_x\mathscr{F}_{i(1)} \cap \cdots \cap T_x\mathscr{F}_{i(p)} = n - p$$

for all $x \in M$ and $\{i(1), \cdots, i(p)\} \subset \{1, \cdots, k\}$ with $p \leqq n$.

When $k = n + 1$, we call $\mathscr{W}$ an *octahedral web* if for all $x \in M$ there is a chart $\phi: U \to \boldsymbol{R}^n$ such that $x \in U$ and $\mathscr{F}_i | U = \phi^*\mathscr{G}_i$ for all i, where $\mathscr{G}_i = \{(x_1, \cdots, x_n) \in \boldsymbol{R}^n | x_i = c\}_{c \in \boldsymbol{R}}$ for $i = 1, \cdots, n$ and $\mathscr{G}_{n+1} = \{(x_1, \cdots, x_n) \in \boldsymbol{R}^n | x_1 + \cdots + x_n = c\}_{c \in \boldsymbol{R}}$. In Nishimori [10] and [11], the author classified almost all the octahedral webs on closed manifolds.

When $k = n$, we call $\mathscr{W}$ a *multifoliation* (or a *total foliation*). Tischler [16] constructed multifoliations on the total spaces of S^1-bundles over closed surfaces, and Silberstein [14] constructed multifoliations on $M \times S^1$ where M is a stably parallelizable manifold. Furthermore Hardorp [5] showed that all the closed orientable manifolds of dimension three admit multifoliations.

When $k = 2$, we see that $\mathscr{W}$ is a pair of transverse foliations. For such $\mathscr{W}$, there exists the study by Tamura and Sato [15]. They regarded a foliated manifold $(M, \mathscr{F})$ as an underlying manifold, and a foliation $\mathscr{G}$ of M transverse to $\mathscr{F}$ as a structure on $(M, \mathscr{F})$. From this point of view, Tamura and Sato characterized codimension-one C^∞ foliations transverse to the Reeb component of $S^1 \times D^2$ or to the Reeb foliation of S^3 and classified them topologically by introducing TS diagrams. From these results and the theorem of Novikov [12], they derived that the foliation of S^3 obtained from a fibered knot with a fiber of non-zero genus has no transversely orientable transverse codimension-one C^∞ foliation. In contrast to this, they remarked that any codimension one C^r foliation of S^3 admits a transverse 2-plane field. Furthermore they raised several problems on transverse foliations. One of them is the following.

PROBLEM A [15, Problem 10]. *Find conditions for C^∞ foliated manifolds to admit transverse foliations.*

From now on, a manifold is always of class C^∞ and a foliation is a codimension one C^∞ foliation, unless stated otherwise.

In Part I of this paper, we generalize the results of Tamura and Sato on the Reeb component to the foliated manifolds $(E(h), \mathscr{F}(h; \sigma))$ introduced as follows. Take a positive integer h and let $\hat{E}(h)$ be a compact manifold obtained from S^2 by deleting h small open 2-disks. Let $E(h) = S^1 \times \hat{E}(h)$. We treat S^1 and $\hat{E}(h)$ as oriented manifolds. Denote by $\hat{\Gamma}(h)$ the set of the connected components of $\partial\hat{E}(h)$ and let $\Gamma(h) = \{S^1 \times \hat{C} | \hat{C} \in \hat{\Gamma}(h)\}$. Note that each $C \in \Gamma(h)$ is diffeomorphic to T^2. Take a continuous map $\sigma: \partial E(h) \to \{1, -1\}$. Frequently we regard σ as a map from $\partial\hat{E}(h)$, $\Gamma(h)$ and $\hat{\Gamma}(h)$ to $\{1, -1\}$ in a canonical way without caution. We turbulize the product foliation

$$\mathscr{F}(h, \mathrm{pr}) = \{\{x\} \times \hat{E}(h) \mid x \in S^1\}$$

of $E(h)$ so that for each $y \in \partial\hat{E}(h)$ the oriented closed path $\sigma(y)(S^1 \times \{y\})$ has an expanding holonomy with respect to the modified foliation $\mathscr{F}(h; \sigma)$. The turbulization will be stated precisely in § 2.

Tamura and Sato decomposed foliations transverse to the Reeb component into three kinds of simple components, namely half Reeb components, foliated I-bundles over $S^1 \times I$ and TS components. In our case we decompose foliations transverse to $\mathscr{F}(h; \sigma)$ into nine kinds of components (see Theorem 3 in § 9). For a foliated manifold $(M, \mathscr{F})$, we denote by $t_1^0(M, \mathscr{F})$ (or simply $t_1^0(\mathscr{F})$) the set of transversely orientable foliations of M transverse to $\mathscr{F}$. We can classify the foliations in $t_1^0(E(h), \mathscr{F}(h; \sigma))$ with respect to a certain equivalence relation by using generalized TS diagrams (see Theorem 4 in § 14).

In Part II, as an application of the results of Part I we consider Problem A for a certain class of foliated manifolds of dimension three introduced as follows. Roughly speaking, our foliated 3-manifolds are unions of foliated manifolds of the form $(E(h), \mathscr{F}(h; \sigma))$.

First take a connected finite graph Φ and fix an orientation for each side of Φ. Denote by $V(\Phi)$ (or $S(\Phi)$) the set of vertices (or sides) of Φ. For $v \in V(\Phi)$, let $S(\Phi; v) = \{s \in S(\Phi) \mid v \text{ is an end of } s\}$, where we take two copies s^+, s^- of s if the ends of s coincide and are v. Let $h(v) = \sharp S(\Phi; v)$ and $E[v] = E(h(v))$. We fix a bijection $C[v]: S(\Phi; v) \to \Gamma[v] = \Gamma(h(v))$.

Take a map $\Psi: S(\Phi) \to \left\{\begin{pmatrix} k & l \\ m & n \end{pmatrix} \middle| kn - lm = -1,\ k, l, m, n \in \boldsymbol{Z}\right\}$. For each side $s \in S(\Phi)$ with $\partial(s) = (v_1) - (v_2)$, $v_1, v_2 \in V(\Phi)$, we define a diffeomorphism $\Psi^*[s]: C[v_1](s) \to C[v_2](s)$ by

$$\Psi^*[s](x, y) = (kx + ly, mx + ny)\ , \quad x, y \in \boldsymbol{R}/\boldsymbol{Z}\ ,$$

where $\Psi(s) = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$, $C[v_i](s) = S^1 \times \hat{C}_i$, and S^1, $\hat{C}_1$ and $\hat{C}_2$ are identified with $\boldsymbol{R}/\boldsymbol{Z}$. When the ends of s coincide and are v, we use the convention that $C[v_1](s) = C[v](s^+)$ and $C[v_2](s) = C[v](s^-)$. Now we obtain a closed connected manifold $M(\Phi, \Psi)$ from the disjoint union $\bigcup \{E[v] \mid v \in V(\Phi)\}$ by identifying $C[v_1](s)$ with $C[v_2](s)$ by $\Psi^*[s]$ for all $s \in S(\Phi)$.

Take a continuous map $\sigma: \Gamma[\Phi] = \bigcup \{\Gamma[v] \mid v \in V(\Phi)\} \to \{1, -1\}$. Then we have a foliation $\mathscr{F}(\Phi, \Psi; \sigma)$ of $M(\Phi, \psi)$ such that $\mathscr{F}(\Phi, \Psi; \sigma) \mid E[v] = \mathscr{F}(h(v); \sigma \mid \partial E[v])$ for all $v \in V(\Phi)$, where $\mathscr{F}(\Phi, \Psi; \sigma) \mid E[v]$ is the foliation induced from $\mathscr{F}(\Phi, \Psi; \sigma)$ by the canonical immersion $\iota: E[v] \to M(\Phi, \Psi)$.

We denote by $t_1^*(M(\Phi, \Psi), \mathscr{F}(\Phi, \Psi; \sigma))$ (or simply $t_1^*(\mathscr{F}(\Phi, \Psi; \sigma))$) the set of foliations $\mathscr{G}$ of $M(\Phi, \Psi)$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$ such that $\mathscr{G}|E[v]$ is transversely orientable. Clearly $t_1^*(\mathscr{F}(\Phi, \Phi; \sigma)) \supset t_1^0(\mathscr{F}(\Phi, \Psi; \sigma))$. Our main purpose is to investigate whether $t_1^*(\mathscr{F}(\Phi, \Psi; \sigma))$ is empty or not. Note that if Φ is a *tree* (that is, a connected contractible graph) then $\mathscr{F}(\Phi, \Psi; \sigma)$ is transversely orientable for all Ψ and σ and it follows that $t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) = t_1^0(\mathscr{F}(\Phi, \Psi; \sigma))$.

Our criterion for the existence of transverse foliations splits into two stages—an arithmetic criterion and a geometric one. Although the former is stronger than the homotopy theoretic one asking for the existence of transverse 2-plane fields (see Theorem 2 below), we do not know whether it is complete as a criterion or not. The latter is complete and takes the form of a jigsaw puzzle or a tangram (see Theorems 8, 8* and 8** in § 21).

Now we formulate the arithmetic criterion precisely.

DEFINITION 1.1. Let $(\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}} = \{(a, b) \in \boldsymbol{N} \times \boldsymbol{Z} \mid ap + bq = 1$ for some $p, q \in \boldsymbol{Z}\}$, where $\boldsymbol{N}$ is the set of positive integers. Let $(\boldsymbol{N} \times \boldsymbol{Z})^* = (\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}} \cup \{(0, 1), (\infty, \infty)\}$.

DEFINITION 1.2. An *arithmetic model* transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$ is a map $(a, b; r)\colon \Gamma[\Phi] \to (\boldsymbol{N} \times \boldsymbol{Z})^* \times 2\boldsymbol{Z}$, where $2\boldsymbol{Z}$ is the set of even integers, satisfying the following conditions (A1)-(A5).

(A1) Consider $v \in V(\Phi)$ with $h(v) = 1$, and let $\Gamma[v] = \{C\}$. Then $a(C) = 1$.

(A2) Consider $v \in V(\Phi)$ with $h(v) = 2$, and let $\Gamma[v] = \{C_1, C_2\}$. Then

(i) $r(C_2) = -r(C_1)$.

(ii) If $r(C_1) \neq 0$ and $a(C_1) > 0$, then $a(C_2) = a(C_1) \neq \infty$ and $b(C_2) = -b(C_1)$.

(iii) If $r(C_1) \neq 0$ and $a(C_1) = 0$, then $a(C_2) = 0$ and $\sigma(C_2) = -\sigma(C_1)$.

(A3) Consider $v \in V(\Phi)$ with $h(v) > 2$, and let $C \in \Gamma[v]$. If $(a(C), b(C)) \neq (1, 0)$, then $r(C) = 0$.

(A4) (The TS formula). For each $v \in V(\Phi)$,

$$\sum_{C \in \Gamma[v]} a(C) r(C) = 4 - 2h(v) ,$$

where we use the convention $\infty \cdot 0 = 0$.

(A5) (The compatibility condition). Let $s \in S(\Phi)$ with $\partial(s) = (v_1) - (v_2)$ and $C_i = C[v_i](s)$ for $i = 1, 2$.

(i) If $a(C_1) = \infty$, then $a(C_2) = \infty$ and $r(C_1) = r(C_2) = 0$.

(ii) If $a(C_1) \neq \infty$, then

$$\begin{pmatrix} a(C_2) \\ b(C_2) \end{pmatrix} = \gamma_1 \begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} a(C_1) \\ b(C_1) \end{pmatrix} \quad \text{and} \quad r(C_2) = \gamma_2 r(C_1) \ .$$

In the above, we put $\Psi(s) = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$ and

$$\gamma_1 = \begin{cases} \operatorname{sgn}(ka(C_1) + lb(C_1)) & \text{if} \quad a(C_2) > 0 \ , \\ \operatorname{sgn}(ma(C_1) + nb(C_1)) & \text{if} \quad a(C_2) = 0 \ . \end{cases}$$

Furthermore, we put $\delta(0) = 0$ and $\delta(a) = 1$ for $a > 0$, and

$$\gamma_2 = \gamma_1 \cdot \sigma(C_1)^{\delta(a(C_1))} \cdot \sigma(C_2)^{\delta(a(C_2))} \ .$$

We denote by $\operatorname{am}(\Phi, \Psi: \sigma)$ the set of arithmetic models transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$.

Now we can state the arithmetic criterion.

THEOREM 1. *There exists a canonical map*

$$\alpha: t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) \to \operatorname{am}(\Phi, \Psi; \sigma) \ .$$

Roughly speaking, if $\alpha(\mathscr{G}) = (a, b; r)$ for $\mathscr{G} \in t_1^*(\mathscr{F}(\Phi, \Psi; \sigma))$, then $(a(C), b(C))$ represents the homology class of a compact leaf of $\mathscr{G}|C$ and $r(C)$ is the difference of the numbers of the positive Reeb components (cf. Definition 4.1) and negative Reeb components of $\mathscr{G}|C$. The condition (A1) was already known in Davis and Wilson [1] and does not depend on the integrability of $\mathscr{G}|E[v]$. The condition (A3) reflects the integrability of $\mathscr{G}|E[v]$ (see Remark 19.3). The conditions (A4) and (A5) do not depend on the integrability of $\mathscr{G}$, but it is not clear whether (A2) does or not.

The following is a direct consequence of Theorem 1.

THEOREM 1*. (The arithmetic criterion). *If* $\operatorname{am}(\Phi, \Psi; \sigma) = \varnothing$, *then* $t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) = \varnothing$.

It is comparatively easy to see whether $\operatorname{am}(\Phi, \Psi; \sigma)$ is empty or not. We will give some examples in § 19. The following will be proved in § 24.

THEOREM 2. *If* $\operatorname{am}(\Phi, \Psi; \sigma) \neq \varnothing$, *then there is a 2-plane field of* $M(\Phi, \Psi)$ *transverse to* $\mathscr{F}(\Phi, \Psi; \sigma)$.

Our criterion is practical. The algorithm is as follows. First determine whether $\operatorname{am}(\Phi, \Psi; \sigma)$ is empty or not. When $\operatorname{am}(\Phi, \Psi; \sigma) = \varnothing$, we are done. When $\operatorname{am}(\Phi, \Psi; \sigma) \neq \varnothing$, try to construct a TS model (cf. Definition 20.7) transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$. In many cases we find a TS model. So far we did not find any $\mathscr{F}(\Phi, \Psi; \sigma)$ such that $\operatorname{am}(\Phi, \Psi; \sigma) \neq \varnothing$

and $t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) = \emptyset$. We give some examples in § 22. We hope that our criterion will give a hint to constructing a theoretical or general criterion. If a new criterion is found, then we can test it by the examples investigated by our criterion.

We wish to thank Professors I. Tamura and A. Sato for critical and valuable discussions.

PART I

A generalization of the results of Tamura and Sato

2. Turbulization I and Reeb components. We describe the *turbulization* precisely. Let W be a compact manifold with boundary and M a codimension-zero compact submanifold of ∂W. Let $\hat{\Gamma}$ be the set of connected components of M. Choose a small collar $\hat{k}: M \times [0, 1] \to W$ such that

$$\hat{k}(y, 0) = y \quad \text{for} \quad y \in M\,,$$
$$\hat{k}(y, t) \in \partial W \quad \text{for} \quad y \in \partial M \quad \text{and} \quad t \in [0, 1]\,.$$

Let $k: S^1 \times M \times [0, 1] \to S^1 \times W$ be the collar of $S^1 \times M$ defined by $k(x, y, t) = (x, \hat{k}(y, t))$ for $x \in S^1$, $y \in M$ and $t \in [0, 1]$. Let $W^\circ = \mathrm{Cl}\,(W - \hat{k}(M \times [0, 1]))$. Take a C^∞ function f: $]0, 1] \to]-\infty, 0]$ such that

(f1) $f(t) = 0$ for all $t \in [1/2, 1]$,

(f2) $\lim_{t\to 0} f(t) = -\infty$,

(f3) $df/dt > 0$ in $]0, 1/2[$,

(f4) the submanifolds $\boldsymbol{R} \times \{0\}$ and $F_c(f) = \{(f(t) + c, t) \mid t \in]0, 1]\}$, $c \in \boldsymbol{R}$, of $\boldsymbol{R} \times [0, 1]$ are leaves of a foliation of $\boldsymbol{R} \times [0, 1]$.

Take a continuous map $\sigma: M \to \{1, -1\}$. Let $\mathscr{F}$ be a foliation of $S^1 \times W$ such that $\mathscr{F} \mid k(S^1 \times M \times [0, 1]) = \{\{x\} \times \hat{k}(M \times [0, 1]) \mid x \in S^1\}$. Then the foliation $T[\mathscr{F}, M, \sigma]$ obtained by *turbulizing* $\mathscr{F}$ *around* M *in the direction of* σ is defined so that $T[\mathscr{F}, M, \sigma] \mid S^1 \times W^\circ = \mathscr{F} \mid S^1 \times W^\circ$ and that $T[\mathscr{F}, M, \sigma] \mid k(S^1 \times M \times [0, 1])$ consists of compact leaves $S^1 \times \hat{C}$ for $\hat{C} \in \hat{\Gamma}$ and non-compact leaves

$$\{k([\sigma(y)f(t)] + x, y, t) \mid y \in \hat{C}, t \in [0, 1]\}$$

for $x \in S^1 = \boldsymbol{R}/\boldsymbol{Z}$ and $\hat{C} \in \hat{\Gamma}$, where $[z]$ means z mod 1.

Now consider $E(h) = S^1 \times \hat{E}(h)$, $\Gamma(h)$ and $\sigma: \partial E(h) \to \{1, -1\}$ as in § 1. Let $\hat{E}(h)$ (or $\partial\hat{E}(h)$) play the role of W (or M respectively). (Below we omit the word "respectively" in the similar description.) Then we have the turbulized foliation $\mathscr{F}(h; \sigma)$ in § 1:

$$\mathscr{F}(h; \sigma) = T[\mathscr{F}(h, \mathrm{pr}), \partial\hat{E}(h), \sigma]\,.$$

Therefore $\mathscr{F}(h;\sigma)$ consists of compact leaves $C \in \Gamma(h)$ and non-compact leaves

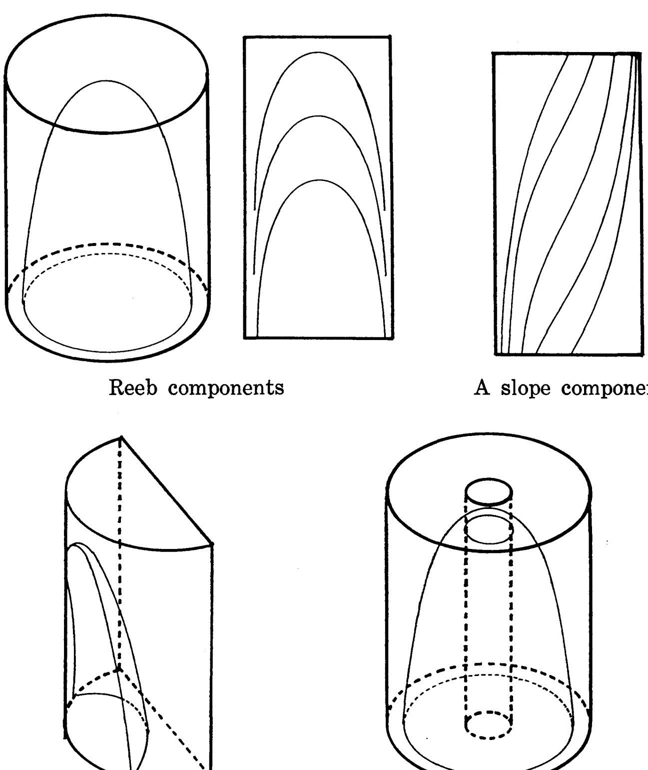
$$F^x = S^1 \times \hat{E}(h)^\circ \cup k(\{([\sigma(y)f(t)] + x,\, y,\, t) \,|\, y \in \partial\hat{E}(h),\, t \in [0, 1]\})$$

for $x \in S^1$, where $\hat{E}(h)^\circ$ and $k\colon \partial E(h) \times [0, 1] \to E(h)$ are constructed as above.

We recall some definitions.

DEFINTION 2.1. Let $\mathscr{F}_{\mathrm{pr}}^{n+1}$ be the product foliation $\{\{x\} \times D^n \,|\, x \in S^1\}$ of $S^1 \times D^n$, and $\sigma\colon \partial D^n \to \{1, -1\}$ be a constant map. Then the turbulized foliation $\mathscr{F}_R^{n+1}(\sigma) = T[\mathscr{F}_{\mathrm{pr}}^{n+1}, \partial D^n, \sigma]$ is called a *standard Reeb component* of $S^1 \times D^n$. We called $\mathscr{F}_R^{n+1}(1)$ *plus* and $\mathscr{F}_R^{n+1}(-1)$ *minus* in Tamura-Sato [15].

DEFINITION 2.2. Let $\sigma\colon \partial D^1 \to \{1, -1\}$ be a bijection. Then $T[\mathscr{F}_{\mathrm{pr}}^2, \partial D^1, \sigma]$ is called a *standard slope component of* $S^1 \times D^1$.



Reeb components A slope component

A half Reeb component A tunneled Reeb component

FIGURE 2.1 The top and the bottom are to be glued

DEFINITION 2.3. Let $D^n_+=\{(x_1,\cdots,x_n)\in D^n\,|\,x_n\geqq 0\}$. Then $T[\mathscr{F}^{n+1}_{\mathrm{pr}}\,|\,S^1\times D^n_+,\ D^{n-1}\times\{0\},\ \pm 1]$ is called a *standard half Reeb component* of $S^1\times D^n_+$.

The following is a new component appearing in our decomposition theorem but not being contained in foliations transverse to the Reeb component $\mathscr{F}^3_R(1)$ of $S^1\times D^2$.

DEFINITION 2.4. Let $D^n_{[1/2,1]}=\{(x_1,\cdots,x_n)\in R^n\,|\,1/4\leqq x_1^2+\cdots+x_n^2\leqq 1\}$. Then $T[\mathscr{F}^{n+1}_{\mathrm{pr}}\,|\,S^1\times D^n_{[1/2,1]},\ \partial D^n,\ \pm 1]$ is called a *standard tunneled Reeb component* of $S^1\times D^n_{[1/2,1]}$.

DEFINITION 2.5. Let $(M_1,\mathscr{F}_1)$ and $(M_2,\mathscr{F}_2)$ be C^r foliated manifolds. We say that $\mathscr{F}_1$ is *C^r isomorphic* to $\mathscr{F}_2$ if there is a C^r diffeomorphism $\phi\colon M_1\to M_2$ with $\mathscr{F}_1=\phi^*\mathscr{F}_2$.

DEFINITION 2.6. A foliation $\mathscr{F}$ is called a *Reeb* (or *slope, half Reeb, tunneled Reeb*, etc.) *component* if $\mathscr{F}$ is C^0 isomorphic to a standard Reeb (or slope, half Reeb, tunneled Reeb, etc.) component.

For better understanding, we give some figures in Figure 2.1.

3. Turbulization II and several components I. For some foliations of $S^1\times S^1\times[0,1]$, we can introduce a somewhat sophisticated type of turbulizations, as follows. The foliations thus obtained will appear in the decomposition theorem for $\mathscr{G}\in t^0_1(\mathscr{F}(h;\sigma))$.

As the data, we take a transversely orientable foliation $\mathscr{G}_0$ of $S^1\times S^1$ without Reeb components, a positive integer μ_0, a map $\sigma\colon\{1,\cdot\cdot,\mu_0\}\to\{1,-1\}$ and an element $(a,b)\in(N\times Z)^{\mathrm{coprime}}$ such that there is a closed transversal L intersecting all the leaves of $\mathscr{G}_0$ with $[L]=a[S^1\times\{*\}]+b[\{*\}\times S^1]$ in $H_1(S^1\times S^1;Z)$. What we will turbulize is the product foliation $\mathscr{G}_0\times I$, where I means the interval $[0,1]$.

Put $\mu=a\cdot\mu_0$. Let $M_i=\{[y]\in S^1=R/Z\,|\,(i-1)/\mu\leqq y\leqq(2i-1)/2\mu\}$ for $i=1,\cdots,\mu$, and $M=M_1\cup\cdots\cup M_\mu$. Let $M^*=\{([at],[bt]+y)\in S^1\times S^1\,|\,t\in R,\ y\in M\}$. Then $M^*\cap\{[0]\}\times S^1=M$, and M^* has μ_0 connected components. Let M_i^* be the connected component of M^* containing $\{[0]\}\times M_i$ for $i=1,\cdots,\mu_0$.

We can construct a diffeomorphism $\alpha\colon S^1\times S^1\times I\to S^1\times S^1\times I$ satisfying the following conditions (1)-(3).

(1) $\alpha(S^1\times S^1\times\{t\})=S^1\times S^1\times\{t\}$ for all $t\in I$.

(2) $\alpha\,|\,S^1\times S^1\times[2/3,1]=\mathrm{id}$.

(3) There is a neighborhood U of M^* in $S^1\times S^1$ such that the leaves of $\alpha^*(\mathscr{G}_0\times I)\,|\,U\times[0,1/3]$ are connected components of $(\{([-bt],[at]+y)\,|\,t\in R\}\cap U)\times[0,1/3]$ for some $y\in S^1$. (See Figure 3.1.)

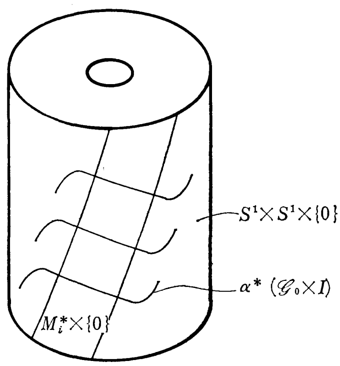


FIGURE 3.1

Bend $S^1 \times S^1 \times I$ along $\partial M^* \times \{0\}$ so that $\partial M^* \times \{0\}$ is a corner. Choose a small collar $k: M^* \times I \to U \times [0, 1/3]$ such that $k(\partial M^* \times I) \subset U \times \{0\}$ and that the leaves of $k^*\alpha^*(\mathscr{G}_0 \times I)$ are connected components of $(\{([-bt], [at] + y) | t \in \boldsymbol{R}\} \cap M^*) \times I$ for $y \in S^1$. Then the turbulized foliation $T[\mathscr{G}_0; \mu_0, \sigma; a, b]$ is defined so that $T[\mathscr{G}_0; \mu_0, \sigma; a, b] = \alpha^*(\mathscr{G}_0 \times I)$ on $S^1 \times S^1 \times I - k(M^* \times I)$ and that $T[\mathscr{G}_0; \mu_0, \sigma; a, b] | k(M^* \times I)$ consists of compact leaves $M_i^* \times \{0\}$, $i = 1, \cdots, \mu$, and non-compact leaves

$$\{k([a\sigma(i)f(t/3)] + x, [b\sigma(i)f(t/3)] + y, t) | t \in]0, 1], (x, y, 0) \in F\}$$

for leaves F of $k^*\alpha^*(\mathscr{G}_0 \times I) | M_i^* \times \{0\}$, $i = 1, \cdots, \mu_0$, where we use $f:]0, 1] \to]-\infty, 0]$ in § 2. (See Figure 3.2.)

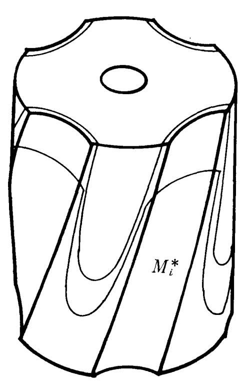


FIGURE 3.2

DEFINITION 3.1. We call $T[\mathscr{G}_0; \mu_0, \sigma; a, b]$ a *standard gear component* if σ is constant.

The definitions below are not used until § 9, and it is possible to omit them till then.

When two standard gear components $\mathscr{F}_1 = T[\mathscr{G}_1; \mu_1, \sigma_1; a_1, b_1]$ and $\mathscr{F}_2 = T[\mathscr{G}_2; \mu_2, \sigma_2; a_2, b_2]$ with $\mathscr{G}_1 = \mathscr{G}_2$ are given, we can glue $\mathscr{F}_1$ and $\mathscr{F}_2$ by identifying $\mathscr{G}_1 \times \{1\}$ and $\mathscr{G}_2 \times \{1\}$, and obtain a foliation of a manifold homeomorphic to $S^1 \times S^1 \times I$.

DEFINITION 3.2. The foliation obtained above is called a *standard double gear component* if the values of σ_1 and σ_2 are different.

When $\mathscr{G}_0$ is the foliation $\mathscr{F}_{\mathrm{pr}} = \{\{x\} \times S^1 | x \in S^1\}$, we can glue the foliation $\{\{x\} \times D^2 | x \in S^1\}$ to $T[\mathscr{F}_{\mathrm{pr}}; \mu_0, \sigma; a, b]$, and obtain a foliation $T^*[\mathscr{F}_{\mathrm{pr}}; \mu_0, \sigma; a, b]$.

DEFINITION 3.3. We call $T^*[\mathscr{F}_{\mathrm{pr}}; \mu_0, \sigma; a, b]$ a *standard arcade component* if $\mu_0 > 1$ and σ is constant.

DEFINITION 3.4. Let q_1 and q_2 be non-negative integers with $q_1 + q_2 > 0$. We call $T^*[\mathscr{F}_{\mathrm{pr}}; \mu_0, \sigma; 1, b]$ a *standard* TS′ *component of type* (q_1, q_2) if $\mu_0 = q_1 + q_2 + 2$ and

$$\sigma(j) = \begin{cases} 1 & \text{for } j = 1, \cdots, q_1 + 1\,, \\ -1 & \text{for } j = q_1 + 2, \cdots, \mu_0\,. \end{cases}$$

REMARK 3.5. A TS′ component of type $(0, q)$ is a TS component of type q defined in Tamura-Sato [15].

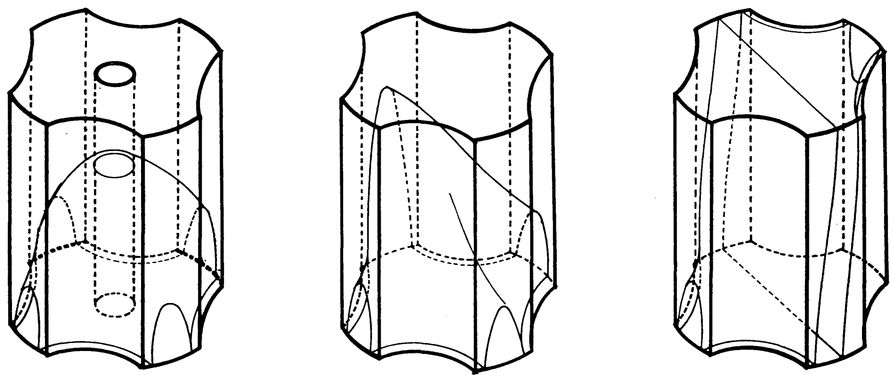
When the leaves of $\mathscr{G}_0$ are all compact, we can turbulize $T[\mathscr{G}_0; \mu_0, \sigma; a, b]$ around $S^1 \times S^1 \times \{1\}$ in the directions orthogonal to $\mathscr{G}_0$.

DEFINITION 3.6. The foliation obtained above by turbulization is called a *standard turbulized gear component* if σ is constant and the turbulization around $S^1 \times S^1 \times \{1\}$ is performed in the direction of $-\sigma(1)(a'[S^1 \times \{*\}] + b'[\{*\} \times S^1])$, where $(a', b') \in (\boldsymbol{N} \times \boldsymbol{Z})^{\mathrm{coprime}} \cup \{(0, 1)\}$ with $aa' + bb' = 0$.

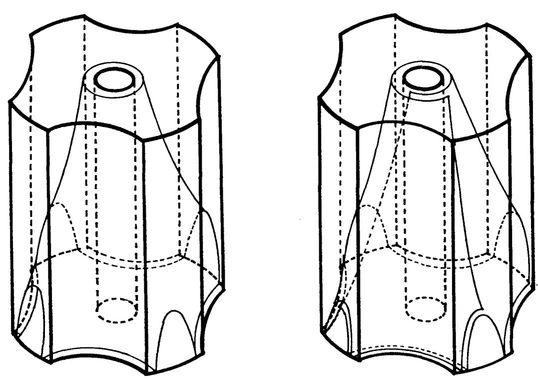
Let $\mathscr{G}$ be a standard turbulized gear component obtained from $T[\mathscr{G}_0; \mu_0, \sigma; a, b]$. We may suppose that $K = \{([at], [bt + (1/4\mu)]) | t \in \boldsymbol{R}\} \times I$ is transverse to $\mathscr{G}$. Then $\mathscr{G} | K$ is a slope component. Therefore $\mathscr{G} | K$ admits a smooth S^1 action (see Imanishi-Yagi [6], Fukui-Ushiki [3] and Fukui [2]) if the turbulization is carefully performed. Let $\beta\colon K \to K$ be a diffeomorphism such that β maps each non-compact leaf of $\mathscr{G} | K$ to a different leaf of $\mathscr{G} | K$. Cutting $S^1 \times S^1 \times I$ along K and pasting by β, we have a foliation $\mathscr{G}'$ of manifold homeomorphic to $S^1 \times S^1 \times I$.

DEFINITION 3.7. The foliation $\mathscr{G}'$ obtained above is called a *standard perturbed gear component*.

We give some figures.



A gear component An arcade component A TS′ component



A turbulized gear component A perturbed gear component

FIGURE 3.3

4. Preliminaries, a lemma of Tamura and Sato, and the TS formula. Let $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$. We give some remarks on $\mathscr{G} | C$ for $C \in \Gamma(h)$. Note that C is diffeomorphic to T^2. When $\mathscr{G} | C$ has a compact leaf L, the homology class $[L] \in H_1(C; \boldsymbol{Z})$ depends only on $\mathscr{G} | C$. We call C *vertical* if $\mathscr{G} | C$ has no compact leaf homologous to $\{*\} \times \hat{C}$, and otherwise *horizontal*. If there is an immersion $g: S^1 \times D^1 \to C$ such that $g | \mathrm{Int}\,(S^1 \times D^1)$ is an imbedding and $g^* \mathscr{G}$ is an Reeb component, then g is an imbedding, and $\mathscr{G} | C$ contains an even number of Reeb components, since $\mathscr{G} | C$ is transversely orientable. As in Tamura-Sato [15], we can construct a C^∞ isotopy $\{\phi_t\}_{t \in \boldsymbol{R}} \subset \mathrm{Diff}\,(E(h))$ satisfying the following conditions (E1)-(E5).

(E1) $\phi_t = \mathrm{id}$ for $t \leqq 0$, and $\phi_t = \phi_1$ for $t \geqq 1$.

(E2) $\phi_t^* \mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$ for all $t \in \boldsymbol{R}$.

(E3) When $\mathscr{G} | C$ has no compact leaf for $C \in \Gamma(h)$, each leaf of $\phi_1^* \mathscr{G} | C$ is transverse to $\{x\} \times \hat{C}$ and $S^1 \times \{y\}$ for all $x \in S^1$ and $y \in \hat{C}$.

(E4) When C is horizontal, each compact leaf L of $\phi_1^*\mathscr{G}|C$ has the form $L = \{x\} \times \hat{C}$ for some $x \in S^1$.

(E5) When C is vertical and $\mathscr{G}|C$ has a compact leaf, each compact leaf L has the form $L = \{([at] + x, [bt]) | t \in \boldsymbol{R}\}$ for some $a \in \boldsymbol{N}$ and $b \in \boldsymbol{Z}$, and for each Reeb component $\mathscr{R}$ contained in $\phi_1^*\mathscr{G}$ there is a circle $\Sigma(\mathscr{R}) \subset \mathrm{Int}\,|\mathscr{R}|$ such that $\phi_1^*\mathscr{G}$ is tangent to the curves $\{x\} \times \hat{C}$, $x \in S^1$, at and only at $\Sigma(\mathscr{R})$.

Since it is sufficient for our purpose to consider $\phi_1^*\mathscr{G}$ instead of $\mathscr{G}$, *hereafter we treat* $\phi_1^*\mathscr{G}$ *and denote it by* $\mathscr{G}$ *for simplicity.*

Since $\mathscr{G}$ is transverse to $\partial E(h)$, there is $\varepsilon > 0$ such that $\mathscr{G}$ is transverse to $k(\partial E(h) \times \{t\})$ for all $t \in [0, \varepsilon]$, where $k: \partial E(h) \times [0, 1] \to E(h)$ is the collar used in the definition of $\mathscr{F}(h; \sigma)$. Let $A = E(h) - k(\partial E(h) \times [0, \varepsilon[)$ and $A^x = F^x \cap A$ for $x \in S^1$, where F^x is the non-compact leaf defined in § 2. Let $\partial_C A = k(C \times \{\varepsilon\})$ and $\partial_C A^x = F^x \cap \partial_C A$ for $C \in \Gamma(h)$. Then $\partial A = \bigcup \{\partial_C A | C \in \Gamma(h)\}$. We have a diffeomorphism $\gamma: \partial E(h) \to \partial A$ such that $z \in \partial E(h)$ and $\gamma(z)$ belong to the same leaf of $\mathscr{G}$. We may assume the following conditions.

(E4)′ If $C \in \Gamma(h)$ is horizontal, then $\partial_C A^{[0]}$ is a compact leaf of $\mathscr{G}|\partial A$, where $[0] \in S^1 = \boldsymbol{R}/\boldsymbol{Z}$ means $0 \bmod 1$.

(E5)′ If $C \in \Gamma(h)$ is vertical, then for each Reeb component $\mathscr{R}$ contained in $\mathscr{G}|\partial_C A$ there is a circle $\Sigma(\mathscr{R}) \subset \mathrm{Int}\,|\mathscr{R}|$ such that $\mathscr{G}|\partial_C A$ is tangent to the curves $\partial_C A^x$, $x \in S^1$, at and only at $\Sigma(C) = \bigcup \{\Sigma(\mathscr{R}) | \mathscr{R}$ is a Reeb component contained in $\mathscr{G}|\partial_C A\}$, where $|\mathscr{R}|$ means the underlying manifold of $\mathscr{R}$. (See Figure 4.1.)

In order to recall a lemma in Tamura-Sato [15], we make preperations. Let $C \in \Gamma(h)$ be vertical. When $\mathscr{G}|C$ has a compact leaf L, we appoint the orientation of L so that $[L] = a[S^1 \times \{*\}] + b[\{*\} \times \hat{C}]$ in $H_1(C; \boldsymbol{Z})$ for some $a \in \boldsymbol{N}$ and $b \in \boldsymbol{Z}$.

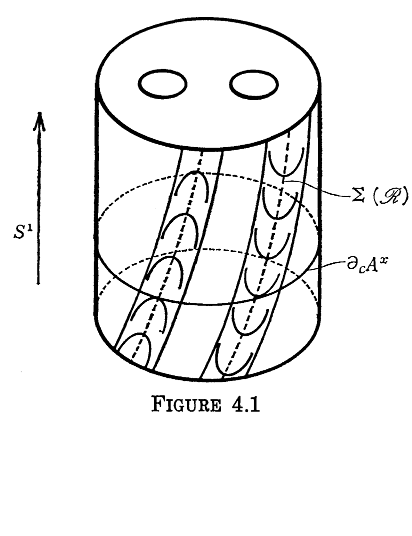


FIGURE 4.1

DEFINITION 4.1. In the above situation, a Reeb component $\mathscr{R}$ contained in $\mathscr{G}\,|\,C$ is called *positive* (or *negative*) if a compact leaf L of $\mathscr{R}$ has an expanding (or contracting) holonomy in the direction of $\sigma(C)\cdot L$. A Reeb component $\mathscr{R}$ contained in $\mathscr{G}\,|\,\partial_C A$ is called *positive* (or *negative*) if $\gamma^*\mathscr{R}$ is positive (or negative).

The lemma which we need is the following.

LEMMA 4.2 [15, Lemma 1]. *Let $\mathscr{R}$ be a Reeb component contained in $\mathscr{G}\,|\,\partial_C A$ and $\Sigma(\mathscr{R}) \cap A^x = \{z\}$. If $\mathscr{R}$ is positive, then $\mathscr{G}\,|\,A^x$ forms a family of concentric half circles with center z in a neighborhood of z. If $\mathscr{R}$ is negative, then $\mathscr{G}\,|\,A^x$ forms a family of confocal parabolas in a neighborhood of z.* (*See Figure* 4.2.)

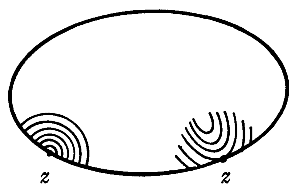


FIGURE 4.2

Now we introduce the TS formula for $\mathscr{G}$. When $C \in \Gamma(h)$ is vertical and $\mathscr{G}\,|\,C$ has no compact leaf, let $a(C) = b(C) = \infty$ and $r(C) = 0$. When $C \in \Gamma(h)$ is vertical and $\mathscr{G}\,|\,C$ has a compact leaf L, define $(a(C), b(C)) \in (\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}}$ by

$$[L] = a(C)[S^1 \times \{*\}] + b(C)[\{*\} \times \hat{C}] \quad \text{in} \quad H_1(C; \boldsymbol{Z}) \, ,$$

and let $r(C) = p(C) - q(C)$, where $p(C)$ (or $q(C)$) is the number of positive (or negative) Reeb components of $\mathscr{G}\,|\,C$. When $C \in \Gamma(h)$ is horizontal, let $(a(C), b(C)) = (0, 1)$ and define $r(C)$ in the same way as above but by replacing $\mathscr{G}\,|\,C$ by $c^*(\mathscr{G}\,|\,C)$, where $c\colon S^1 \times \hat{C} \to S^1 \times \hat{C}$ is defined by $c(x, y) = (y, x)$ for $x \in S^1$, $y \in \hat{C} = S^1$.

PROPOSITION 4.3 (The TS formula). *In the above situation,*

$$\sum_{C \in \Gamma(h)} a(C) r(C) = 4 - 2h \, ,$$

where we use the convention that $\infty \cdot 0 = 0$.

PROOF. Regard $\mathscr{G}\,|\,A^{[0]}$ as the set of orbits of a vector field Y by giving an orientation. By patching two copies of $A^{[0]}$ along $\partial A^{[0]}$, we have a closed manifold W. We obtain a vector field $\tilde{Y}$ on W from $Y \cup (-Y)$. By Lemma 4.2, the vector field $\tilde{Y}$ has p (or q) singular points of index

1 (or -1), where

$$p = \sum_{C \in \Gamma(h)} a(C)p(C) \qquad \text{and} \qquad q = \sum_{C \in \Gamma(h)} a(C)q(C) .$$

Since the Euler number of W equals $4-2h$, we have the formula.

5. The characteristic diffeomorphism of $\mathscr{G} \in t_1^0(\mathscr{F}(h;\sigma))$ and the projection of a leaf of $\mathscr{G}|C$ to F^x. Let $\mathscr{G} \in t_1^0(\mathscr{F}(h;\sigma))$. Take a vector field X of $E(h)$ tangent to $\mathscr{G}$ and transverse to $\mathscr{F}(h;\sigma)$ such that X is inward (or outward) at $y \in \partial E(h)$ with $\sigma(y)=1$ (or -1). We may suppose that $y \in \partial E(h)$ and $\gamma(y)$ is on the same orbit of X. For $x \in S^1$, let F^x be the non-compact leaf of $\mathscr{F}(h;\sigma)$ defined in § 2. Since F^x is proper, for each $y \in F^x$ there is the first point $\psi_x(y)$, of the orbits of X starting from y, intersecting F^x. Then $\psi_x(y)$'s give rise to a diffeomorphism $\psi_x\colon F^x \to F^x$.

DEFINITION 5.1. We call ψ_x above the *characteristic diffeomorphism* of $\mathscr{G}$ with respect to X for F^x.

DEFINITION 5.2. For a subset B of $\partial E(h)$, the *real projection* $\mathrm{RP}_x(B)$ of B to F^x along X is the set of $z \in F^x$ such that the orbit of X passing through z intersects B. For a leaf L of $\mathscr{G}|\partial E(h)$, the *projection* $\mathrm{P}_x(L)$ of L to F^x along X is the saturation of $\mathrm{RP}_x(L)$ with respect to $\mathscr{G}|F^x$. We denote by $\mathrm{P}_x^*(L)$ the set of leaves of $\mathscr{G}|F^x$ contained in $\mathrm{P}_x(L)$.

Clearly $\psi_x^*(\mathscr{G}|F^x) = \mathscr{G}|F^x$ and $\psi_x(\mathrm{P}_x(L)) = \mathrm{P}_x(L)$. The set $\mathrm{RP}_x(L)$ is open in $\mathrm{P}_x(L)$. For disjoint subsets B and B' of $\partial E(h)$, it follows that $\mathrm{RP}_x(B) \cap \mathrm{RP}_x(B') = \varnothing$ if $\sigma|B \cup B'$ is constant. Furthermore ψ_x and $\mathrm{P}_x(L)$ have the following useful properties.

PROPOSITION 5.3. *Let L be a leaf of $\mathscr{G}|\partial E(h)$.*

(1) *The group $\{\psi_x^n \mid n \in \boldsymbol{Z}\}$ acts transitively on the set $\mathrm{P}_x^*(L)$.*

(2) *Let L' be another leaf of $\mathscr{G}|\partial E(h)$. If $\mathrm{P}_x(L) \cap \mathrm{P}_x(L') \neq \varnothing$, then $\mathrm{P}_x(L) = \mathrm{P}_x(L')$.*

(3) *If L is a compact leaf and $C \in \Gamma(h)$ with $L \subset C$ is vertical, then $\#\mathrm{P}_x^*(L) = a(C)$, where $a(C)$ was defined in § 4.*

(4) *If $\#\mathrm{P}_x^*(L) < \infty$, then $C \in \Gamma(h)$ with $L \subset C$ is vertical and L is a compact leaf or a non-compact leaf of a negative Reeb component contained in $\mathscr{G}|C$.*

PROOF. (1) Let K_1 and K_2 be leaves of $\mathscr{G}|F^x$ contained in $\mathrm{P}_x(L)$. By definition of $\mathrm{P}_x(L)$, there are points y_1 and $y_2 \in L$ such that the orbit of X passing through y_i intersects K_i at some point z_i, $i=1, 2$. Since L is connected, there is a path $\omega\colon I \to L$ with $\omega(0)=y_1$ and $\omega(1)=y_2$. Transporting ω along X, we have a path $\bar{\omega}\colon I \to K$ such that $\bar{\omega}(0)$ equals

z_1 and $\omega(t)$ and $\bar{\omega}(t)$ are on the same orbit of X. Then $\bar{\omega}(1) \in K_1$, $y_2 = \omega(1)$ and $z_2 \in K_2$ are on the same orbit of X. This implies that $\psi_x^n(\omega(1)) = z_2$ for some $n \in \boldsymbol{Z}$. Then $\psi_x^n(K_1) = K_2$.

(2) Suppose that $\mathrm{P}_x(L) \cap \mathrm{P}_x(L') \neq \emptyset$. Then $\mathrm{P}_x^*(L) \cap \mathrm{P}_x^*(L') \neq \emptyset$. Let $K_0 \in \mathrm{P}_x^*(L) \cap \mathrm{P}_x^*(L')$. Then for each $K \in \mathrm{P}_x^*(L)$ there is $n \in \boldsymbol{Z}$ with $\psi_x^n(K_0) = K$ by (1). Since $\psi_x^n(\mathrm{P}_x^*(L')) = \mathrm{P}_x^*(L')$ by (1), it follows that that $K \in \mathrm{P}_x^*(L')$. This implies that $\mathrm{P}_x(L) \subset \mathrm{P}_x(L')$. In the same way we have $\mathrm{P}_x(L') \subset \mathrm{P}_x(L)$.

In order to prove (3) and (4), we make preparations. Let $H = \{[0]\} \times C$. Then $\mathrm{RP}_x(H)$ consists of an infinite number of circles and we can number them so that $\mathrm{RP}_x(H) = \{H_i \mid i \in \boldsymbol{Z}\}$ and that if $i < j$ then H_i is between H and H_j in $E(h)$. With respect to the topology of F^x, the set $\bigcap_{n \in \boldsymbol{Z}} \mathrm{Cl}\,(\bigcup_{i<n} H_i)$ is empty. For each i, the connected component F_i of $F^x - H_i$ containing H_{i-1} is diffeomorphic to $S^1 \times \boldsymbol{R}$ and the closure of F_i in F^x is not compact. Furthermore the decreasing sequence $F_0, F_{-1}, \cdots$ determines an end ε of F^x with $L_\varepsilon(F^x) = C$, where $L_\varepsilon(F^x)$ is the ε-limit set of F^x (see Nishimori [8]).

LEMMA 5.4. *Suppose that C is vertical.*

(1) *If L is a leaf of $\mathscr{G} \mid C$ contained in $C - \bigcup \{\mathrm{Int} \mid \mathscr{R} \mid \mid \mathscr{R}$ is a Reeb component contained in $\mathscr{G} \mid C\}$, then each connected component of $\mathrm{RP}_x(L)$ intersects H_i at exactly one point for all $i \in \boldsymbol{Z}$.*

(2) *For a Reeb component $\mathscr{R}$ contained in $\mathscr{G} \mid C$, the real projection $\mathrm{RP}_x(|\mathscr{R}|)$ of $|\mathscr{R}|$ intersects H_i for all $i \in \boldsymbol{Z}$, and $\mathscr{G} \mid (\mathrm{RP}_x(|\mathscr{R}|) \cap F_i)$ is as in Figure 5.1.*

PROOF. (1) is clear and (2) follows from Lemma 4.2.

PROOF OF PROPOSITION 5.3 CONTINUED. (3) Suppose that L is compact and C is vertical. Since $H \cap L$ is finite, so is the set $H_i \cap \mathrm{RP}_x(L)$ for all $i \in \boldsymbol{Z}$. By Lemma 5.4 (1), we have $\sharp \mathrm{P}_x^*(L) < \infty$.

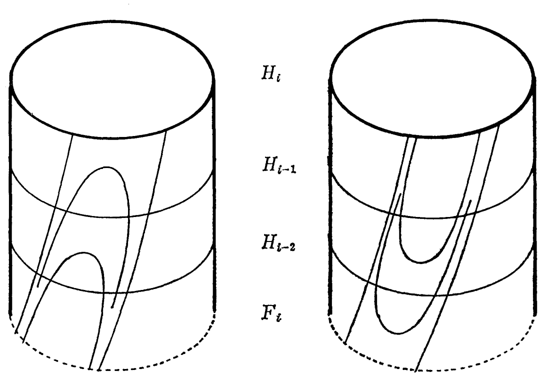


FIGURE 5.1

(4) Suppose that $\# \mathrm{P}_x^*(L) < \infty$. Then C is vertical. For, otherwise, we may suppose that $H = \{[0]\} \times \hat{C}$ is a compact leaf and it is easy to see that for each leaf L' of $\mathscr{G}|C$ the intersection $\mathrm{P}_x(L') \cap (F_i - F_{i-1})$ consists of exactly one leaf of $\mathscr{G}|F^x$. Therefore $\# \mathrm{P}_x^*(L) = \infty$, which is a contradiction.

We see that L is not a non-compact leaf contained in $C - \bigcup\{|\mathscr{R}|\,|\mathscr{R}$ is a Reeb component contained in $\mathscr{G}|C\}$, as follows. Suppose the contrary. Then L intersects H infinitely many times, and $\#(H_i \cap \mathrm{RP}_x(L)) = \infty$ for each $i \in \boldsymbol{Z}$. Since the foliation $\mathscr{G}|C$ is orientable, the curves $K \cap \mathrm{RP}_x(L)$ for $K \in \mathrm{P}_x^*(L)$ cross the circle H_0 in the same direction when $\mathscr{G}|F^x$ is oriented. Futhermore the closure of each connected component of $\mathrm{RP}_x(L)$ with respect to the topology of F^x is non-compact by Lemma 5.4 (1). Therefore each $K \in \mathrm{P}_x^*(L)$ intersects $H_0 \cap \mathrm{RP}_x(L)$ at exactly one point. Thus we have $\# \mathrm{P}_x^*(L) = \infty$, which is contradiction.

We see that L is not a non-compact leaf of a positive Reeb component contained in $\mathscr{G}|C$, as follows. Suppose the contrary. Then $\#(H_i \cap \mathrm{RP}_x(L)) = \infty$ for each $i \in \boldsymbol{Z}$, By Lemma 5.4 (2), the closure of each connected component of $\mathrm{RP}_x(L)$ in F^x is non-compact. Therefore each $K \in \mathrm{P}_x^*(L)$ intersects $H_0 \cap \mathrm{RP}_x(L)$ at at most two points. We have a contradiction as above. This completes the proof of Proposition 5.3.

6. Negative Reeb cycles. In this section we investigate negative Reeb cycles defined below, which can be regarded as a preparation for the next section. Let $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$.

DEFINITION 6.1. A *negative Reeb chain* of $\mathscr{G}$ is a finite ordered set $\mathscr{C} = (\mathscr{N}_1, \cdots, \mathscr{N}_n)$, $n \geqq 1$, of negative Reeb components contained in $\mathscr{G}|\partial E(h)$ such that

(1) $\mathrm{P}_x(N_i^2) = \mathrm{P}_x(N_{i+1}^1)$ for $i = 1, \cdots, n-1$, where N_i^1 and N_i^2 are the compact leaves of $\mathscr{N}_i$, and

(2) $\mathrm{Int}|\mathscr{N}_i|$ and $\mathrm{Int}|\mathscr{N}_{i+1}|$ are in the same side of the compact leaf G_i of $\mathscr{G}$ containing N_i^2 and N_{i+1}^1, for $i = 1, \cdots, n-1$.

We denote N_1^1, N_n^2, $\mathscr{N}_1$ and $\mathscr{N}_n$ by $o(\mathscr{C})$, $e(\mathscr{C})$, $\mathscr{N}_o(\mathscr{C})$ and $\mathscr{N}_e(\mathscr{C})$, respectively.

DEFINITION 6.2. A *negative Reeb cycle* of $\mathscr{G}$ is a negative Reeb chain $\mathscr{C}$ of $\mathscr{G}$ with $e(\mathscr{C}) = o(\mathscr{C})$. A negative Reeb cycle $\mathscr{C}$ is called *strange* if there are a vertical $C \in \Gamma(h)$ and a compact leaf L of $\mathscr{G}|C$ with $\mathrm{P}_x(L) \cap \mathrm{RP}_x(\mathrm{Int}\,|\mathscr{N}_o(\mathscr{C})|) \neq \emptyset$.

Note that if $\mathscr{G}$ contains a gear component $\mathscr{G}_0$ then $\mathscr{G}_0$ contains a negative Reeb cycle.

First we have the following.

PROPOSITION 6.3. *Let* $\mathscr{C} = (\mathscr{N}_1, \cdots, \mathscr{N}_n)$ *be a negative Reeb chain. Then* $\sigma|(|\mathscr{N}_1| \cup \cdots \cup |\mathscr{N}_n|)$ *is constant.*

PROOF. Let G_i be the compact manifold of $\mathscr{G}$ containing N_i^2 and N_{i+1}^2, as in Definition 6.1. A consideration on the holonomy of G_i on the side of $|\mathscr{N}_i|$ tells us that $\sigma|(|\mathscr{N}_i| \cup |\mathscr{N}_{i+1}|)$ is constant. Then the proposition follows.

Let $\mathscr{C} = (\mathscr{N}_1, \cdots, \mathscr{N}_n)$ be a negative Reeb chain. We use the notations in Definition 6.1. Let us consider the holonomy of leaves in $\mathrm{P}_x^*(N_i^j)$ with respect to $\mathscr{G}|F^x$. For each i, there is a unique bijection $\alpha_i\colon \mathrm{P}_x^*(N_i^1) \to \mathrm{P}_x^*(N_i^2)$ such that $M \in \mathrm{P}_x^*(N_i^1)$ and $\alpha_i(M)$ intersect the same connected component of $\mathrm{RP}_x(|\mathscr{N}_i|)$. For each $M \in \mathrm{P}_x^*(N_i^j)$, take a small line segment $T^j(M)$ in $\mathrm{RP}_x(|\mathscr{N}_i|)$ transverse to $\mathscr{G}|F^x$ with an endpoint $z^j(M)$ in M. A *local homeomorphism* $\phi\colon (X, x_0) \to (Y, y_0)$ means a homeomorphism from a neighborhood of x_0 in X to a neighborhood of y_0 in Y with $\phi(x_0) = y_0$. For each $M \in \mathrm{P}_x^*(N_i^1)$, there is a local homeomorphism $h[M]\colon (T^1(M), z^1(M)) \to (T^2(\alpha_i(M)), z^2(\alpha_i(M)))$ such that $z \in \mathrm{Dom}\,(h[M])$ and $h[M](z)$ are on the same leaf of $\mathscr{G}|\mathrm{RP}_x(|\mathscr{N}_i|)$, where $\mathrm{Dom}\,(h[M])$ is the domain of $h[M]$. Furthermore there is a local homeomorphism $k[M]\colon (T^2(M), z^2(M)) \to (T^1(M), z^1(M))$ such that $w \in \mathrm{Dom}\,(k[M])$ and $k[M](w)$ are on the same leaf of $\mathscr{G}|W$, where W is a sufficiently small neighborhood of M in F^x. (See Figure 6.1.)

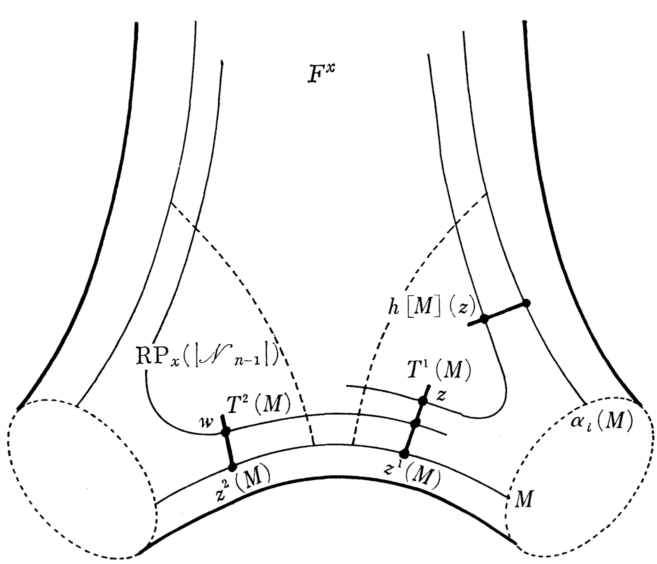


FIGURE 6.1

Suppose that $\mathscr{C}$ is a negative Reeb cycle, that is, $N_n^2 = N_1^1$. Let $\alpha = \alpha_n \circ \cdots \circ \alpha_1 \colon \mathrm{P}_x^* (N_1^1) \to \mathrm{P}_x^* (N_1^1)$. Since $\sharp \mathrm{P}_x^* (N_1^1) < \infty$ by Proposition 5.3, there is a minimal positive integer ν with $\alpha^\nu = \mathrm{id}$. Let $\bar{\eta}[M] = k[\alpha(M)] \circ h[\alpha_{n-1} \circ \cdots \circ \alpha_1(M)] \circ \cdots \circ k[\alpha_1(M)] \circ h[M] \colon (T^1(M), z^1(M)) \to (T^1(\alpha(M)), z^1(\alpha(M)))$ and $\eta[M] = \bar{\eta}[\alpha^{\nu-1}(M)] \circ \cdots \circ \bar{\eta}[M] \colon (T^1(M), z^1(M)) \to (T^1(M), z^1(M))$.

The goal of this section is to prove the following.

PROPOSITION 6.4. *Let $\mathscr{C} = (\mathscr{N}_1, \cdots, \mathscr{N}_n)$ be a strange negative Reeb cycle. Then there exists a torus $S(\mathscr{C})$ imbedded in* $\mathrm{Int}\, E(h)$ *satisfying the following conditions.*

(1) $S(\mathscr{C}) \cap F^x \subset \mathrm{RP}_x (|\mathscr{N}_1|) \cup \cdots \cup \mathrm{RP}_x (|\mathscr{N}_n|) \cup U$ *for all* $x \in S^1$, *where U is an arbitrarily small neighborhood of* $\mathrm{P}_x (N_1^1) \cup \cdots \cup \mathrm{P}_x (N_n^2)$.

(2) $S(\mathscr{C})$ *is transverse to $\mathscr{G}$ and to* $\mathscr{F}(h; \sigma)$.

(3) $S(\mathscr{C}) \cap F^x$ *consists of $a(C_1)/\nu$ circles for all $x \in S^1$, where $C_1 \in \Gamma(h)$ contains $|\mathscr{N}_1|$ and ν is as above.*

DEFINITION 6.5. We call $S(\mathscr{C})$ in Proposition 6.4 a *separating torus* of $\mathscr{C}$.

REMARK 6.6. In § 11, we see that $\mathscr{G}|D$ is a gear component, where D is the closure of the domain surrounded by $|\mathscr{N}_1|, \cdots, |\mathscr{N}_n|, G_1, \cdots, G_n$ and $S(\mathscr{C})$. In § 14, we show that $S(\mathscr{C}) \cap F^x$ consists of exactly one circle, that is, $\nu = a(C_1)$.

PROOF OF PROPOSITION 6.4. Since $\mathscr{C}$ is strange, there are a vertical $C \in \Gamma(h)$ and a compact leaf L of $\mathscr{G}|C$ with $\mathrm{P}_x(L) \cap \mathrm{RP}_x(\mathrm{Int}\, |\mathscr{N}_1|) \neq \emptyset$. This is true for $x = [0]$. For simplicity, we omit the suffix [0] from $F^{[0]}$,

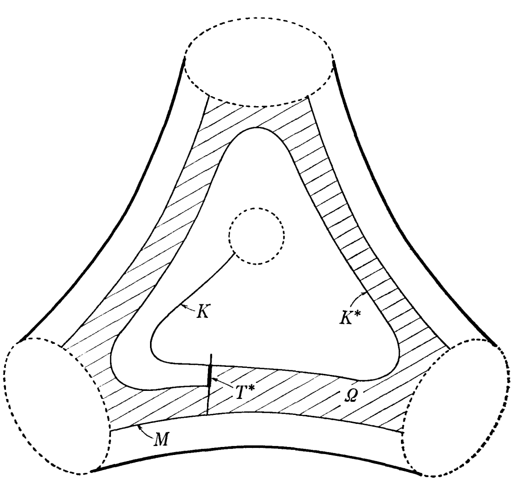


FIGURE 6.2

$\psi_{[0]}$, etc. We use the notations as above for $x = [0]$ and consider $M \in \mathrm{P}^*(N_1^1)$. Then there is $K \in \mathrm{P}^*(L)$ intersecting the connected component B of $\mathrm{RP}(|\mathscr{N}_1|)$ containing M. Let K_0 be a connected component of $K \subset B$. Put $a_1 = a(C_1)$. Since $\psi^{a_1}(M) = M$, the confocal parabolas $\psi^{na_1}(K_0)$ approach M when n moves to ∞ or $-\infty$. This implies that K intersects $T^1(M)$ at infinitely many points converging to $z^1(M)$, since $\psi^{na_1a(C)}(K) = K$ for all $n \in \boldsymbol{Z}$. Therefore the local homeomorphism $\eta[M]: (T^1(M), z^1(M)) \to (T^1(M), z^1(M))$ has no fixed point. Take $z \in K \cap \mathrm{Dom}(\eta[M])$. Let Ω be the domain in F surrounded by M, $\alpha_1(M), \cdots, \alpha_{n-1} \circ \cdots \circ \alpha_1 \circ \alpha^{\nu-1}(M)$, the closed interval T^* in $T^1(M)$ between z and $\eta[M](z)$, and the closed interval K^* in K between z and $\eta[M](z)$. (See Figure 6.2.)

We can take a closed transversal S_c of $\mathscr{G} \mid F$ in Ω with $S_c' = \psi^{a_1/\nu}(S_c) \subset \Omega$. Denote by S_a the union of the intervals of orbits of X between some point $y \in S_c$ and $\psi^{a_1/\nu}(y)$. Then $S_a \cap F^x$ passes near each $M' \in \mathrm{P}_x^*(N_1^1)$ exactly once for all $x \in S^1 - \{[0]\}$. We see that S_c and S_c' intersect K at exactly one point. Therefore S_c and S_c' intersect each leaf of $\mathscr{G} \mid \Omega$ at exactly one point. Then there is a diffeomorphism $\xi: S_c \to S_c'$ such that $y \in S_c$ and $\xi(y)$ are on the same leaf of $\mathscr{G} \mid \Omega$. Now we can modify S_a in $\bigcup \{F^{[t]} \mid 1 - \varepsilon < t \leqq 1\}$ for small $\varepsilon > 0$ by translating each $y \in S_c$ to $\xi(y)$ along a leaf of $\mathscr{G} \mid F$, so that we obtain a torus $S(\mathscr{C})$ transverse to $\mathscr{G}$ and $\mathscr{F}(h; \sigma)$. Clearly $S(\mathscr{C})$ has the desired property.

7. An investigation of leaves of $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$ containing compact leaves of $\mathscr{G} \mid C$ for a vertical $C \in \Gamma(h)$. Let $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$ and fix a vector field X of $E(h)$ as in §5. Let $\mathscr{C}_1, \cdots, \mathscr{C}_\mu$ be the strange negative Reeb cycles of $\mathscr{G}$. For each $\mathscr{C}_i$, take a separating torus $S(\mathscr{C}_i)$. We may suppose that X is tangent to $S(\mathscr{C}_i)$ for all i.

For a vertical $C \in \Gamma(h)$, we have the following proposition. Proposition 7.1 is a generalization of Lemma 3 in Tamura-Sato [15], and its proof can be regarded as a new proof of the lemma.

PROPOSITION 7.1. *Let $C \in \Gamma(h)$ be vertical. Let L be a compact leaf of $\mathscr{G} \mid \partial E(h)$, and G the leaf of $\mathscr{G}$ containing L. Then one of the following occurs.*

(1) *G is a compact leaf diffeomorphic to $S^1 \times I$.*

(2) *G is a non-compact leaf diffeomorphic to $S^1 \times [0, \infty[$ and the limit set of G consists of a compact leaf of $\mathscr{G}$ diffeomorphic to T^2.*

(3) *$G \cap S(\mathscr{C}_i) \neq \emptyset$ for some i, and the closure of the connected component of $G - (S(\mathscr{C}_1) \cup \cdots \cup S(\mathscr{C}_\mu))$ containing L is diffeomorphic to $S^1 \times I$.*

PROOF. Let $A = E(h) - k(\partial E(h) \times [0, \varepsilon[)$, $A^x = F^x \cap A$, etc., be as in

§4. We fix [0] as $x \in S^1$ and omit the suffix [0] except from $A^{[0]}$ for simplicity. Thus $F = F^{[0]}$, $\psi = \psi_{[0]}$, $\mathrm{P}(L) = \mathrm{P}_{[0]}(L)$ and so forth. We may suppose that $S(\mathscr{C}_i) \subset \mathrm{Int}\, A$ for all i.

Take a leaf $K \in \mathrm{P}^*(L)$ and a point $y_0 \in K \cap \mathrm{RP}(L) \cap \partial A^{[0]}$. When the connected component K^* of $K \cap A^{[0]}$ containing y_0 is not compact, the limit set of K^* is a circle S contained in $\mathrm{Int}\, A^{[0]}$ by the Poincaré-Bendixson theorem. Since $a = \sharp \mathrm{P}^*(L) < \infty$ and $\psi(\mathrm{P}(L)) = \mathrm{P}(L)$, it follows that $\psi^a(S) = S$. It is easy to check that the case (2) occurs. (Consequently $\mathrm{P}(L) \cap \mathrm{RP}(\mathrm{Int}\,|\mathscr{N}|) = \varnothing$ for all Reeb components $\mathscr{N}$ contained in $\mathscr{G}\,|\,\partial E(h)$.)

When K^* is compact, the endpoint y_1 of K^* with $y_1 \neq y_0$ belongs to $\partial A^{[0]}$. Then there is a leaf L' of $\mathscr{G}\,|\,\partial E(h)$ with $L' \neq L$ and $y_1 \in \mathrm{RP}(L')$. Since $\sharp \mathrm{P}^*(L') = \sharp \mathrm{P}^*(L) < \infty$, the leaf L' is

(i) a compact leaf,

or

(ii) a non-compact leaf of some negative Reeb component $\mathscr{N}_1$ contained in $\mathscr{G}\,|\,\partial E(h)$,

by Proposition 5.3. In the case (i), we see easily that the case (1) occurs. (Consequently $\mathrm{P}(L) \cap \mathrm{RP}(\mathrm{Int}\,|\mathscr{N}|) = \varnothing$ for all Reeb components $\mathscr{N}$ of $\mathscr{G}\,|\,\partial E(h)$, again.)

Now consider the case (ii). We need the following.

LEMMA 7.2. *In the case* (ii), *there is an infinite sequence* $L_1 = L$, $L_2, \cdots$ *of compact leaves of* $\mathscr{G}\,|\,\partial E(h)$ *such that for each* i *one of the following occurs.*

(a) L_{i+1} *is a compact leaf of a negative Reeb component* $\mathscr{N}_i$ *contained in* $\mathscr{G}\,|\,\partial E(h)$ *with* $\mathrm{RP}(\mathrm{Int}\,|\mathscr{N}_i|) \cap \mathrm{P}(L_i) \neq \varnothing$.

(b) L_{i+1} *is a compact leaf of a negative Reeb component* $\mathscr{N}_i$ *such that* $L'_i \neq L_i$ *and* $\mathrm{P}(L'_i) = \mathrm{P}(L_i)$, *where* L'_i *is the other compact leaf of* $\mathscr{N}_i$.

PROOF. Let L_2 be a compact leaf of $\mathscr{N}_1$. Then for $i = 1$ the case (a) occurs. Suppose that we have already obtained $L_1, \cdots, L_n$ ($n \geqq 2$) satisfying (a) or (b). Note that $\mathrm{P}(L) \cap \mathrm{RP}(\mathrm{Int}\,|\mathscr{N}_i|) \neq \varnothing$ for all $i = 1, \cdots, n-1$ and that the limit set $\mathscr{L}(K)$ of K in F intersects $\mathrm{P}(L_i)$ for $i = 2, \cdots, n$. Take $K_n \in \mathrm{P}^*(L_n)$ and $y_n \in K_n \cap \mathrm{RP}(L_n) \cap \partial A^{[0]}$. Let K_n^* be the connected component of $K_n \cap A^{[0]}$ containing y_n. We see that K_n^* is compact, as follows. Suppose the contrary. Then $\mathscr{L}(K_n^*)$ is a circle in $\mathrm{Int}\, A^{[0]}$. Since $\mathscr{L}(K) \supset \mathscr{L}(K_n^*)$, the case (2) occurs. Therefore $\mathrm{P}(L) \cap \mathrm{RP}(\mathrm{Int}\,|\mathscr{N}|) = \varnothing$ for all Reeb components $\mathscr{N}$ contained in $\mathscr{G}\,|\,\partial E(h)$, which is a contradiction.

Let z_n be the other endpoint of K_n^*. Then $z_n \in \partial A^{[0]}$ and there is a leaf L_n' of $\mathscr{G}\,|\,\partial E(h)$ with $L_n' \neq L_n$ and $\mathrm{P}\,(L_n') = \mathrm{P}\,(L_n)$. Therefore $\sharp\mathrm{P}\,(L_n') < \infty$. When L_n' is a non-compact leaf of a negative Reeb component $\mathscr{N}_n$ contained in $\mathscr{G}\,|\,\partial E(h)$, let L_{n+1} be a compact leaf of $\mathscr{N}_n$. Then L_{n+1} satisfies the condition (a).

Now suppose that L_n' is a compact leaf. Since $\mathscr{L}(K)$ intersects $\mathrm{P}\,(L_n) = \mathrm{P}\,(L_n')$, there is a leaf L'' of $\mathscr{G}\,|\,\partial E(h)$ passing arbitrarily near L_n' with $\mathrm{P}\,(L'') = \mathrm{P}\,(L)$. Since $\mathrm{P}\,(L) \cap \mathrm{RP}\,(\mathrm{Int}\,|\mathscr{N}_1|) \neq \varnothing$, the leaf L'' cannot be compact by the remark before Lemma 7.2. (For, otherwise the case (1) occurs.) Therefore L_n' is a compact leaf of a negative Reeb component $\mathscr{N}_n$ contained in $\mathscr{G}\,|\,\partial E(h)$. Let L_{n+1} be the other compact leaf of $\mathscr{N}_n$. Then L_{n+1} satisfies the condition (b). This completes the proof of Lemma 7.2.

PROOF OF PROPOSITION 7.1 CONTINUED. We see that (a) in Lemma 7.2 occurs for only a finite number of i's, as follows. Suppose the contrary. Since $\mathscr{G}\,|\,\partial E(h)$ contains only a finite number of negative Reeb components, there is a sequence $i(1) < i(2) < \cdots$ with $L_{i(1)} = L_{i(2)} = \cdots$. Note that for j, k with $j < k$ the limit set of each leaf $\in \mathrm{P}^*\,(L_{i(j)})$ contains a leaf $\in \mathrm{P}^*\,(L_{i(k)})$. Since $\sharp\mathrm{P}^*\,(L_{i(1)}) < \infty$, there is $M \in \mathrm{P}^*\,(L_{i(1)})$ with $\mathscr{L}(M) \supset M$. Since all leaves of $\mathscr{G}\,|\,F$ are proper by the Poincaré-Bendixson theorem, we get a contradiction.

Since $\mathscr{G}\,|\,\partial E(h)$ contains only a finite number of negative Reeb components, there are i, j with $i < j$ such that $\mathscr{N}_i = \mathscr{N}_j$. Thus we obtain a negative Reeb cycle $\mathscr{C} = (\mathscr{N}_i, \cdots, \mathscr{N}_{j-1})$. Easily we see that the case (3) occurs. (Consequently we see that (a) occurs only for $i = 1$.) This completes the proof of Proposition 7.1.

8. An investigation of $\mathscr{G} \in t_1^0(\mathscr{F}(h;\sigma))$ near a horizontal $C \in \Gamma(h)$.

Let $\mathscr{G} \in t_1^0(\mathscr{F}(h;\sigma))$ and fix a vector field X on $E(h)$ as in §5. For a horizontal $C \in \Gamma(h)$, we have the following.

PROPOSITION 8.1. *Let $C_0 \in \Gamma(h)$ be horizontal. Then one of the following occurs.*

(1) *$h = 2$, the other $C_1 \in \Gamma(h) - \{C_0\}$ is horizontal, $\sigma(C_1) = -\sigma(C_0)$, and $\mathscr{G}$ is isomorphic to the product foliation $(\mathscr{G}\,|\,C_0) \times [0, 1]$.*

(2) *All leaves of $\mathscr{G}\,|\,C_0$ are compact, all leaves of $\mathscr{G}$ intersecting C_0 intersect no $C \in \Gamma(h) - C_0$, and $\mathscr{G}\,|\,\mathrm{Cl}\,(\mathrm{Sat}\,(C_0))$ is a tunneled Reeb component, where* Sat ( ) *means the saturation with respect to $\mathscr{G}$.*

(3) *The foliation $\mathscr{G}\,|\,C_0$ has no Reeb component, all leaves of $\mathscr{G}$ intersecting C_0 intersect a vertical $C \in \Gamma(h)$, and if $C \cap \mathrm{Sat}\,(C_0) \neq \varnothing$ for $C \in \Gamma(h) - C_0$, then C is vertical and $\sigma(C) = -\sigma(C_0)$. Furthermore*

$\mathscr{G}\,|\,\mathrm{Cl}\,(\mathrm{Sat}\,(C_0))$ *is a gear component.*

PROOF. For simplicity, suppose that $\sigma(C_0) = 1$. Hence X is inward at C_0. We omit the suffix [0] except from $A^{[0]}$, as in §7.

Take a compact leaf L of $\mathscr{G}\,|\,C_0$. Since C_0 is horizontal, we can number the leaves in $\mathrm{P}^*(L)$ so that $\mathrm{P}^*(L) = \{\cdots, L_{-1}, L_0, L_1, \cdots\}$, and $\psi(L_j) = L_{j+1}$ for all $j \in \boldsymbol{Z}$. Note that $[L_j] = [L_k]$ in $H_1(\mathrm{RP}\,(C_0); \boldsymbol{Z})$ for all $j, k \in \boldsymbol{Z}$. Consider $\mathscr{L} = \bigcap_{n \in \boldsymbol{Z}} \mathrm{Cl}_F\,(\bigcup_{j>n} L_j)$, where $\mathrm{Cl}_F(\ \ )$ means the closure with respect to the topology of F. Note that $\mathscr{L} = \mathrm{Cl}_F\,(\mathrm{RP}\,(C_0)) - \mathrm{RP}\,(C_0)$.

(i) Suppose that $\mathscr{L}$ is empty. Then $F = \mathrm{RP}\,(C_0)$ and $h = 2$. Let $\Gamma(h) - \{C_0\} = \{C_1\}$. Since $\mathrm{RP}\,(C_0) \cap \mathrm{RP}\,(C_1) \neq \varnothing$, it follows that $\sigma(C_1) = -\sigma(C_0)$. It is easy to check that $\mathscr{G}$ is isomorphic to $(\mathscr{G}\,|\,C_0) \times I$. Thus we have the case (1).

(ii) Suppose that $\mathscr{L}$ is non-empty and compact. Then we may suppose that $\mathscr{L} \subset A^{[0]}$. Since $\mathrm{Cl}_F\,(\bigcup_{j>n} L_j)$ is saturated with respect to $\mathscr{G}\,|\,F$, so is $\mathscr{L}$. If $\mathscr{L}$ contains a non-compact leaf K of $\mathscr{G}\,|\,F$, then the limit set $\mathscr{L}(K)$ consists of exactly two compact leaves in $\mathscr{L}$ because $A^{[0]}$ can be regarded as a subspace of D^2. If $\mathscr{L}$ contains at least two compact leaves K_1 and K_2, then $\mathrm{RP}\,(C_0)$ must contain a one-sided neighborhood of K_i in F for $i = 1, 2$, and $\mathrm{RP}\,(C_0)$ has at least three isolated ends. This is a contradiction since $\mathrm{RP}\,(C_0)$ is homeomorphic to $S^1 \times \boldsymbol{R}$. Therefore $\mathscr{L}$ consists of exactly one compact leaf L^* of $\mathscr{G}\,|\,F$. Since $\psi(L^*) = L^*$, the leaf G^* of $\mathscr{G}$ containing L^* is diffeomorphic to T^2. Clearly $G^* \subset \mathrm{Int}\,E(h)$.

We see that all leaves of $\mathscr{G}\,|\,C_0$ are compact, as follows. Suppose

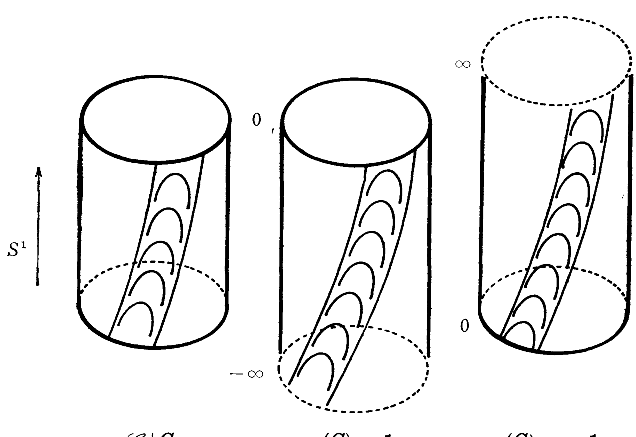


FIGURE 8.1

the contrary. Then L^* has a non-trivial holonomy on the side from which the sequence $L_1, L_2, \cdots$ converges to L^*. Since G^* has a non-trivial holonomy in the direction of orbits of X, too, and $\mathscr{G}$ is of class C^∞, we have a contradiction by the result of Kopell [7]. It is easy to check that $\mathscr{G}|\mathrm{Cl}(\mathrm{Sat}(C_0))$ is a tunneled Reeb component. Thus we have the case (2).

(iii) Suppose that $\mathscr{L}$ is non-empty and non-compact. It follows that $\mathscr{L}$ contains no compact leaf of $\mathscr{G}|F$. Let $F' = F - \mathrm{Int}\, A^{[0]}$. For $C \in \Gamma(h)$, let F'_C be the connected component of F' containing $\partial_C A^{[0]}$. Then F'_C is diffeomorphic to $]-\infty, 0] \times \hat{C}$ (or $[0, \infty[\times \hat{C}$) if $\sigma(C) = 1$ (or -1), and $\mathscr{G}|F'_C$ is isomorphic to the restriction of the covering foliation on $\boldsymbol{R} \times \hat{C}$ of $\mathscr{G}|C$. (See Figure 8.1.)

Since $\mathscr{L}$ is non-empty, it follows that $L_j \cap F'_C = \emptyset$ for all horizontal $C \in \Gamma(h) - \{C_0\}$. Since $\mathscr{L}$ is non-compact, it follows that $L_j \cap F'_C \neq \emptyset$ for a sufficiently large j and some $C \in \Gamma(h) - \{C_0\}$. If $L_j \cap F'_c \neq \emptyset$ for $C \in \Gamma(h) - \{C_0\}$, then L_j is tangent to the curves in $\mathrm{RP}(\{x\} \times C)$, $x \in S^1$, at some point by (E5) in §4, and $\sigma(C) = -\sigma(C_0) = -1$ by the remark before Proposition 5.3. Furthermore in this case it follows that $L_{j'} \cap F'_C \neq \emptyset$ for $j' > j$, and $L_j \in \mathrm{P}^*(L')$ for a non-compact leaf L' of a negative Reeb component of $\mathscr{G}|C$. We denote by $\mathscr{C}$ the set of negative Reeb components $\mathscr{N}$ contained in $\mathscr{G}|\partial E(h)$ with $\mathrm{RP}(\mathrm{Int}\,|\mathscr{N}|) \cap \mathrm{P}(L) \neq \emptyset$. Then $\mathscr{L} = \bigcup \{\mathrm{P}(N) | N$ is a compact leaf of some $\mathscr{N} \in \mathscr{C}\}$ by the above arguments. Since $\mathrm{RP}(C_0)$ has exactly two ends, we can give an order to $\mathscr{C}$ so that $\mathscr{C}$ becomes a negative Reeb cycle. Let $\mathscr{C} = (\mathscr{N}_1, \cdots, \mathscr{N}_n)$. We use the notations in Definition 6.1.

We see that $\mathscr{G}|C_0$ contains no Reeb components, as follows. Suppose the contrary. Then the structure of $\mathscr{G}|F$ as a foliation breaks near points in $\mathscr{L}$, which is a contradiction.

Clearly $\mathrm{Cl}(\mathrm{Sat}(C_0))$ is homeomorphic to $S^1 \times S^1 \times I$. And $\mathscr{G}|(\mathrm{Cl}(\mathrm{Sat}(C_0)) - U)$ is C^0 equivalent to the product foliation $(\mathscr{G}|C_0) \times I$, where U is an open tubular neighborhood of $G_1 \cup \cdots \cup G_n$ in $E(h)$. Since N_1^1 is homotopic to a circle S in $|\mathscr{N}_1| \cup \partial U$ and S is transverse to $\mathscr{G}$, the circle N_1^1 is homotopic to a circle transverse to $\mathscr{G}|C_0$ in $\mathrm{Cl}(\mathrm{Sat}(C_0))$. Furthermore S intersects all the leaves of $\mathscr{G}|(\mathrm{Cl}(\mathrm{Sat}(C_0)) - U)$. Now it is easy to see that $\mathscr{G}|\mathrm{Cl}(\mathrm{Sat}(C_0))$ is a gear component. Thus we have the case (3). This completes the proof of Proposition 8.1.

9. The decomposition theorem. The purpose of this section is to state one of the main theorems of Part I. Let $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$. We suppose that $\mathscr{G}$ is not C^0 isomorphic to $(\mathscr{G}|C) \times I$ for any $C \in \Gamma(h)$. When $h \neq 2$, this assumption is automatically satisfied.

Let $\mathscr{C}_1, \cdots, \mathscr{C}_\mu$ be the strange negative Reeb cycles of $\mathscr{G}$. For each $\mathscr{C}_i$, take a separating torus $S(\mathscr{C}_i)$. A *subleaf* of $\mathscr{G}$ (or $\mathscr{G}|A$) is the closure of a connected component of $G - (S(\mathscr{C}_1) \cup \cdots \cup S(\mathscr{C}_\mu))$ for some leaf G of $\mathscr{G}$ (or $\mathscr{G}|A$). Let Ω be the set of compact manifolds obtained from the connected components of $A - (S(\mathscr{C}_1) \cup \cdots \cup S(\mathscr{C}_\mu)) - \bigcup \{G | G$ is a compact subleaf of $\mathscr{G}|A\}$ by attaching the boundary. For $D \in \Omega$, we denote by $^*D$ the image of canonical immersion $\iota[D]: D \to A$. Let $^*\Omega = \{^*D | D \in \Omega\}$.

Let $^*\Theta$ be the set of the closure of connected components of $A - (S(\mathscr{C}_1) \cup \cdots \cup S(\mathscr{C}_\mu)) - \bigcup \{^*D | D \in \Omega\}$. For $^*T \in {}^*\Theta$, the foliation $\mathscr{G}|^*T$ consists of a compact subleaf, or is a bundle foliation over I or S^1. When $\mathscr{G}|^*T$ is a bundle foliation over S^1, let $T = q^*(^*T)$, where $q: [0, 1] \to S^1 = [0, 1]/\{0, 1\}$ is the quotient map, and denote by $\iota[T]: T \to {}^*T$ the canonical immersion (when we fix a bundle structure of $^*T$). In the other cases, let $T = {}^*T$ and $\iota[T] = \mathrm{id}: T \to {}^*T$. Let $\Theta = \{T | ^*T \in {}^*\Theta\}$.

For each $^*D \in {}^*\Omega \cup {}^*\Theta$ and $x \in S^1$, let $^*D^x = {}^*D \cap A^x$. The number of connected components of $^*D^x$ is finite. Let $^*D_1^x, \cdots, {}^*D_{a(D)}^x$ be the connected components of $^*D^x$. Let $D_j^x = \iota[D]^{-1}(^*D_j^x)$. Furthermore let $^*\Omega^x = \{^*D_j^x | D \in \Omega,\ j = 1, \cdots, a(D)\}$, $\Omega^x = \{D_j^x | D \in \Omega,\ j = 1, \cdots, a(D)\}$, $^*\Theta^x = \{^*T_j^x | T \in \Theta,\ j = 1, \cdots, a(T)\}$ and $\Theta^x = \{T_j^x | T \in \Theta,\ j = 1, \cdots, a(T)\}$.

DEFINITION 9.1. We call $\Omega \cup \Theta$ (or $\Omega^x \cup \Theta^x$) the TS *decomposition* of A (or A^x) with respect to $\mathscr{G}$.

For each $D \in \Omega$, we define six non-negative integers, as follows. A *leaf* of $\mathscr{G}|^*D_1^{[0]}$ is a connected component of $G \cap {}^*D_1^{[0]}$ for some leaf G of $\mathscr{G}$. We say a connected component J of $\partial D_1^{[0]}$ is

of type (l) if $^*J$ is equal to $\partial_C A^{[0]}$ for some $C \in \Gamma(h)$ or to a connected component of $A^{[0]} \cap S(\mathscr{C}_i)$ for some i,

of type (m) if $^*J$ is equal to a connected component of $A^{[0]} \cap G$ for some compact leaf G of $\mathscr{G}|\mathrm{Int}\, A$, or

of type (n) if $^*J$ contains a leaf of $\mathscr{G}|^*D_1^{[0]}$ homeomorphic to I,

where $^*J = \iota[D](J)$. Let $l(D)$ (or $m(D)$, $n(D)$) be the number of connected components of $\partial D_1^{[0]}$ of type (l) (or (m), (n)). Since each connected component of $\partial D_1^{[0]}$ is of type (l), (m) or (n) by Propositions 7.1 and 8.1, we see that $l(D) + m(D) + n(D)$ equals the number of the connected components of $\partial D_1^{[0]}$.

Let J be a connected component of $\partial D_1^{[0]}$ of type (n). By Propositions 7.1 and 8.1, $^*J$ contains a finite number of leaves $J_1, \cdots, J_\alpha$ of $\mathscr{G}|^*D_1^{[0]}$ homeomorphic to I, and $^*J - (J_1 \cup \cdots \cup J_\alpha)$ consists of open intervals contained in $\mathrm{Int}\,|\mathscr{R}|$ for Reeb or slope components $\mathscr{R}$ contained in $\mathscr{G}|\partial^*D$. Furthermore $^*J \cap (S(\mathscr{C}_1) \cup \cdots \cup S(\mathscr{C}_\mu)) = \varnothing$, and $C \in \Gamma(h)$

intersecting $^*J$ is vertical by (E4)′ in §4.

Let $p(D)$ (or $q(D)$, $s(D)$) be the number of positive Reeb (or negative Reeb, slope) components contained in $\mathscr{G}|\partial^*D$ intersecting $\partial_n{}^*D_1^{[0]}$, where $\partial_n{}^*D_1^{[0]}$ is the union of $^*J$ for connected components J of $D_1^{[0]}$ of type (n). Note that $p(D)+q(D)+s(D)$ equals the number of leaves of $\mathscr{G}|D_1^{[0]}$ contained in the connected components of $\partial D_1^{[0]}$ of type (n), where $\mathscr{G}|D_1^{[0]} = (\iota[D]|D_1^{[0]})^*\mathscr{G}$.

DEFINITION 9.2. We call $\mathrm{ch}(D) = (l(D), m(D), n(D); p(D), q(D), s(D))$ the *characteristic hexad* of D.

Now we can state the following.

THEOREM 3 (The decomposition theorem). *Let* $\mathscr{G} \in t_0^1(\mathscr{F}(h;\sigma))$. *Suppose that* $\mathscr{G}$ *is not* C^0 *isomorphic to* $(\mathscr{G}|C)\times I$ *for any* $C \in \Gamma(h)$. *Let* $\Omega \cup \Theta$ *be the* TS *decomposition of* A *with respect to* $\mathscr{G}$. *Then for each* $D \in \Omega$ *the possibilities for* $\mathscr{G}|D$ *are the cases in the following table, and these cases can occur for some* $\mathscr{G}$.

type	ch (D)	$\mathscr{G}\|D$	
I	$(0,0,1;1,0,0)$	*a half Reeb component*	
II	$(0,0,1;0,0,2)$	*an I-times slope component*$^{(*)}$	
III	$(0,0,1;0,q>0,2)$	*a* TS′ *component*	$[\sigma]$
IV	$(0,0,1;1,q>0,0)$	*an arcade component*	$[\sigma]$
V	$(0,0,2;0,q>1,2)$	*a double gear component*	
VI	$(1,0,1;0,q>0,0)$	*a gear component*	$[\sigma]$
VII	$(0,1,1;0,q>0,0)$	(1) *a turbulized gear component*	$[\sigma]$
		(2) *a perturbed gear component*	
VIII	$(1,1,0;0,0,0)$	(1) *a tunneled Reeb component*	
		(2) *a rational rifle component*$^{(*)}$	
		(3) *an irrational rifle component*$^{(*)}$	
IX	$(0,2,0;0,0,0)$	(1) *an* S^1*-times slope component*$^{(*)}$	
		(2) *an* S^1*-times Reeb component*$^{(*)}$	
		(3) *a twisted* S^1*-times Reeb component*$^{(*)}$	

The terms with $(*)$ *in the table will be defined in the next section. The mark* $[\sigma]$ *in the table means the existence of the following restrictions to* $\sigma(C)$ *for* $C \in \Gamma(h)$ *concerned.*

(III) $\sigma(C') = \sigma(C)$ (*or* $-\sigma(C)$) *if* $\mathscr{G}|(^*D \cap \partial_C A)$ *and* $\mathscr{G}|(^*D \cap \partial_{C'} A)$ *contain negative Reeb components, and* $^*D \cap \partial_C A$ *and* $^*D \cap \partial_{C'} A$ *belong to the same* (*or different*) *connected components of* $\partial^*D - \mathrm{Int}(|\mathscr{S}_1| \cup |\mathscr{S}_2|)$, *where* $\mathscr{S}_1$ *and* $\mathscr{S}_2$ *are the slope components contained in* $\mathscr{G}|\partial^*D$.

(IV) $\sigma(C') = -\sigma(C)$ *if* $\mathscr{G}|(^*D \cap \partial_C A)$ *contains a positive Reed component and* $\mathscr{G}|(^*D \cap \partial_{C'} A)$ *contains a negative Reeb components.*

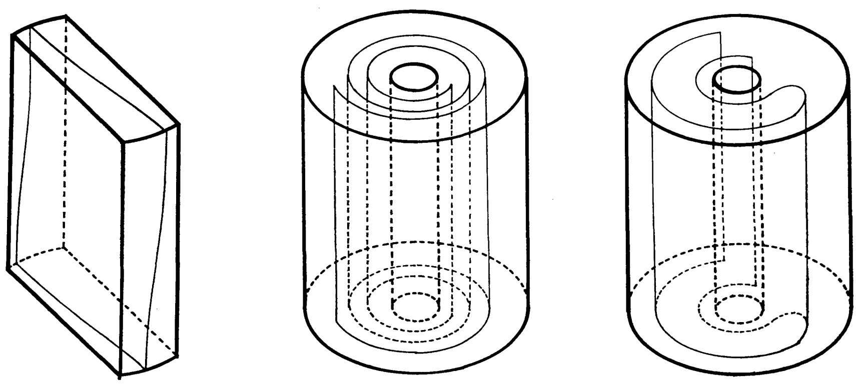
(VI) $\sigma(C') = -\sigma(C)$ *if* $C \in \Gamma(h)$ *with* $\partial_C A \subset {}^*D$ *is horizontal,* ${}^*D \cap \partial_{C'} A \neq \emptyset$ *and* $C' \neq C$.

(VII) $\sigma(C') = \sigma(C)$ *if* ${}^*D \cap \partial_C A \neq \emptyset$ *and* ${}^*D \cap \partial_{C'} A \neq \emptyset$.

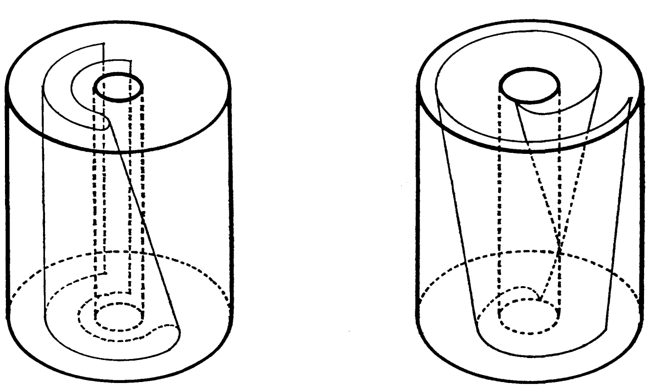
REMARK 9.3. In the case $h = 1$, Theorem 3 corresponds to Theorem 1 in Tamura-Sato [15], where the possibilities for $\mathscr{G}|D$ are types I, II and III, and there exist no negative Reeb cycles.

10. Several components II. We give the definition for the components appearing in the decomposition theorem (Theorem 3) and not yet defined. Recall Definition 2.6.

DEFINITION 10.1. *A standard I-times* (or *S^1-times*) *slope component* is the product foliation $I \times \mathscr{G}_0$ (or $S^1 \times \mathscr{G}_0$), where $\mathscr{G}_0$ is a standard slope component of $S^1 \times I$. A *standard S^1-times Reeb component* is the product foliation $S^1 \times \mathscr{F}_R^2(\pm 1)$, where $\mathscr{F}_R^2(\pm 1)$ is a standard Reeb component of $S^1 \times D^1$.



An I-times slope component An S^1-times slope component An S^1-times Reeb component



A twisted S^1-times Reeb component A rational (or irrational) rifle component

FIGURE 10.1

Consider $I \times \mathscr{F}_R^2(\pm 1)$ and take a diffeomorphism $\phi\colon S^1 \times D^1 \to S^1 \times D^1$ such that ϕ maps each non-compact leaf of $\mathscr{F}_R^2(\pm 1)$ to a different leaf of $\mathscr{F}_R^2(\pm 1)$, as in §3. Then we have a foliation $\mathscr{F}_\phi$ of a compact manifold diffeomorphic to $S^1 \times S^1 \times D^1$ from $I \times \mathscr{F}_R^2(\pm 1)$ by attaching the top and bottom of $I \times S^1 \times D^1$ by ϕ.

DEFINITION 10.2. The foliation $\mathscr{F}_\phi$ constructed above is called a *standard twisted S^1-times Reeb component.*

Take $\alpha \in R$. Let $\mathscr{F}_\alpha$ be the foliation of $\boldsymbol{R}^2 \times I$ consisting of a leaf $\boldsymbol{R} \times \{0\}$ and leaves

$$\{(x + f(t),\ -\alpha f(t),\ t) \mid t \in I\}$$

for $x \in \boldsymbol{R}$, where we use the function $f\colon]0, 1] \to]-\infty, 0]$ introduced in §2. Since the canonical action of $\boldsymbol{Z} \oplus \boldsymbol{Z}$ to $R^2 \times I$ preserves $\mathscr{F}_\alpha$, we obtain the quotient foliation $\mathscr{F}_\alpha/\boldsymbol{Z} \oplus \boldsymbol{Z}$ of $S^1 \times S^1 \times I = R^2 \times I/\boldsymbol{Z} \oplus \boldsymbol{Z}$.

DEFINITION 10.3. We call $\mathscr{F}_\alpha/\boldsymbol{Z} \oplus \boldsymbol{Z}$ a *standard rational* (or *irrational*) *rifle component* if α is rational (or irrational).

We give some figures. (See Figure 10.1.)

11. The proof of the decomposition theorem. The purpose of this section is to prove Theorem 3. Let $D \in \Omega$ and $\mathrm{ch}\,(D) = (l, m, n; p, q, s)$. Construct the double W of $D_1^{[0]}$ by pasting two copies of $D_1^{[0]}$ along $\partial_n D_1^{[0]} - \bigcup \{K \mid K$ is a compact leaf of $\mathscr{G} \mid D_1^{[0]}$ contained in $\partial_n D_1^{[0]}\}$, where $\partial_n D_1^{[0]}$ is the union of connected components of $\partial D_1^{[0]}$ of type (n). Then we have a vector field Y on W whose orbits are the leaves of $(\mathscr{G} \mid D_1^{[0]} \cup \mathscr{G} \mid D_1^{[0]})$. The index of Y equals $p - q$, as in §4. Since W is obtained from a closed surface of genus $n - 1$ by deleting $2(l + m) + p + q + s$ open two disks, we have

$$p - q = 4 - 2(l + m + n) - p - q - s\,.$$

Therefore we have an equation

$$(*) \qquad 2(l + m + n + p) + s = 4\,.$$

The solutions of $(*)$ are

$$(0, 0, 1; 1, q, 0)\,,\quad (0, 0, 1; 0, q, 2)\,,\quad (0, 0, 2; 0, q'', 0)\,,\quad (1, 0, 1; 0, q', 0)\,,$$
$$(0, 1, 1; 0, q', 0)\,,\quad (2, 0, 0; 0, 0, 0)\,,\quad (1, 1, 0; 0, 0, 0)\,,\quad (0, 2, 0; 0, 0, 0)\,,$$

where $q \geqq 0$, $q' \geqq 1$ and $q'' \geqq 2$. Now let us examine the solutions of $(*)$ one by one.

The case $\mathrm{ch}\,(D) = (0, 0, 1; 1, q, 0)$ where $q \geqq 0$. Suppose that $q = 0$. Then $\mathscr{G} \mid D_1^{[0]}$ consists of concentric half circles with center z_0 at $\partial D_1^{[0]}$.

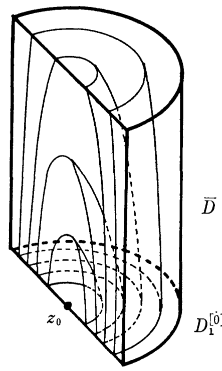


FIGURE 11.1

We denote by $\bar{D}$ the compact manifold obtained from $D - D_1^{[0]}$ by attaching two copies of $D_1^{[0]}$ as the boundary. Then $\bar{D}$ is diffeomorphic to $I \times D_1^{[0]}$. (See Figure 11.1.)

Regard the center z_0 as a point in the bottom of $\bar{D}$. Then the leaves of $\mathscr{G}\,|\,\bar{D}$ passing sufficiently near z_0 are all homeomorphic to D^2. Using the local stability theorem for simply connected compact leaves (see Reeb [13], Haefliger [4]), we see that all the leaves of $\mathscr{G}\,|\,\bar{D}$ except $\{z_0\}$ are homeomorphic to D^2. Now it is easy to construct an orientation preserving homeomorphism $\phi\colon D \to S^1 \times D^2_+$ with $\mathscr{G}\,|\,D = \phi^* T[\mathscr{F}^3_{\mathrm{pr}}\,|\,S^1 \times D^2_+, D^1 \times \{0\}, \sigma(C)]$, where $C \in \Gamma(h)$ intersects $^*D$. Therefore $\mathscr{G}\,|\,D$ is a half Reeb component. Thus we have the case (I).

Suppose that $q > 0$. By a consideration on the holonomy of compact leaves of $\mathscr{G}$ contained in $\partial^* D$, we see that $\sigma(C') = -\sigma(C)$, where $\mathscr{G}\,|\,(^*D \cap \partial_C A)$ has a positive Reeb component and $\mathscr{G}\,|\,(^*D \cap \partial_{C'} A)$ has a negative Reeb component. By the same arguments as above, we see that all the leaves of $\mathscr{G}\,|\,\bar{D}$ except the one point leaves are homeomorphic to D^2. Then it is easy to show that $\mathscr{G}\,|\,D$ is an arcade component. We omit the details. Thus we have the case (IV).

The case $\mathrm{ch}\,(D) = (0, 0, 1; 0, q, 2)$ where $q \geqq 0$. If $q = 0$, then $\mathscr{G}\,|\,D_1^{[0]}$ is isomorphic to the foliation $\{\{x\} \times I\}_{x \in I}$ of $I \times I$, and $\mathscr{G}\,|\,D$ is an I-times slope component, which is the case (II). Suppose that $q > 0$. Let $\mathscr{S}_1$, $\mathscr{S}_2$ be the slope component of $\mathscr{G}\,|\,(^*D \cap \partial A)$, and $\mathscr{N}_1, \cdots, \mathscr{N}_q$ the negative Reeb components of $\mathscr{G}\,|\,(^*D \cap \partial A)$. Considering the holonomy of compact leaves of $\mathscr{G}$ contained in $\partial^* D$, we see the following: a connected component $\partial(1)$ of $\partial^* D - \mathrm{Int}\,(|\mathscr{S}_1| \cup |\mathscr{S}_2|)$ has the property that if $|\mathscr{N}_j| \subset \partial(1)$ and $|\mathscr{N}_j| \subset \partial_C A$ then $\sigma(C) = 1$. On the other hand, the other component $\partial(-1)$ has the property that if $|\mathscr{N}_j| \subset \partial(-1)$ and $|\mathscr{N}_j| \subset \partial_C A$ then $\sigma(C) = -1$. Using the arguments on $\mathscr{G}\,|\,\bar{D}$ as above, we see that $\mathscr{G}|D$ is a TS$'$ component, which is the case (III).

The case $\mathrm{ch}(D) = (0, 0, 2; 0, q, 0)$ where $q \geqq 2$. Let $\mathscr{N}_1', \cdots, \mathscr{N}_n'$ be the negative Reeb components contained in $\mathscr{G}|\partial A$ intersecting a connected component $\partial(1)$ of $\partial^* D_1^{[0]}$ and $\mathscr{N}_{n+1}', \cdots, \mathscr{N}_{n+n'}'$ the ones intersecting the other connected component $\partial(-1)$. Let $\mathscr{N}_j = \gamma^* \mathscr{N}_j'$, where $\gamma\colon \partial E(h) \to A$ is as in § 4. Then $\mathscr{C} = (\mathscr{N}_1, \cdots, \mathscr{N}_n)$ and $\mathscr{C}' = (\mathscr{N}_{n+1}, \cdots, \mathscr{N}_{n+n'})$ are negative Reeb cycles. We use the notations for $\mathscr{C}$ in § 6. Take $M \in P^*(N_1^1)$ and consider the local homeomorphism $\eta[M]$: $(T^1(M), z^1(M)) \to (T^1(M), z^1(M))$.

Suppose that $\eta[M]$ has a fixed point z. Let K be the leaf of $\mathscr{G}|F$ passing through z. Then K is a circle. Let $K_i = \psi^i(K)$ for $i \in \boldsymbol{Z}$. For simplicity, suppose that $\sigma(y) = 1$ for $y \in |\mathscr{N}_1| \cup \cdots \cup |\mathscr{N}_n|$. Since $\mathscr{G}|\mathrm{Int}\, {}^*D$ has no compact leaf, the set $(\bigcap_{j \in Z} \mathrm{Cl}_F (\bigcup_{i>j} K_i)) \cap A^{[0]}$ must be the union of leaves of $\mathscr{G}|{}^*D_1^{[0]}$ contained in $\partial(-1)$. This implies that the vector field X is outward at $|\mathscr{N}_i|$ for $i = n+1, \cdots, n+n'$. Therefore $\sigma(y) = -1$ for $y \in |\mathscr{N}_{n+1}| \cup \cdots \cup |\mathscr{N}_{n+n'}|$. We see that $\mathscr{G}|D$ is C^0 isomorphic to a standard double gear component constructed from $\mathscr{G}_0$ of $S^1 \times S^1$ containing a compact leaf homologous to $\{*\} \times S^1$. Thus we have the case (V).

When $\eta[M]$ has no fixed point, we can take a torus imbedded in $\mathrm{Int}\, A$ transverse to $\mathscr{G}$ and to $\mathscr{F}(h; \sigma)$ as in Proposition 6.4. Then we see that $\mathscr{G}|D$ is a double gear component, and have the case (V) again. We omit the details.

The case $\mathrm{ch}(D) = (1, 0, 1; 0, q, 0)$ where $q > 0$. By similar arguments, we see that $\mathscr{G}|D$ is a gear component, and can check the condition on σ. Thus we have the case (VI).

The case $\mathrm{ch}(D) = (0, 1, 1; 0, q, 0)$ where $q > 0$. Let G be the compact leaf of $\mathscr{G}|\mathrm{Int}\, A$ contained in $\partial^* D$. Then $K = G \cap {}^*D_1^{[0]}$ is diffeomorphic to S^1, and there is $a \in \boldsymbol{N}$ with $\psi^a(K) = K$. If $\mathscr{G}|\mathrm{Int}\, {}^*D_1^{[0]}$ has a compact leaf K', then $\psi^{ja}(K')$ converges to K as j moves to ∞ or $-\infty$, and the leaves of $\mathscr{G}|{}^*D_1^{[0]}$ are all compact by the usual arguments by means of the theorem of Kopell [7]. In this case, we see that $\mathscr{G}|D$ is a turbulized gear component, and have the case (VII-1). When $\mathscr{G}|\mathrm{Int}\, {}^*D_1^{[0]}$ has no compact leaf, we see that $\mathscr{G}|D$ is a perturbed gear component, and have the case (VII-2). The condition on σ is easily checked.

The case $\mathrm{ch}(D) = (2, 0, 0; 0, 0, 0)$. It follows that $\partial^* D \subset \partial A$. Therefore ${}^*D$ is a non-empty closed open subset of A. Since A is connected, it follows that ${}^*D = A$ and $h = 2$. Furthermore $\mathscr{G}$ is C^0 isomorphic to $(\mathscr{G}|C) \times I$ for some $C \in \Gamma(h)$, which is a contradiction. Threfore this case does not occur.

Finally we see that the cases $\mathrm{ch}(D) = (1, 1, 0; 0, 0, 0)$, $(0, 2, 0; 0, 0, 0)$ imply the cases (VIII), (IX) respectively. We omit the details. The construction of several components in the decomposition theorem is indicated in §§2, 3, 10. This completes the proof of Theorem 3.

12. Regular TS pieces. In this and next sections, we define a regular TS diagram as a generalization of a TS diagram introduced in Tamura-Sato [15] for $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$ not C^0 isomorphic to $(\mathscr{G} | C) \times I$ for any $C \in \Gamma(h)$. The construction of a regular TS diagram is like a jigsaw puzzle or a tangram. The pieces admitted in our puzzle are regular TS pieces defined below. In order to classify $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$, we will attach a regular TS piece to each $D_j^{[0]} \in \Omega^{[0]} \cup \Theta^{[0]}$, where $\Omega^{[0]} \cup \Theta^{[0]}$ is the TS decomposition of $A^{[0]}$ with respect to $\mathscr{G}$. For $\mathscr{G}$ isomorphic to $(\mathscr{G} | C) \times I$ for some $C \in \Gamma(h)$, we will define a singular TS piece and a singular TS diagram in §20.

We make some preparations.

DEFINITION 12.1. A TS *block* Δ is a compact oriented C^∞ manifold homeomorphic to D^2 or $S^1 \times I$ and possibly with an even number of corner points on each connected component of $\partial \Delta$.

DEFINITION 12.2. Let Δ be a TS block. When Δ has no corner, let $\mathscr{J}(\Delta) = \emptyset$. When Δ has corner points, take and fix a set $\mathscr{J}(\Delta)$ of disjoint closed intervals of $\partial \Delta$, whose endpoints are in the corner $\angle \Delta$ of Δ, such that $2 \# \mathscr{J}(\Delta) = \# \angle \Delta$. We denote by $\mathscr{K}(\Delta)$ the set of connected components of the closure of $\partial \Delta - \bigcup \{J | J \in \mathscr{J}(\Delta)\}$.

DEFINITION 12.3. An orientation of $K \in \mathscr{K}(\Delta)$ is *sympathetic* (or *antipathetic*) if it coincides with that of $\partial \Delta$ as the boundary.

DEFINITION 12.4. Let $\mathscr{S}$ be the set of five symbols $\bigcirc$, $\bullet$, $\vee$, $\wedge$, $\|$. Let TYPE = {I, II, III, IV, V, VI, VII, VIII, IX}.

Now we can define TS pieces for $\mathscr{G}$ not isomorphic to $(\mathscr{G} | C) \times I$ for any $C \in \Gamma(h)$, as follows.

DEFINITION 12.5. A *regular* TS *piece* is a quadruplet $P = (\Delta, \nu, s\colon \mathscr{J}(\Delta) \to \mathscr{S}, \omega\colon \mathscr{K} \to \{1, -1\})$, where Δ is a TS block and ν belongs to TYPE, and $\mathscr{K}$ is a subset of $\mathscr{K}(\Delta)$, satisfying the following conditions.

(P0) If $\nu \notin \{\mathrm{VI}, \mathrm{VIII}\}$, then $\mathscr{K} = \mathscr{K}(\Delta)$.

(P1) If $\nu = \mathrm{I}$, then $\Delta \simeq D^2$, $\# \mathscr{J}(\Delta) = 1$ and $s(J) = \bigcirc$, where $\mathscr{J}(\Delta) = \{J\}$.

(P2) If $\nu = \mathrm{II}$, then $\Delta \simeq D^2$, $\# \mathscr{J}(\Delta) = 2$, $s(\mathscr{J}(\Delta)) = \{\|\}$ or $\{\vee, \wedge\}$, and $\omega(\mathscr{K}(\Delta)) = \{1, -1\}$.

(P3) If $\nu = \mathrm{III}$, then $\Delta \simeq D^2$, $\# \mathscr{J}(\Delta) > 2$, $s(J_1) = \vee$ and $s(J_2) = \wedge$

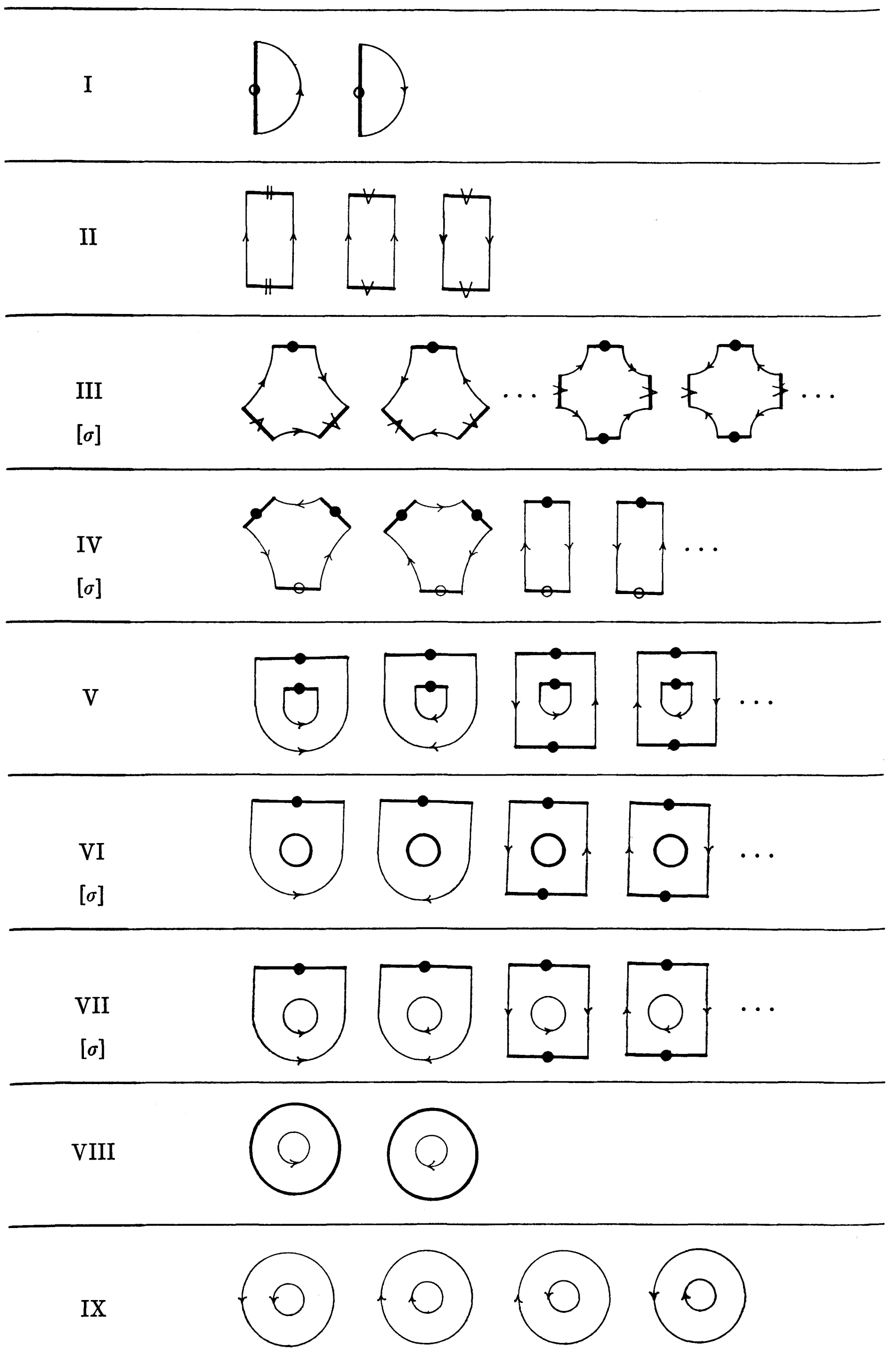


FIGURE 12.1 Regular TS pieces

for some $J_1, J_2 \in \mathscr{J}(\Delta)$, $s(J) = \bullet$ for all $J \in \mathscr{J}(\Delta) - \{J_1, J_2\}$, and $\omega(K_2) = -\omega(K_1)$ if $K_1, K_2 \in \mathscr{K}$ are contained in different connected components of $\partial\Delta - \operatorname{Int}(J_1 \cup J_2)$.

(P4) If $\nu = \mathrm{IV}$, then $\Delta \simeq D^2$, $\#\mathscr{J}(\Delta) > 1$, $s(J_1) = \bigcirc$ for some $J_1 \in \mathscr{J}(\Delta)$, $s(J) = \bullet$ for all $J \in \mathscr{J}(\Delta) - \{J_1\}$, and ω is constant.

(P5) If $\nu = \mathrm{V}$, then $\Delta \simeq S^1 \times I$, $s(\mathscr{J}(\Delta)) = \{\bullet\}$, $\mathscr{K}(\Delta)$ contains no circle, and $\omega(K_2) = -\omega(K_1)$ if $K_1, K_2 \in \mathscr{K}$ are contained in different connected components of $\partial\Delta$.

(P67) If $\nu = \mathrm{VI}$ or VII, then $\Delta \simeq S^1 \times I$, $s(\mathscr{J}(\Delta)) = \{\bullet\}$, $\mathscr{K}(\Delta)$ contains exactly one circle K_0, an ω is constant. If $\nu = \mathrm{VI}$, then $\mathscr{K} = \mathscr{K}(\Delta) - \{K_0\}$.

(P89) If $\nu = \mathrm{VIII}$ or IX, then $\Delta \simeq S^1 \times I$ and $\mathscr{J}(\Delta) = \varnothing$. If $\nu = \mathrm{VIII}$, then $\#\mathscr{K} = 1$.

We call Δ, ν, s, ω the *underlying block, type, symbol map, orienting map* of P respectively. We denote Δ (or ν) by $|P|$ (or type(P)) sometimes.

REMARK 12.6. A regular TS piece corresponds to a component of the same type in Theorem 3, and the symbols $\bigcirc$ (or $\bullet$) to a positive (or negative) Reeb component of $\mathscr{G}|\partial E(h)$. The symbols $\vee$ and $\wedge$ correspond to a slope component, and the symbol $||$ to a trivial component.

REMARK 12.7. The map $\omega\colon \mathscr{K} \to \{1, -1\}$ means the choice of orientations of $K \in \mathscr{K}$ (cf. the proof of Theorem 4 in §14). Precisely we give $K \in \mathscr{K}$ the sympathetic (or antipathetic) orientation if $\omega(K) = 1$ (or -1).

In order to make regular TS pieces more understandable, we picturize them in Figure 12.1. In figures, we use the convension that $\vee$ (or $\wedge$) is put inward (or outward) to underlying TS blocks (see Figure 12.1, II, III for example). When it is not necessary to distinguish between $\vee$ and $\wedge$, we will use $\times$ in their place. The bold (or fine) lines mean the elements of $\mathscr{J}(\Delta) \cup (\mathscr{K}(\Delta) - \mathscr{K})$ (or $\mathscr{K}$). All the TS blocks are oriented as $\circlearrowleft$. The mark $[\sigma]$ means the existence of conditions on σ for constructing regular TS diagrams in the next section.

13. Regular TS diagrams. For the future use, we define regular TS diagrams in a more general setting. Let $\Sigma_g(h)$ be the compact oriented manifold obtained from the closed surface of genus g by deleting $h(>0)$ small open two disks, and denote by $\Gamma_g(h)$ the set of connected components of $\partial\Sigma_g(h)$. Clearly $E(h) = \Sigma_0(h)$ and $\hat{\Gamma}(h) = \Gamma_0(h)$. Take a continuous map $\sigma\colon \partial\Sigma_g(h) \to \{1, -1\}$. As before we regard σ as a map from $\Gamma_g(h)$ sometimes.

DEFINITION 13.1. A *pre* TS *diagram* of $\Sigma_g(h)$ is a triad $(S, \{P_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda: |P_\lambda| \to \Sigma_g(h)\}_{\lambda\in\Lambda})$ satisfying the following conditions.

(PR1) S is the union of a finite number of disjoint circles $S_1, \cdots, S_n$ contained in $\operatorname{Int} \Sigma_g(h)$.

(PR2) $P_\lambda = (\Delta_\lambda, \nu_\lambda, s: \mathscr{J}_\lambda \to \mathscr{S}, \omega: \mathscr{K}_\lambda \to \{1, -1\})$ is a regular TS piece, $\#\{\lambda \in \Lambda | \nu_\lambda \notin \{\mathrm{II}, \mathrm{IX}\}\} < \infty$, and $\iota_\lambda: \Delta_\lambda \to \Sigma_g(h)$ is an orientation preserving C^∞ immersion such that $\iota_\lambda | \operatorname{Int} \Delta_\lambda$ is an imbedding.

We denote $\iota_\lambda(\ \ )$ by $*(\ \ )$ sometimes for simplicity.

(PR3) $*(\operatorname{Int} \Delta_\lambda)$'s are disjoint, and $\Sigma_g(h)$ is the closure of $\bigcup \{*\Delta_\lambda | \lambda \in \Lambda\}$. For each S_i, there are $\lambda, \lambda' \in \Lambda$, $K \in \mathscr{K}_\lambda$, $K' \in \mathscr{K}_{\lambda'}$ and $C, C' \in \Sigma_g(h)$ such that

(1) $\nu_\lambda = \mathrm{VI}$, $*K = S_i$, $*\Delta_\lambda \cap C \neq \emptyset$,

(2) $*K' \cap S_i \neq \emptyset$, $*K' \cap C' \neq \emptyset$.

(We call P_λ the TS piece of Type VI *separated by* S_i.)

(PR4) $*J \subset S \cup \partial\Sigma_g(h)$ for $J \in \mathscr{J}_\lambda$, $*J \subset \partial\Sigma_g(h)$ if $s(J) = \bigcirc$ or $\bullet$, $*K \subset S \cup \partial\Sigma_g(h)$ for $K \in \mathscr{K}(\Delta_\lambda) - \mathscr{K}_\lambda$, and $\operatorname{Int} *K \subset \operatorname{Int} \Sigma_g(h) - S$ for $K \in \mathscr{K}_\lambda$.

(PR5) If $K \neq K'$ and $*K = *K'$ for $K \in \mathscr{K}_\lambda$ and $K' \in \mathscr{K}_{\lambda'}$, then $\omega(K') = -\omega(K)$. (Hence we can give $*K$ an orientation such that $\iota_\lambda | K$ and $\iota_{\lambda'} | K'$ are orientation preserving.)

DEFINITION 13.2. Let $\hat{\mathscr{T}} = (S, \{P_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\}_{\lambda\in\Lambda})$ and $\hat{\mathscr{T}}' = (S', \{P'_\lambda\}_{\lambda\in\Lambda'}, \{\iota'_\lambda\}_{\lambda\in\Lambda'})$ be pre TS diagrams of $\Sigma_g(h)$. A homeomorphism $\phi: \Sigma_g(h) \to \Sigma_g(h)$ is called an *isomorphism* from $\hat{\mathscr{T}}$ to $\hat{\mathscr{T}}'$ if $\phi(S) = S'$ and if there are bijections

$$\rho: \Lambda \to \Lambda'\ ,\quad ': \mathscr{J}_\lambda \to \mathscr{J}'_{\rho(\lambda)} \quad \text{and} \quad ': \mathscr{K}_\lambda \to \mathscr{K}'_{\rho(\lambda)}$$

such that

(1) $\nu'_{\rho(\lambda)} = \nu_\lambda$, $\phi(*\Delta_\lambda) = *\Delta'_{\rho(\lambda)}$,

(2) $\phi(*J) = *J'$ and $s(J') = s(J)$ for $J \in \mathscr{J}_\lambda$,

(3) $\phi(*K) = *K'$ and $\omega(K') = \omega(K)$ for $K \in \mathscr{K}_\lambda$,

where $P_\lambda = (\Delta_\lambda, \nu_\lambda, s: \mathscr{J}_\lambda \to \mathscr{S}, \omega: \mathscr{K}_\lambda \to \{1, -1\})$ and $P'_\lambda = (\Delta'_\lambda, \nu'_\lambda, s: \mathscr{J}'_\lambda \to \mathscr{S}, \omega: \mathscr{K}'_\lambda \to \{1, -1\})$.

Now we can define regular TS diagrams as follows.

DEFINITION 13.3. A *regular* TS *diagram* of $(\Sigma_g(h); \sigma)$ is a triad $\mathscr{T} = (\hat{\mathscr{T}}, \{\phi_t: \Sigma_g(h) \to \Sigma_g(h)\}_{t\in I}, (a, b; r): \Gamma_g(h) \to (\boldsymbol{N} \times \boldsymbol{Z})^* \times 2\boldsymbol{Z})$ satisfying the following conditions.

(R1) $\hat{\mathscr{T}} = (S, \{P_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\}_{\lambda\in\Lambda})$ is a pre TS diagram of $\Sigma_g(h)$, and $\{\phi_t\}_{t\in I}$ is a C^∞ isotopy of diffeomorphisms such that ϕ_0 is the identity and ϕ_1 is an isomorphism from $\hat{\mathscr{T}}$ to $\hat{\mathscr{T}}$.

(R2) Let $C \in \Gamma_g(h)$ and put $p(C)$ (or $q(C)$) $= \#\{J | J \in \mathscr{J}_\lambda$ for some $\lambda \in \Lambda$, $*J \subset C$, $s(J) = \bigcirc$ (or $\bullet$)$\}$.

(i) If $(a(C), b(C)) = (0, 1)$ or (∞, ∞), then $r(C) = 0$ and there are $\lambda \in \Lambda$ and $K \in \mathscr{K}(\Delta_\lambda) - \mathscr{K}_\lambda$ with $*K = C$.

(ii) If $(a(C), b(C)) = (a, b) \in (\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}}$, then $r(C) = (p(C) - q(C))/a$, there are $\lambda \in \Lambda$ and $J \in \mathscr{J}_\lambda$ with $*J \subset C$, the map $(\phi_1 | C)^a$ is the identity, $(\phi_1 | C)^{a'}$ has no fixed point for $0 < a' < a$, and the degree of η: $[0, a]/\{0, a\} \to C$ equals b, where η is defined by $\eta([t]) = \phi_{t'}(\phi_1^k(y_0))$ for $t = k + t'$, $k \in \boldsymbol{Z}$, $0 \leqq t < 1$ and a fixed point $y_0 \in C$.

(s) Let S_i be a circle in S, and $C, C' \in \Gamma_g(h)$ be as in Definition 13.1 (PR3). Then $(a(C'), b(C')) \neq (a(C), -b(C))$.

(R3) (The conditions on σ). Below, J and J' are the elements of $\mathscr{J}_\lambda$ with $*J, *J' \subset \partial \Sigma_g(h)$.

(iii) If $\nu_\lambda = \mathrm{III}$, then $\sigma(*y') = \sigma(*y)$ (or $-\sigma(*y)$) for $y \in J$ and $y' \in J'$ such that J and J' are contained in the same (or different) connected component of $\partial \Delta_\lambda - \bigcup \{J'' | J'' \in \mathscr{J}_\lambda, s(J'') = \vee \text{ or } \wedge\}$.

(iv) If $\nu_\lambda = \mathrm{IV}$, then $\sigma(*y') = -\sigma(*y)$ for $y \in J$ and $y' \in J'$ such that $s(J) = \bigcirc$ and $s(J') = \bullet$.

(vi) If $\nu_\lambda = \mathrm{VI}$ and $(a(C), b(C)) = (0, 1)$ for $C \supset K \in \mathscr{K}(\Delta_\lambda) - \mathscr{K}_\lambda$, then $\sigma(*y') = -\sigma(*y)$ for $y \in K$ and $y' \in J$.

(vii) If $\nu_\lambda = \mathrm{VII}$, then $\sigma(*y') = \sigma(*y)$ for $y \in J$ and $y' \in J'$.

In (R2) and (R3), we used the description $P_\lambda = (\Delta_\lambda, \nu_\lambda, s: \mathscr{J}_\lambda \to \mathscr{S}, \omega: \mathscr{K}_\lambda \to \{1, -1\})$.

Definition 13.4. For a regular TS piece $P = (\Delta, \nu, s: \mathscr{J} \to \mathscr{S}, \omega: \mathscr{K} \to \{1, -1\})$, let $-P = (\Delta, \nu, s, -\omega)$. For a pre TS diagram $\hat{\mathscr{T}} = (S, \{P_\lambda\}_{\lambda \in \Lambda}, \{\iota_\lambda\}_{\lambda \in \Lambda})$, let $-\hat{\mathscr{T}} = (S, \{-P_\lambda\}_{\lambda \in \Lambda}, \{\iota_\lambda\}_{\lambda \in \Lambda})$. For a regular TS diagram $\mathscr{T} = (\hat{\mathscr{T}}, \{\phi_t\}_{t \in I}, (a, b; r))$, let $-\mathscr{T} = (-\hat{\mathscr{T}}, \{\phi_t\}_{t \in I}, (a, b; r))$.

We introduce an equivalence relation on regular TS diagrams of $(\Sigma_g(h); \sigma)$ as follows.

Definition 13.5. Let $\mathscr{T} = (\hat{\mathscr{T}}, \{\phi_t\}_{t \in I}, (a, b; r))$ and $\mathscr{T}' = (\hat{\mathscr{T}}', \{\phi_t\}_{t \in I}, (a', b'; r'))$ be regular TS diagrams of $(\Sigma_g(h); \sigma)$. Then $\mathscr{T}$ is *isomorphic* to $\mathscr{T}'$ if there exists a C^0 isotopy of homeomorphisms $\{h_t: \Sigma_g(h) \to \Sigma_g(h)\}_{t \in I}$, such that $\phi'_t = h_t \circ \phi_t$ for $t \in I$, h_0 is an isomorphism from $\hat{\mathscr{T}}$ to $\hat{\mathscr{T}}'$ or $-\hat{\mathscr{T}}'$, h_0 is an isotopic to the identity, and $(a', b'; r') = (a, b; r)$. We denote by $\mathrm{RTS}(\Sigma_g(h); \sigma)$ the set of isomorphism classes of regular TS diagrams of $(\Sigma_g(h); \sigma)$.

14. The classification theorem. Usng regular TS diagrams, we can generalize the classification theorem of Tamura-Sato [15]. In this paper we give only the following.

Theorem 4. *Let* $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$. *If* $\mathscr{G}$ *is not* C^0 *isomorphic to*

$(\mathscr{G}|C)\times I$ *for any horizontal* $C\in\Gamma(h)$, *then an element of* RTS $(\hat{E}(h);\sigma)$ *is canonically attached to* $\mathscr{G}$.

PROOF. Take an orientation of $\mathscr{G}|F^{[0]}$. Let $\mathscr{C}_1,\cdots,\mathscr{C}_n$ be the strange negative Reeb cycles of $\mathscr{G}$ and S_i^* a separating torus of $\mathscr{C}_i$. Define a map $(a, b; r)\colon \Gamma(h)\to(N\times Z)^*\times 2Z$ as in §4, and let $(\bar{a},\bar{b};\bar{r})=(a, b; r)\circ\zeta$, where $\zeta\colon\hat{\Gamma}(h)\to\Gamma(h)$ is defined by $\zeta(\hat{C})=S^1\times C$.

We construct a C^∞ vector field Z on A transverse to A^x for all $x\in S^1$ and tangent to $S_1^*,\cdots,S_n^*$ and ∂A. First we define Z on ∂A as follows. Let $C\in\Gamma(h)$. When C is horizontal or $\mathscr{G}|\partial_C A$ has no compact leaf, let $Z=\partial/\partial t$ on $\partial_C A$, where t is the coordinate of the factor S^1. When C is vertical and $\mathscr{G}|C$ has a compact leaf, take as $Z|\partial_C A$ a vector field transverse to A^x for all $x\in S^1$, tangent to the compact leaves of $\mathscr{G}|\partial_C A$, and having no non-closed orbit. By Proposition 7.1, the foliation $\mathscr{G}|S_i^*$ has a compact leaf. Take as $Z|S_i^*$ a vector field transverse to A^x for all $x\in S^1$, tangent to the compact leaves of $\mathscr{G}|S_i^*$, and having no non-closed orbits. Then we can take as $Z|G$ a vector field on G for each compact subleaf G of $\mathscr{G}|A$ in a consistent way by Propositions 7.1 and 8.1. Furthermore we can extend the vector field thus obtained over all A by the decomposition theorem.

We define a C^∞ isotopy $\{\phi_t\colon A^{[0]}\to A^{[0]}\}_{t\in I}$ as follows. Let $C(A, A^{[0]})$ be the compact manifold obtained from A by cutting along $A^{[0]}$. Denote by A^0 (or A^1) the bottom (or top) of $C(A, A^{[0]})$. Let $A^t=A^{[t]}$ for $t\in\,]0,1[$. Now define $\bar{\phi}_t(z)$ for $z\in A^{[0]}$ as the intersection point of A^t and the orbit of Z passing through z, and define $\phi_t(z)$ as the intersection point of $A^{[0]}$ and the fiber of the projection: $S^1\times\hat{E}(h)\to S^1$ passing through $\bar{\phi}_t(z)$.

In a canonical way, we can construct a pre TS diagram $\hat{\mathscr{T}}=(S,\{P_\lambda\}_{\lambda\in\Lambda},\{\iota_\lambda\}_{\lambda\in\Lambda})$ of $A^{[0]}$ satisfying the following conditions.

(1) $S=A^{[0]}\cap(S_1^*\cup\cdots\cup S_n^*)$.

(2) $\{|P_\lambda|\}_{\lambda\in\Lambda}$ (or $\{^*|P_\lambda|\}_{\lambda\in\Lambda}$) coincides with $\Omega^{[0]}\cup\Theta^{[0]}$ (or $^*\Omega^{[0]}\cup{}^*\Theta^{[0]}$) except some compact subleaf $\in\Theta^{[0]}$ (or $^*\Theta^{[0]}$).

(3) To an element $D_j^{[0]}\in\Omega^{[0]}$ corresponds a regular TS piece of the same type as D.

(4) To an element of $\Theta^{[0]}$ corresponds a regular TS piece of type II with symbol $\|$ or a regular TS piece of type IX.

(5) The symbols ○, ● and $\|$ correspond to the components stated in Remark 12.7.

(6) The symbol $\vee$ (or $\wedge$) for $J\subset\partial A^{[0]}$ corresponds to a slope component $\mathscr{C}$ contained in $\mathscr{G}|\partial A^{[0]}$ such that a connected component of $\partial|\mathscr{C}|$ has an expanding holonomy with respect to $\mathscr{C}$ in the same (or

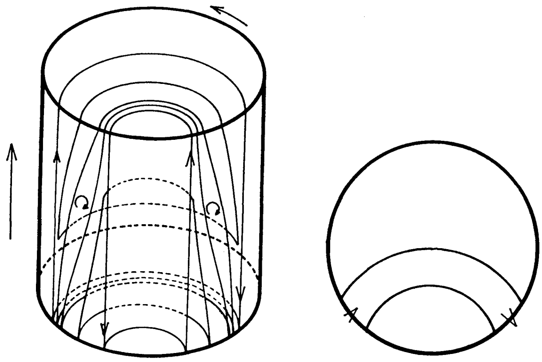


FIGURE 14.1

opposite) direction as the orientation of $\partial|\mathscr{E}|$. (See Figure 14.1.)

(7) For $K \in \mathscr{K}_\lambda$, the orientation of $^*K$ determined by ω coincides with that of $\mathscr{G}|F^{[0]}$.

Now take a diffeomorphism $\xi: A^{[0]} \to \hat{E}(h)$ isotopic to the identity. Transforming $(\hat{\mathscr{T}}, \{\phi_t\}_{t\in I})$ by ξ, we have a triad $\mathscr{T}' = (\hat{\mathscr{T}}', \{\phi_t': \hat{E}(h) \to \hat{E}(h)\}_{t\in I}, (\bar{a}, \bar{b}; \bar{r}))$. Then $\mathscr{T}'$ is a regular TS diagram of $(\hat{E}(h); \sigma)$ and the isomorphish class of $\mathscr{T}'$ depends only on $\mathscr{G}$. Since the check of the details is tedious, we omit it except for the condition (R2-s) in Definition 13.3. For checking (R2-s), we need the following.

PROPOSITION 14.1. *Let $\hat{\mathscr{T}} = (S, \{P_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\}_{\lambda\in\Lambda})$ be a pre* TS *diagram and $\{\phi_t: \Sigma_g(h) \to \Sigma_g(h)\}_{t\in I}$ an isotopy satisfying the condition* (R1) *in Definition* 13.3. *Let S_i be a circle in S. Then $\phi_1(S_i) = S_i$.*

PROOF. Suppose that $\phi_1(S_i) \neq S_i$. Since $\phi_1(S) = S$, it follows that $S_j = \phi_1(S_i) \subset S$. Take $\lambda, \lambda' \in \Lambda$, $K \in \mathscr{K}_\lambda$, $K' \in \mathscr{K}_{\lambda'}$, and $C, C' \in \Gamma_g(h)$ satisfying the conditions in Definition 13.1 (RP3). Since $\phi_1(C) = C$ and $\phi_1(C') = C'$, it follows that $\phi_1(^*\Delta_\lambda) \cap C \neq \varnothing$, $\phi_1(^*K') \cap S_j \neq \varnothing$ and $\phi_1(^*K') \cap C' \neq \varnothing$. Take a path $\pi: I \to {}^*\Delta_\lambda$ with $\pi(0) \in C$ and $\pi(1) \in {}^*K' \cap S_i$. Consider the circle

$$L = \pi(I) \cup {}^*K' \cup \{\phi_t(\pi(0)) \,|\, t \in I\} \cup \phi_1(\pi(I) \cup {}^*K') \cup \{\phi_t(y) \,|\, t \in I\}\,,$$

where $\{y\} = {}^*K' \cap C'$. Then the intersection number $L \cdot S_i$ equals ± 1. On the other hand, L is the boundary of the degenerate disk $\{\phi_t(\pi(I) \cup {}^*K') \,|\, t \in I\}$ as a singular chain. Hence $[L] = 0$ in $H_1(\Sigma_g(h); \boldsymbol{Z})$, which is a contradiction. This completes the proof of Proposition 14.1.

THE CHECK OF (R2-s). By Proposition 14.1, we see that $S_i^* \cap A^{[0]}$ consists of exactly one circle. Furthermore a compact leaf of $\mathscr{G}|S_i^*$ is isotopic to a compact leaf in $\mathscr{G}|C'$ if for $C' \in \Gamma(h)$ there is a compact subleaf of $\mathscr{G}$ intersecting S_i^* and C'. The same argument as in the proof of Proposition 8.1 implies that $(a(C'), b(C')) \neq (a(C), -b(C))$, where $C \supset |\mathscr{N}|$ for some negative Reeb component $\mathscr{N}$ in $\mathscr{C}_i$. This completes the proof of Theorem 4.

PART II

Existence problem of transverse foliations

15. An investigation of regular TS diagrams in the case $h = 2$. In order to check the conditions (A2) and (A3) for $\mathscr{G} \in t_1^0(\mathscr{F}(h; \sigma))$, we investigate regular TS diagrams thoroughly in this and next sections. This is the most essential part of the proof of Theorem 1 and can be regarded also as an addendum to Part I.

Let $\mathscr{T} = (\hat{\mathscr{T}}, \{\phi_t\}_{t \in I}, (a, b; r))$ be a regular TS diagram of $(\hat{E}(h); \sigma)$. Let $\hat{\mathscr{T}} = (S, \{P_\lambda\}_{\lambda \in \Lambda}, \{\iota_\lambda\}_{\lambda \in \Lambda})$ and $P_\lambda = (\Delta_\lambda, \nu_\lambda, s: \mathscr{J}_\lambda \to \mathscr{S}, \omega: \mathscr{K}_\lambda \to \{1, -1\})$. Transforming $(a, b; r)$ by the canonical bijection $\hat{}: \Gamma(h) \to \hat{\Gamma}(h)$, we can regard it as a map from $\Gamma(h)$ to $(\boldsymbol{N} \times \boldsymbol{Z})^* \times 2\boldsymbol{Z}$.

Note the following, which is a direct consequence of Definition 13.3 (R2). We omit the proof.

LEMMA 15.1. *Let $(a, b; r): \Gamma(h) \to (\boldsymbol{N} \times \boldsymbol{Z})^* \times 2\boldsymbol{Z}$ be in a regular* TS *diagram.*

(1) *If $r(C) \neq 0$, then $(a(C), b(C)) \in (\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}}$.*

(2) *If $a(C)r(C) = 0$, then $r(C) = 0$.*

Let $\hat{E}_0$ be the closure of a connected component of $\hat{E}(h) - S$. Clearly $\hat{E}_0$ is diffeomorphic to $\hat{E}(h_0)$ for some $h_0 \in \boldsymbol{N}$. Let $\Gamma_0 = \{C \in \Gamma(h) \mid \hat{C} \subset \partial \hat{E}_0\}$. Then we have the following.

PROPOSITION 15.2 (The TS formula for regular TS diagrams).

(1) $\sum_{C \in \Gamma_0} a(C)r(C) = 4 - 2h_0$.

(2) $\sum_{C \in \Gamma(h)} a(C)r(C) = 4 - 2h$.

PROOF. We can construct a vector field Y on $\hat{E}(h)$ satisfying the following conditions.

(1) For each $J \in \mathscr{J}_\lambda$, $\lambda \in \Lambda$, with $s(J) = \bigcirc$ (or $\bullet$), there is a singular point $z(J) \in \operatorname{Int} {}^*J$ of Y such that the orbits of Y near $z(J)$ are concentric half circles (or confocal parabolas). (See Figure 4.2.) Furthermore Y has no other singular point.

(2) Y is tangent to ${}^*K$ for each $K \in \mathscr{K}_\lambda$, $\lambda \in \Lambda$, and the direction of Y coincides with the orientation of ${}^*K$ determined by $\omega(K)$.

Now we can prove Proposition 15.2 in the same way as Proposition 4.3.

Hereafter we suppose that $h=2$. The goal of this section is the following, which corresponds to (A2).

THEOREM 5. *Let $\mathscr{T}=(\hat{\mathscr{T}}, \{\phi_t\}_{t\in I}, (a, b; r))$ be a regular* TS *diagram of $(\hat{E}(2); \sigma)$. Let $\Gamma(h)=\{C, C'\}$. Then*

(1) $r(C')=-r(C)$.

(2) *If $r(C)\neq 0$, then $(a(C'), b(C'))=(a(C), -b(C))$.*

In order to prove Theorem 5, we prove three lemmas. Let $(a(C), b(C); r(C))=(a, b; r)$ and $(a(C'), b(C'); r(C))=(a', b'; r')$. The first lemma is the following.

LEMMA 15.3. *If there are $\lambda\in\Lambda$ and $K\in\mathscr{K}_\lambda$ such that $^*K\cap\hat{C}\neq\varnothing$ and $^*K\cap\hat{C}'\neq\varnothing$, then $(a', b'; r')=(a, -b; -r)$.*

PROOF. Since $^*K\cap\hat{C}\neq\varnothing$ and $^*K\cap\hat{C}'\neq\varnothing$, it follows that (a, b), $(a', b')\in(\boldsymbol{N}\times\boldsymbol{Z})^{\text{coprime}}$ by Definition 13.3 (R2-i). Let $^*K\cap\hat{C}=\{y\}$ and $^*K\cap\hat{C}'=\{y'\}$. Then $\phi_1^a(y)=y$ and $\phi_1^k(y)\neq y$ for $0<k<a$ by (R2-ii). Similarly $\phi_1^{a'}(y)=y'$ and $\phi_1^k(y')\neq y'$ for $0<k<a'$. Since ϕ_1 is an isomorphism from $\hat{\mathscr{T}}$ to $\hat{\mathscr{T}}$ by (R1), it follows that $\phi_1^a(^*K)=\phi_1^{a'}(^*K)={}^*K$ and $\phi_1^k(^*K)\neq{}^*K$ for $0<k<a$ or $0<k<a'$. This implies that $a'=a$.

Modifying $\{\phi_t\}_{t\in I}$ if necessary, we may suppose that $(\phi_1|{}^*K)^a=\mathrm{id}$ since $^*K$ is diffeomorphic to I. Define a map $F\colon S_a^1\times{}^*K\to\hat{E}(h)$ by $F([t], z)=\phi_{t'}(\phi_1^k(z))$ for $[t]\in S_a^1=[0, a]/\{0, a\}$, $t=k+t'$, $k\in\boldsymbol{Z}$, $0\leqq t'<1$ and $z\in{}^*K$. Then F can be regarded as a homotopy from $F|S_a^1\times\{y\}$ to $F|S_a^1\times\{y'\}$. By (R2-ii), it follows that

$$b[\hat{C}]=F_*([S_a^1\times\{y\}])=F_*([S_a^1\times\{y'\}])=b'[\hat{C}']$$

in $H_1(\hat{E}(2); \boldsymbol{Z})$. Since $[\hat{C}']=-[\hat{C}]\neq 0$, we have $b'=-b$. Since $ar+ar'=0$ by Proposition 15.2, it follows that $r'=-r$, which completes the proof of Lemma 15.3.

The second is the following.

LEMMA 15.4. *If $S\neq\varnothing$, then $(a', b')\neq(a, -b)$ and $r'=r=0$.*

PROOF. By the condition (R2-s) in Definition 13.3, it follows that $(a', b')\neq(a, -b)$. Since $\hat{E}(2)$ is an annulus, a circle in $\hat{E}(2)$ bounds a disk or is isotopic to $\hat{C}$ in $\hat{E}(2)$. By the condition (PR1) in Definition 13.1, a circle in S bounds no disk. Furthermore we see that S is connected. By Proposition 15.2 (1), it follows that $ar=a'r'=0$. By Lemma 15.1 (2), we have $r=r'=0$. This completes the proof of Lemma 15.4.

Now Theorem 5 follows directly from Lemmas 15.3 and 15.4 and the following.

LEMMA 15.5. *Suppose that* $S=\emptyset$ *and that for any* $\lambda\in\Lambda$ *and* $K\in\mathscr{K}_\lambda$ *with* $^*K\cap\hat{C}\neq\emptyset$ *it holds that* $^*K\cap\hat{C}'=\emptyset$. *Then* $r=r'=0$.

PROOF. Let $V=\bigcup\{^*\Delta_\lambda\,|\,^*\Delta_\lambda\cap\hat{C}\neq\emptyset\}$. When $V\cap\hat{C}'=\emptyset$, there is a circle in ∂V separating $\hat{C}$ and $\hat{C}'$. Then we can show that $ar=a'r'=0$ in the same way as in the proof of Proposition 15.2. Hence $r=r'=0$. When $V\cap\hat{C}'\neq\emptyset$, there is exactly one regular TS piece P_λ of type V or VI. Taking a circle in Int $^*\Delta_\lambda$ separating $\hat{C}$ and $\hat{C}'$, we see that $ar=a'r'=0$ as above. Hence $r=r'=0$. This completes the proof of Lemma 15.5 and Theorem 5.

16. An investigation of regular TS diagrams in the case $h>2$. The purpose of this section is to prove the following, which corresponds to (A3).

THEOREM 6. *Let* $\mathscr{T}=(\hat{\mathscr{T}},\{\phi_t\}_{t\in I},(a,b;r))$ *be a regular* TS *diagram of* $(\hat{E}(h);\sigma)$ *and suppose that* $h>2$. *If* $r(C)\neq 0$ *for* $C\in\Gamma(h)$, *then* $(a(C),b(C))=(1,0)$.

We use the same notations as in §15. First we prove the following.

LEMMA 16.1. *Suppose that* $h>2$ *and let* $C\in\Gamma(h)$. *If there are* $\lambda\in\Lambda$ *and* $K\in\mathscr{K}_\lambda$ *such that* $^*K\cap\hat{C}\neq\emptyset$ *and* $^*K\cap\hat{C}'\neq\emptyset$ *for some* $C'\in\Gamma(h)-\{C\}$, *then* $(a(C),b(C))=(1,0)$.

PROOF. Let $(a(C),b(C))=(a,b)$ and $(a(C'),b(C'))=(a',b')$. Then we see that $a=a'$ and $b[\hat{C}]=b'[\hat{C}']$ in $H_1(\hat{E}(h);\boldsymbol{Z})$ as in the proof of Lemma 15.3. Since $h>2$, the homology classes $[\hat{C}]$ and $[\hat{C}']$ have no linear relation. Therefore $b=b'=0$. Since $(a,b)\in(\boldsymbol{N}\times\boldsymbol{Z})^{\text{coprime}}$, it follows that $a=1$. This completes the proof of Lemma 16.1.

Now Theorem 6 follows directly from Lemma 16.1 and the following.

LEMMA 16.2. *Suppose that* $h>2$. *Let* $C\in\Gamma(h)$ *and* $K(C)=\bigcup\{^*K\,|\,\lambda\in\Lambda,K\in\mathscr{K}_\lambda,{}^*K\cap\hat{C}\neq\emptyset\}$. *If* $K(C)\cap\hat{C}'=\emptyset$ *for all* $C'\in\Gamma(h)-\{C\}$, *then* $(a(C),b(C))=(1,0)$ *or* $r(C)=0$.

PROOF. Suppose that $r(C)\neq 0$. Let $(a(C),b(C))=(a,b)$. We are going to prove that $(a,b)=(1,0)$. By Lemma 15.1 (1), it follows that $(a,b)\in(\boldsymbol{N}\times\boldsymbol{Z})^{\text{coprime}}$.

When $K(C)\cap S\neq\emptyset$, we see that $(a,b)=(1,0)$, as follows. Since $r(C)\neq 0$, the connected component of $\hat{E}(h)-S$ containing $\hat{C}$ is homeomorphic to $\hat{E}(h')$ for some $h'>2$ by Proposition 15.2 (1). Then the arguments in the proof of Lemma 16.1 implies that $(a,b)=(1,0)$.

Hereafter suppose that $K(C)\cap S=\emptyset$. Let $V=\bigcup\{^*\Delta_\lambda\,|\,^*\Delta_\lambda\cap\hat{C}\neq\emptyset\}$

and $\partial_0 V = \partial V - \{\hat{C}\}$. Denote by B the set of connected components of $\partial_0 V$. Then we can write

$$B = \{\hat{C}_1, \cdots, \hat{C}_\alpha, S_1, \cdots, S_\beta, {}^*K_1, \cdots, {}^*K_\gamma\},$$

where $C_j \in \Gamma(h)$, $S_j \subset S$ and $K_j \in \mathscr{K}_{\lambda(j)}$ for $\lambda(j) \in \Lambda$. Clearly $P_{\lambda(j)}$ is of type VII and ${}^*\Delta_{\lambda(j)} \subset V$. For each C_j, there is $\mu(j) \in \Lambda$ such that ${}^*\Delta_{\mu(j)} \subset V$, $\nu_{\mu(j)} = \mathrm{VI}$ and $\hat{C}_j = {}^*L_j$ for $L_j \in \mathscr{K}(\Delta_{\mu(j)}) - \mathscr{K}_{\mu(j)}$. For each S_j, there is $\rho(j) \in \Lambda$ such that ${}^*\Delta_{\rho(j)} \subset V$, $\nu_{\rho(j)} = \mathrm{VI}$ and $S_j = {}^*M_j$ for $M_j \in \mathscr{K}(\Delta_{\rho(j)}) - \mathscr{K}_{\rho(j)}$.

We see that $\sharp B = \alpha + \beta + \gamma > 1$ as follows. Suppose that $\sharp B = 1$. Then V is homeomorphic to $\hat{E}(2)$. Since the TS formula can be obtained for V, we have $r(C) = 0$ from it. Thus we have a contradiction.

Applying the arguments in the proof of Proposition 14.1, we see that $\phi_1({}^*K_j) = {}^*K_j$. Therefore ϕ_1 fixes all the elements of B. Since $\sharp B > 1$, the map ϕ_1 must fix all $J \in \mathscr{J}_{\mu(1)} \cup \cdots \cup \mathscr{J}_{\mu(\alpha)} \cup \mathscr{J}_{\rho(1)} \cup \cdots \cup \mathscr{J}_{\rho(\beta)} \cup \mathscr{J}_{\lambda(1)} \cup \cdots \cup \mathscr{J}_{\lambda(\gamma)}$. This implies that $\alpha = 1$ and $\phi_1 | \hat{C} = \mathrm{id}$.

Since $\sharp B > 1$, there are $\lambda \in \Lambda$ and $K \in \mathscr{K}_\lambda$ such that ${}^*K \subset K(C)$ and $[{}^*K] \neq 0$ in $H_1(V, \hat{C}; \boldsymbol{Z})$. Let L be the union of ${}^*K$ and a connected component of $\hat{C} - {}^*K$. Let $y_0 \in \hat{C} \cap {}^*K$. Then $\hat{C}$ and L determine elements ξ_1 and ξ_2 of $\pi_1(V, y_0)$, respectively. Adding adequate elements $\xi_3, \cdots, \xi_{\alpha+\beta+\gamma}$, we can regard $\pi_1(V, y_0)$ as the free group generated by $\xi_1, \cdots, \xi_{\alpha+\beta+\gamma}$. Modifying $\{\phi_t\}_{t\in I}$ in a neighborhood of ${}^*K$, we may suppose that $\phi_1 | L = \mathrm{id}$. Then we can define a map $\eta: S^1 \times L \to V$ by $\eta([t], y) = \phi_t(y)$ for $t \in I$ and $y \in L$. The paths $\eta | S^1 \times \{y_0\}$ and $\eta | \{[0]\} \times L$ represent ξ_1^b and ξ_2, respectively. Since $\pi_1(S^1 \times L, ([0], y_0))$ is abelian, it follows that $\xi_1^b \cdot \xi_2 = \xi_2 \cdot \xi_1^b$. Therefore $b = 0$. This completes the proof of Lemma 16.2 and Theorem 6.

17. The proof of Theorem 1. Let $\mathscr{G} \in t_1^*(\mathscr{F}(\Phi, \Psi; \sigma))$, where Φ, Ψ and σ be in § 1. We use the notations in § 1. For each vertex $v \in V(\Phi)$, consider $\mathscr{G} | E[v]$ and regard it as an element of $t_1^0(\mathscr{F}(\Phi, \Psi; \sigma) | E[v]) = t_1^0(\mathscr{F}(h(v); \sigma))$. For each $C \in \Gamma[v] = \Gamma(h(v))$, define $(a(C), b(C); r(C))$ by using $\mathscr{G} | E[v]$ as in § 4. Then we have a map $(a, b; r): \Gamma[\Phi] \to (\boldsymbol{N} \times \boldsymbol{Z})^* \times \boldsymbol{Z}$. We are going to show that $(a, b; r)$ is an arithmetic model transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$.

The transverse orientability of $\mathscr{G} | E[v]$ for each $v \in V(\Phi)$ implies that Image $r \in 2\boldsymbol{Z}$. The condition (A4) in Definition 1.2 holds by Proposition 4.3. The condition (A1) follows from (A4).

When $\mathscr{G} | \mathrm{E}[v]$ is not C^0 isomorphic to $(\mathscr{G} | C) \times I$ for any $C \in \Gamma[v]$, we can attach to $\mathscr{G} | E[v]$ a regular TS diagram with $(a, b; r) | \Gamma[v]$ by Theorem 4. Then the conditions (A2) and (A3) are guaranteed by

Theorems 5 and 6, respectively. When $\mathscr{G}\,|\,E[v]$ is C^0 isomorphic to $(\mathscr{G}\,|\,C)\times I$ for some $C\in\Gamma[v]$, we have $h(v)=2$. Let $\Gamma[v]=\{C, C'\}$. Then we see the following.

(i) If $(a(C), b(C))=(\infty, \infty)$, then $(a(C'), b(C'))=(\infty, \infty)$ and $r(C)=r(C')=0$.

(ii) If $(a(C), b(C))=(0, 1)$, then $(a(C'), b(C'))=(0, 1)$ and $r(C')=-r(C)$.

(iii) If $(a(C), b(C))\in(\boldsymbol{N}\times\boldsymbol{Z})^{\text{coprime}}$, then $(a(C'), b(C'))=(a(C), -b(C))$ and $r(C')=-r(C)$.

Therefore the condition (A2) holds.

Finally we check the condition (A5), as follows. Let $s\in S(\Phi)$ with $\partial(s)=(v_1)-(v_2)$, $v_1, v_2\in V(\Phi)$ and $C_i=C[v_i](s)$. Since $(\Psi^*[s])^*(\mathscr{G}\,|\,C_2)=\mathscr{G}\,|\,C_1$, we see that if $a(C_1)=\infty$ then $a(C_2)=\infty$. When $a(C_1)\neq\infty$, the foliation $\mathscr{G}\,|\,C_1$ has a compact leaf L and we have

$$(\Psi^*[s])_*[L]=(ka(C_1)+lb(C_1))[S^1\times\{*\}]+(ma(C_1)+nb(C_1))[\{*\}\times\hat{C}_2]\,.$$

Then it follows that

$$\begin{pmatrix}a(C_2)\\ b(C_2)\end{pmatrix}=\gamma_1\begin{pmatrix}k & l\\ m & n\end{pmatrix}\begin{pmatrix}a(C_1)\\ b(C_1)\end{pmatrix}\quad\text{and}\quad r(C_2)=\gamma_2 r(C_1)\,,$$

where γ_1 and γ_2 are as in Definition 1.2. This completes the proof of Theorem 1.

18. Some remarks on arithmetic models. In this section, we investigate the properties of arithmetic models. Let Φ, Ψ and σ be as in §1. First we obtain some informations on a side $s\in S(\Phi)$ such that $h(v_1)>2$ and $h(v_2)>2$, where $\partial(s)=(v_1)-(v_2)$, $v_1, v_2\in V(\Phi)$. The following is useful.

DEFINITION 18.1. (1) A side $s\in S(\Phi)$ is called *longitude preserving* if $\Psi(s)=\begin{pmatrix}1 & l\\ 0 & -1\end{pmatrix}$ or $\begin{pmatrix}-1 & l\\ 0 & 1\end{pmatrix}$ for some $l\in\boldsymbol{Z}$, and otherwise *longitude twisting*.

(2) Let $s\in S(\Phi)$ with $\partial(s)=(v_1)-(v_2)$ and $\Psi(s)=\begin{pmatrix}k & l\\ m & n\end{pmatrix}$. For s with $k=1$ or -1, let

$$\xi(s)=k\cdot\sigma(C[v_1](s))\cdot\sigma(C[v_2](s))\,.$$

We call $\xi(s)$ the *glueing sign* of s.

Now we have the following.

PROPOSITION 18.2. *Let* $\mathscr{A}=(a, b; r)\in\mathrm{am}(\Phi, \Psi; \sigma)$. *Let* $s\in S(\Phi)$ *with* $\partial(s)=(v_1)-(v_2)$ *and* $C_j=C[v_j](s)$, $j=1, 2$. *Suppose that* $h(v_1)>2$ *and* $h(v_2)>2$.

(1) *If s is longitude twisting, then* $r(C_1)=r(C_2)=0$.

(2) *If s is longitude preserving, then $\mathscr{A}'$ obtained from $\mathscr{A}$ by changing $(a(C_1), b(C_1))$ and $(a(C_2), b(C_2))$ for $(1, 0)$ is also an arithmetic model transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$.*

PROOF. (1) Suppose that $r(C_1)\neq 0$. Then $r(C_2)\neq 0$ by (A5) in Definition 1.3. By (A3), it follows that $(a(C_j), b(C_j))=(1, 0)$ for $j=1, 2$. Let $\Psi(s)=\begin{pmatrix} k & l \\ m & n \end{pmatrix}$. By (A5), we have $\begin{pmatrix} k & l \\ m & n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\operatorname{sgn}(k)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore $m=0$ and $k=-n=1$ or -1.

(2) When $r(C_1)\neq 0$, we have $r(C_2)\neq 0$ and $\mathscr{A}'=\mathscr{A}$ by (A5) and (A3). Suppose that $r(C_1)=0$. Then $r(C_2)=0$ by (A5). It is easy to see that $\mathscr{A}'\in \mathrm{am}(\Phi, \Psi; \sigma)$. This completes the proof of Proposition 18.2.

By Proposition 18.2, we have simpler equations to determine whether $\mathrm{am}(\Phi, \Psi; \sigma)$ is empty or not in the case $h(v)>2$ for all $v\in V(\Phi)$, as follows. We omit the proof.

THEOREM 7. *Suppose that $h(v)>2$ for all $v\in V(\Phi)$. Then $\mathrm{am}(\Phi, \Psi; \sigma)\neq \emptyset$ if and only if there is a map $r\colon \Gamma[\Phi]\to 2\boldsymbol{Z}$ satisfying the following conditions.*

(A3)$'$ *If s is longitude twisting, then $r(C[v_1](s))=r(C[v_2](s))=0$, where $\partial(s)=(v_1)-(v_2)$.*

(A4)$'$ $\sum_{C\in\Gamma[v]} r(C)=4-2h(v)$ *for all $v\in V(\Phi)$.*

(A5)$'$ *If s is longitude preserving, then $r(C[v_2](s))=\xi(s)r(C[v_1](s))$, where $\partial(s)=(v_1)-(v_2)$.*

For $v\in V(\Phi)$ with $h(v)=1$, we have the following.

PROPOSITION 18.3. *Let $s\in S(\Phi)$ with $\partial(s)=(v_1)-(v_2)$. If $h(v_1)>2$, $h(v_2)=1$ and $\mathrm{am}(\Phi, \Psi; \sigma)\neq\emptyset$, then $\Psi(s)=\begin{pmatrix} 1 & l \\ m & n \end{pmatrix}$ or $\begin{pmatrix} -1 & l \\ m & n \end{pmatrix}$ for some $l, m, n\in \boldsymbol{Z}$.*

PROOF. Let $(a, b; r)\in \mathrm{am}(\Phi, \Psi; \sigma)$. By (A1) and (A4), we have $a(C_2)=1$ and $r(C_2)=2$. Then $r(C_1)\neq 0$ by (A5) and $(a(C_1), b(C_1))=(1, 0)$ by (A3). Let $\Psi(s)=\begin{pmatrix} k & l \\ m & n \end{pmatrix}$. Since $\begin{pmatrix} k & l \\ m & n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\operatorname{sgn}(k)\begin{pmatrix} 1 \\ b(C_2) \end{pmatrix}$, it follows that $k=1$ or -1 and $m=kb(C_2)$.

When the graph Φ is a tree, we have the following.

PROPOSITION 18.4. *Suppose that Φ is a tree and that $h(v)\neq 2$ for all $v\in V(\Phi)$. If $(a_1, b_1; r_1), (a_2, b_2; r_2)\in \mathrm{am}(\Phi, \Psi; \sigma)$, then $r_1=r_2$.*

In order to prove Proposition 18.4, we need the following lemma. The proof is easy and we omit it.

LEMMA 18.5. *Suppose that Φ is a tree. Then there exists a sequence $\Gamma_0, \cdots, \Gamma_\rho$, $\rho = \sharp S(\Phi)$, of subsets of $\Gamma[\Phi]$ satisfying the following conditions.*

(1) $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_\rho = \Gamma[\Phi]$.

(2) *For each $j = 1, \cdots, \rho$, there is $s_j \in S(\Phi)$ with $\Gamma_j - \Gamma_{j-1} = \{C[v_j](s_j), C[v'_j](s_j)\}$, where $\partial(s_j) = \pm((v_j) - (v'_j))$.*

(3) *For each $j = 1, \cdots, \rho - 1$, define a subgraph Φ_j of Φ by*

$$V(\Phi_j) = \{v \in V(\Phi) \mid \Gamma[v] \not\subset \Gamma_j\} \quad \text{and} \quad S(\Phi_j) = S(\Phi) - \{s_1, \cdots, s_j\} ,$$

where $V(\Phi_j)$ (or $S(\Phi_j)$) is the set of vertices (or sides) of Φ_j. Then Φ_j is a tree, and $\sharp(\Gamma[v_j] - \Gamma_{j-1}) = 1$.

PROOF OF PROPOSITION 18.4. We use the sequence $\Gamma_0, \cdots, \Gamma_\rho$ in Lemma 18.5. Let $C_j = C[v_j](s_j)$ and $C'_j = C[v'_j](s_j)$. We prove

$$((j)) \qquad r_1(C) = r_2(C) \quad \text{for} \quad C \in \Gamma_j ,$$

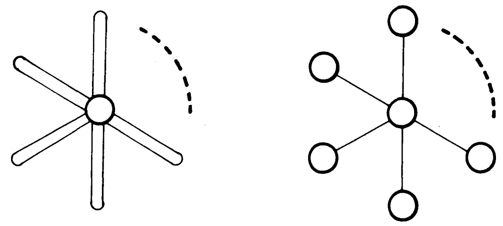
by induction on j. First we see that $h(v_1) = 1$. Then we have $a_1(C_1) = a_2(C_1) = 1$ and $r_1(C_1) = r_2(C_1) = 2$. Since $r_1(C'_1) \neq 0$ and $r_2(C'_1) \neq 0$ by (A5), it follows that $(a_1(C'_1), b_1(C'_1)) = (a_2(C'_1), b_2(C'_1)) = (1, 0)$ by (A3). Then we have $b_1(C_1) = b_2(C_1)$ by (A5). Using (A5) once more, we get $r_1(C'_1) = r_2(C'_1)$. Therefore ((1)) holds.

Now suppose $((j))$. Since $\sharp(\Gamma[v_{j+1}] - \Gamma_j) = 1$, we see that $a_1(C_{j+1})r_1(C_{j+1}) = a_2(C_{j+1})r_2(C_{j+1})$ by (A4) and the fact that $a_1(C) = a_2(C) = 1$ if $r_1(C) \neq 0$ for $C \in \Gamma_j$. When $h(v_{j+1}) = 1$, we verify $((j + 1))$ as above. Suppose that $h(v_{j+1}) > 2$. If $a_1(C_{j+1})r_1(C_{j+1}) = 0$, we have $r_1(C_{j+1}) = r_2(C_{j+1}) = 0$ by Lemma 15.1 (2). By (A5), it follows that $r_1(C'_{j+1}) = r_2(C'_{j+1}) = 0$. Now consider the case $a_1(C_{j+1})r_1(C_{j+1}) \neq 0$. Since $r_1(C_{j+1}) \neq 0$ and $r_2(C_{j+1}) \neq 0$, it follows that $r_1(C'_{j+1}) \neq 0$ and $r_2(C'_{j+1}) \neq 0$ by (A5). Then we have $(a_1(C), b_1(C)) = (a_2(C), b_2(C)) = (1, 0)$ for $C = C_{j+1}, C'_{j+1}$ by (A3). Therefore $r_1(C) = r_2(C)$ for $C = C_{j+1}, C'_{j+1}$. Thus $((j + 1))$ holds. This completes the proof of Proposition 18.4.

19. Some application of the arithmetic criterion. The purpose of this section is to determine whether $\mathrm{am}(\Phi, \Psi; \sigma)$ is empty or not for some $\mathscr{F}(\Phi, \Psi; \sigma)$. First consider the graphs in Figure 19.1.

PROPOSITION 19.1. *Suppose that $V(\Phi) = \{v\}$ and $\sharp S(\Phi) > 1$. (See Figure 19.1 (a).) Then $\mathrm{am}(\Phi, \Psi; \sigma) \neq \emptyset$ if and only if there is a longitude preserving side $s \in S(\Phi)$ with $\xi(s) = 1$.*

PROOF. Suppose that $\mathrm{am}(\Phi, \Psi; \sigma) \neq \emptyset$. By Proposition 18.1, we have an arithmetic model $(a, b; r)$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$ satisfying the following.



(a) (b)

FIGURE 19.1

(i) If $s \in S(\Phi)$ is longitude twisting, then $r(C[v](s^+)) = r(C[v](s^-)) = 0$.

(ii) If $s \in S(\Phi)$ is longitude preserving, then $(a(C[v](s^+)), b(C[v](s^+))) = (a(C[v](s^-)), b(C[v](s^-))) = (1, 0)$.

Since $r(C[v](s^-)) = \xi(s)r(C[v](s^+))$ for a longitude preserving side $s \in S(\Phi)$, it follows that

$$\sum_{C \in \Gamma[v]} a(C)r(C) = 2\sum_{C \in \Gamma} r(C) = 4 - 2h(v) ,$$

where $\Gamma = \{C[v](s^+) \mid s \in S(\Phi)$ is longitude preserving and $\xi(s) = 1\}$. Since $4 - 2h(v) = 4(1 - \#S(\Phi)) \neq 0$, it follows that $\Gamma \neq \emptyset$. Therefore there is a longitude preserving side $s \in S(\Phi)$ with $\xi(s) = 1$.

Conversely suppose that there is a longitude preserving side $s \in S(\Phi)$ with $\xi(s) = 1$. Let $r(C) = 0$ for $C \in \Gamma[v] - \{C[v](s^+), C[v](s^-)\}$, and $r(C[v](s^+)) = r(C[v](s^-)) = 2 - h(v)$. Then $r: \Gamma[\Phi] \to 2Z$ satisfies (A3)$'$, (A4)$'$ and (A5)$'$ in Theorem 7. By Theorem 7, it follows that $\mathrm{am}(\Phi, \Psi; \sigma) \neq \emptyset$. This completes the proof of Proposition 19.1.

PROPOSITION 19.2. *Suppose that* $V(\Phi) = \{v_0, \cdots, v_\mu\}$, $\mu > 2$, *and that* $S(\Phi) = \{s_1, \cdots, s_\mu\}$ *and* $\partial(s_j) = (v_0) - (v_j)$ *for all* j. *(See Figure* 19.1 (b).) *Then* $\mathrm{am}(\Phi, \Psi; \sigma) \neq \emptyset$ *if and only if* $\Psi[s_j] = \begin{pmatrix} 1 & l \\ m & n \end{pmatrix}$ *or* $\begin{pmatrix} -1 & l \\ m & n \end{pmatrix}$ *for all* j *and* $\#\{s \in S(\Phi) \mid \xi(s) = 1\} = 1$.

PROOF. Suppose that $\mathrm{am}(\Phi, \Psi; \sigma) \neq \emptyset$ and let $(a, b; r)$ be an arithmetic model. By the proof of Proposition 18.2, we have

(i) $\Psi[s_j] = \begin{pmatrix} 1 & l \\ m & n \end{pmatrix}$ or $\begin{pmatrix} -1 & l \\ m & n \end{pmatrix}$ for all j,

(ii) $a(C) = 1$ for all $C \in \Gamma[\Phi]$,

(iii) $r(C) = 2$ for all $C \in \Gamma[v_1] \cup \cdots \cup \Gamma[v_\mu]$.

Therefore we see that $r(C[v_0](s_j)) = 2\xi(s_j)$. Since $2(\xi(s_1) + \cdots + \xi(s_\mu)) = 4 - 2h(v_0) = 2(2 - \mu)$ and $\xi(s_j) = 1$ or -1, it follows that $\#\{s \in S(\Phi) \mid \xi(s) = 1\} = 1$.

Conversely suppose that $\Psi[s_j] = \begin{pmatrix} 1 & l \\ m & n \end{pmatrix}$ or $\begin{pmatrix} -1 & l \\ m & n \end{pmatrix}$ for all j and $\#\{s \in S(\Phi) \mid \xi(s) = 1\} = 1$. We may suppose that $\xi(s_1) = 1$ and $\xi(s_2) = \cdots =$

$\xi(s_\mu) = -1$. Let $r(C[v_0](s_1)) = 2$, $r(C[v_0](s_j)) = -2$ for $j = 2, \cdots, \mu$, and $r(C[v_j](s_j)) = -2$ for $j = 1, \cdots, \mu$. Let $a(C) = 1$ for all $C \in \Gamma[\Phi]$ and $b(C[v_0](s_j)) = 0$ for all j. Determining $b(C[v_j](s_j))$ by (A5) for all j, we have an arithmetic model $(a, b; r)$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$. This completes the proof of Proposition 19.2.

REMARK 19.3. Consider the graphs Φ in Proposition 19.2. Let $\Psi[s_1] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\Psi[s_j] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for $j = 2, \cdots, \mu$. Then $M(\Phi, \Psi)$ is diffeomorphic to S^3. By Proposition 19.2, it follows that $\mathrm{am}(\Phi, \Psi; \sigma) = \varnothing$ for an arbitrary σ. This implies that $t_1^0(\mathscr{F}(\Phi, \Psi; \sigma)) = \varnothing$ by Theorem 1. On the other hand, the foliation $\mathscr{F}(\Phi, \Psi; \sigma)$ admits a transverse 2-plane field as proved in Tamura-Sato [15]. Since (A1) does not depend on the integrability of transverse foliations, this means that (A4) reflects the integrability.

Hereafter we consider $\mathscr{F}(\Phi, \Phi; \sigma)$ such that $h(v) > 2$ for all $v \in V(\Phi)$. Let Φ' be the subgraph of Φ such that $V(\Phi') = V(\Phi)$ and $S(\Phi') = \{s \in S(\Phi) \mid s$ is longitude preserving$\}$. The following is the direct consequence of Theorem 7 and we omit the proof.

PROPOSITION 19.4. *If Φ' has an isolated vertex, then* $\mathrm{am}(\Phi, \Psi; \sigma) = \varnothing$.

Suppose that Φ' is a tree. Take a vertex $v_0 \in V(\Phi)$ and fix it. For each $v \in V(\Phi) - \{v_0\}$, there are a unique sequence $S(v, v_0) = (s_1, \cdots, s_{l(v)})$ in $S(\Phi')$ and a unique sequence $V(v, v_0) = (v_1 = v, v_2, \cdots, v_{l(v)+1} = v_0)$ in $V(\Phi)$ such that $\partial(s_j) = \pm((v_j) - (v_{j+1}))$ for $j = 1, \cdots, l(v)$. Let $\xi(v, v_0) = (-1)^{l(v)}\xi(s_1) \cdots \xi(s_{l(v)})$ and $\xi(v_0, v_0) = 1$. Then we have the following.

PROPOSITION 19.5. *Suppose that Φ' is a tree. Then* $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$ *if and only if $\sum_{v \in V(\Phi)} \xi(v, v_0)(4 - 2h(v)) = 0$ for some $v_0 \in V(\Phi)$.*

PROOF. Suppose that $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$. Then there is a map $r\colon \Gamma[\Phi] \to 2\boldsymbol{Z}$ satisfying (A3)′, (A4)′ and (A5)′ by Theorem 7. By Lemma 18.5, we have a sequence $\varnothing = \Gamma_0, \Gamma_1, \cdots, \Gamma_\rho$, $\rho = \# S(\Phi')$, satisfying the conditions corresponding to (1), (2) and (3) in Lemma 18.5. We use the notations in Lemma 18.5. Let $C_j = C[v_j](s_j)$ and $C_j' = C[v_j'](s_j)$. By induction on j, we prove

$$[[j]] \qquad r(C_j) = 4 - 2h(v_j) - \sum_{v \in V(j)} \xi(v, v_j)(4 - 2h(v)) ,$$

where $V(j) = \{v \in V(\Phi) - \{v_j\} \mid S(v, v_j) \subset \{s_1, \cdots, s_{j-1}\}\}$.

Since $V(1) = \varnothing$ and $r(C) = 0$ for $C \in \Gamma[v_1] - \{C_1\}$, the condition [[1]] follows from (A4)′. Now suppose that [[i]] holds for $i \leqq j$. For each $C \in \Gamma[v_{j+1}]$, there is a unique side $s \in S(\Phi)$ with $C = C[v_{j+1}](s)$. If s is longitude twisting, then $r(C)=0$ by (A3)$'$. If s is longitude preserving, then there is $j(C)\in\{1, \cdots, j+1\}$ with $s=s_{j(C)}$. When $j(C)<j+1$, we have

$$r(C)=\xi(s_{j(C)})\left\{4-2h(v_i)-\sum_{v\in V(j(C))}\xi(v, v_{j(C)})(4-2h(v))\right\} .$$

Since $V(j+1)=\bigcup\{V(j(C))\cup\{v_{j(C)}\} \mid C\in\Gamma[v_{j+1}]$ and $j(C)<j+1\}$, the condition $[[j+1]]$ follows from (A4)$'$ and the above formula. This completes the induction.

Since $V(\Phi)=V(\rho)\cup\{v_\rho\}$ and $r(C)=0$ for $C\in\Gamma[v_\rho]-\{C_\rho\}$, we have

$$\sum_{v\in V(\Phi)}\xi(v, v_\rho)(4-2h(v))=0$$

by $[[\rho]]$ and (A4)$'$.

Conversely suppose that $\sum_{v\in V(\Phi)}\xi(v, v_0)(4-2h(v))=0$ for some $v_0\in V(\Phi)$. We can take a sequence $\Gamma_0, \cdots, \Gamma_\rho$ as above and we may suppose that $v_0=v_\rho$. For $s\in S(\Phi)-S(\Phi')$ with $\partial(s)=(v)-(v')$, let $r(C[v](s))=r(C[v']))=0$. Using induction on j, define $r(C[v_j](s_j))$ by the formula $[[j]]$. It is easy to check that $r\colon \Gamma[\Phi]\to 2\boldsymbol{Z}$ satisfies (A3)$'$, (A4)$'$ and (A5)$'$. This completes the proof of Proposition 19.5.

REMARK 19.6. Suppose that Φ' is not connected but consists of trees containing at least two vertices. Then $\mathrm{am}(\Phi, \Psi; \sigma)\neq\emptyset$ if and only if the corresponding formula holds for each tree contained in Φ'.

20. TS models transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$. Using TS diagrams, we describe a necessary and sufficient condition under which $t_1^*(\Phi, \Psi; \sigma)$ becomes non-empty in this and the next section.

First we must define a TS diagram for $\mathscr{G}\in t_1^0(\mathscr{F}(h; \sigma))$ C^0 isomorphic to $(\mathscr{G}\mid C)\times I$ for some $C\in\Gamma(h)$. Note that the existence of such $\mathscr{G}$ implies that $h=2$ and that if C_1 is horizontal with respect to $\mathscr{G}$ then $\sigma(C_2)=-\sigma(C_1)$.

DEFINITION 20.1. A *singular* TS *diagram* of $(\hat{E}(2); \sigma)$ is a quadruplet $\mathscr{T}=(\{J_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\colon J_\lambda\to S^1\}_{\lambda\in\Lambda}, \{s(J_\lambda)\in\mathscr{S}\}_{\lambda\in\Lambda}, (a, b; r)\in(\boldsymbol{N}\times\boldsymbol{Z})^*\times 2\boldsymbol{Z})$ satisfying the following conditions.

(S1) J_λ is a copy of I for all $\lambda\in\Lambda$.

(S2) If $\#\Lambda>1$, then $\iota_\lambda\colon J_\lambda\to S^1$ is an imbedding for each $\lambda\in\Lambda$, $\mathrm{Int}\,{}^*J_\lambda\cap\mathrm{Int}\,{}^*J_{\lambda'}=\emptyset$ for $\lambda\neq\lambda'$, and S^1 is the closure of $\bigcup\{{}^*J_\lambda\mid\lambda\in\Lambda\}$, where ${}^*(\ )$ means $\iota_\lambda(\ )$ for an appropriate $\lambda\in\Lambda$. If $\#\Lambda=1$, then $\iota_\lambda|\mathrm{Int}\,J_\lambda$ is an imbedding and ${}^*J_\lambda=S^1$, where $\Lambda=\{\lambda\}$.

(S3) $\#\{\lambda\in\Lambda\mid s(J)=\bigcirc$ or $\bullet\}<\infty$.

(S4) If σ is constant, then $(a, b)\in(\boldsymbol{N}\times\boldsymbol{Z})^{\mathrm{coprime}}\cup\{(\infty, \infty)\}$. If $(a, b)=$

(∞, ∞), then $\# \Lambda = 1$ and $r = \#\{\lambda \in \Lambda \mid s(J) = \bigcirc \text{ or } \bullet\} = 0$. If $(a, b) \in (\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}} \cup \{(0, 1)\}$, then $r = \#\{\lambda \in \Lambda \mid s(J_\lambda) = \bigcirc\} - \#\{\lambda \in \Lambda \mid s(J_\lambda) = \bullet\}$.

We call a triad $(J_\lambda, \iota_\lambda, s(J_\lambda))$, in a singular TS diagram, a *sigular* TS *piece*.

We introduce an equivalence relation on the set of singular TS diagrams as follows.

DEFINITION 20.2. Let $\mathscr{T} = (\{J_\lambda\}_{\lambda \in \Lambda}, \{\iota_\lambda\}_{\lambda \in \Lambda}, \{s(J)\}_{\lambda \in \Lambda}, (a, b; r))$ and $\mathscr{T}' = (\{J'_\mu\}_{\mu \in M}, \{\iota'_\mu\}_{\mu \in M}, \{s(J'_\mu\}_{\mu \in M}, (a', b'; r'))$ be singular TS diagrams of $(\hat{E}(2); \sigma)$. Then $\mathscr{T}$ and $\mathscr{T}'$ are *isomorphic* if $(a, b; r) = (a', b'; r')$ and there are a homeomorphism $\phi: S^1 \to S^1$ and a bijection $\rho: \Lambda \to M$ such that $\phi({}^*J_\lambda) = {}^*J'_{\mu(\lambda)}$ and $s(J'_{\rho(\lambda)}) = s(J_\lambda)$ for all $\lambda \in \Lambda$.

We denote by $\mathrm{STS}(\hat{E}(2); \sigma)$ the set of isomorphism classes of singular TS diagrams of $(\hat{E}(2); \sigma)$.

DEFINITION 20.3. We call P a TS *piece* if P is a regular TS piece or a singular TS piece. We call $\mathscr{T}$ a TS *diagram* if $\mathscr{T}$ is a regular TS diagram or a singular TS diagram. Let $\mathrm{TS}(\hat{E}(h); \sigma) = \mathrm{RTS}(\hat{E}(h); \sigma)$ if $h \neq 2$, and $\mathrm{TS}(\hat{E}(2); \sigma) = \mathrm{RTS}(\hat{E}(2); \sigma) \cup \mathrm{STS}(\hat{E}(2); \sigma)$.

As a generalization of Theorem 4, we have the following and we omit the proof.

THEOREM 4*. *There exists a canonical map*

$$\tau: t_1^0(\mathscr{F}(h; \sigma)) \to \mathrm{TS}(\hat{E}(h); \sigma) .$$

Now let us consider $\mathscr{F}(\Phi, \Psi; \sigma)$ as in §1. For each $v \in V(\Phi)$, we obtain a set $\mathrm{TS}[v] = \mathrm{TS}(E[v]; \sigma \mid \Gamma[v])$. We describe the compatibility conditions for a family $\{[\mathscr{T}(v)] \in \mathrm{TS}[v]\}_{v \in V(\Phi)}$ to correspond to some $\mathscr{G} \in t_1^*(\Phi, \Psi; \sigma)$ as follows.

DEFINITION 20.4. Let $\mathscr{T} = (\hat{\mathscr{T}}, \{\phi_t\}_{t \in I}, (a, b; r))$ be a regular TS diagram of $(\hat{E}(h); \sigma)$. For each $C \in \Gamma(h)$, the *C boundary diagram* $\partial_C \mathscr{T}$ of $\mathscr{T}$ is a quadruplet $(\{J_\mu\}_{\mu \in M}, \{\iota_\mu: J_\mu \to L(C)\}_{\mu \in M}, \{s(J_\mu)\}_{\mu \in M}, (a(C), b(C); r(C)))$ satisfying the following conditions.

(B0) If $a(C) = 0$ or ∞, then $L(C) = S^1 \times \{*\}$. If $0 < a(C) < \infty$, then $L(C) = C/\sim$, where $y \sim y'$ for $y, y' \in \hat{C}$ if and only if $y' = \phi_1^k(y)$ for some $k \in \boldsymbol{Z}$. (Note that $(\phi_1 \mid C)^{a(C)} = \mathrm{id}$.)

(B1) J_μ is a copy of I for all $\mu \in M$.

(B2) If $a(C) = 0$ or ∞, then $\# \Lambda = 1$ and $\iota_\mu \mid \mathrm{Int}\, J_\mu$ is an imbedding and ${}^*J_\mu = L(C)$ for $\mu \in M$, where ${}^*(\ ) = \iota_\mu(\ )$ as before. If $0 < a(C) < \infty$, then $\iota_\mu \mid \mathrm{Int}\, J_\mu$ is an imbedding for $\mu \in M$ and there is a surjection $\xi: \mathscr{J}(C) = \{{}^*J \mid \lambda \in \Lambda,\ J \in \mathscr{J}_\lambda \cup (\mathscr{K}(\Delta_\lambda) - \mathscr{K}_\lambda),\ {}^*J \subset \hat{C}\} \to M$ such that ${}^*J_{\xi({}^*K)} =$

$\pi({}^*K)$ and that $s({}^*J_{\xi({}^*K)}) = s(K)$ if $K \in \mathscr{J}_\lambda$ for some λ, and otherwise $s({}^*J_{\xi({}^*K)}) = \|$, where $\hat{\mathscr{T}}$ contains $\{P_\lambda = (\Delta_\lambda, \iota_\lambda, s: \mathscr{J}_\lambda \to \mathscr{S}, \omega: \mathscr{K}_\lambda \to \{1, -1\})\}_{\lambda\in\Lambda}$ and $\pi: \hat{C} \to L(C)$ is the projection.

DEFINITION 20.5. Suppose that $h = 2$ and let $\mathscr{T} = (\{J_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\}_{\lambda\in\Lambda}, \{s(J_\lambda)\}_{\lambda\in\Lambda}, (a, b; r))$ be a singular TS diagram. For $C \in \Gamma(2)$, the C *boundary diagram* $\partial_C\mathscr{T}$ is defined as follows.

(1) When σ is not constant, let $\sigma(C^+) = 1$ and $\sigma(C^-) = -1$, where $\Gamma(2) = \{C^+, C^-\}$. Then $\partial_{C^+}\mathscr{T}$ is equal to $\mathscr{T}$, and $\partial_{C^-}\mathscr{T}$ to $(\{J_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\}_{\lambda\in\Lambda}, \{s'(J_\lambda)\}_{\lambda\in\Lambda}, (a', b'; -r))$ such that

(i) $s'(J_\lambda) = \bullet, \bigcirc, \wedge, \vee, \|$ if $s(J_\lambda) = \bigcirc, \bullet, \vee, \wedge, \|$ respectively.

(ii) $(a', b') = (a, b)$ ir $(a, b) = (0, 1)$ or (∞, ∞), and $(a', b') = (a, -b)$ if $(a, b) \in (\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}}$.

(2) When σ is constant, we fix an order $<$ on $\Gamma(2)$ and let $C < C'$, $C, C' \in \Gamma(2)$. Then $\partial_C\mathscr{F}$ is equal to $\mathscr{T}$, and $\partial_{C'}\mathscr{T}$ to $(\{J_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\}_{\lambda\in\Lambda}, \{s'(J_\lambda)\}_{\lambda\in\Lambda}, (a', b'; r'))$ such that

(i) $s'(J_\lambda) = \bullet, \bigcirc$ (or $\bigcirc, \bullet$) if $s(J_\lambda) = \bigcirc, \bullet$ and $(a, b) = (0, 1)$ (or $(a, b) \in (\boldsymbol{N} \times \boldsymbol{Z})^{\text{coprime}}$). $s'(J_\lambda) = \wedge, \vee, \|$ if $s(J_\lambda) = \vee, \wedge, \|$.

(ii) $(a', b'; r') = (a, b; -r)$ if $(a, b) = (0, 1)$ or (∞, ∞), and $(a', b'; r') = (a, -b; r)$ if $(a, b) \in (\boldsymbol{N} \times \mathbf{Z})^{\text{coprime}}$.

DEFINITION 20.6. Given a map $(a, b): \Gamma \to (\boldsymbol{N} \times \mathbf{Z})^*$, for each $C \in \Gamma$ let $v(C) = \begin{pmatrix} -b(C) \\ a(C) \end{pmatrix}$ if $0 < a(C) < \infty$, and $v(C) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ if $a(C) = 0$.

DEFINITION 20.7. A TS *model* transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$ is a family $\{[\mathscr{T}(v)] \in \mathrm{TS}[v]\}_{v\in V(\Phi)}$ satisfying the following conditions.

Let $s \in S(\Phi)$ with $\partial(s) = (v) - (v')$. Let $C = C[v](s)$, $C' = C[v'](s)$, $\partial_C\mathscr{T}(v) = (\{J_\lambda\}_{\lambda\in\Lambda}, \{\iota_\lambda\}_{\lambda\in\Lambda}, \{s(J_\lambda)\}_{\lambda\in\Lambda}, (a, b; r))$ and $\partial_{C'}\mathscr{T}(v') = (\{J'_\mu\}_{\mu\in M}, \{\iota'_\mu\}_{\mu\in M}, \{s(J'_\mu)\}_{\mu\in M}, (a', b'; r'))$.

(1) $(a, b; r)$ and $(a', b'; r')$ satisfies the condition corresponding to (A5) in Definition 1.2.

(2) There is a homeomorphism $\phi[s]: L(C) \to L(C')$ such that

(i) if $a(C) = \infty$, then $\phi[s]$ is orientation preserving,

(ii) if $a(C) \neq \infty$ and $\varepsilon > 0$ (or $\varepsilon < 0$), then $\phi[s]$ is orientation preserving (or reversing), where $\varepsilon = {}^{\mathrm{tr}}v(C')\cdot\Psi[s]\cdot v(C)$ (product as matrices).

(3) Furthermore there is a bijection $\rho: \Lambda \to M$ such that for each $\lambda \in \Lambda$,

(i) $\phi[s]({}^*J_\lambda) = {}^*J'_{\rho(\lambda)}$, and

(ii) $s(J'_{\rho(\lambda)}) = \bigcirc, \bullet, \wedge, \vee, \|$ (or $\bullet, \bigcirc, \wedge, \vee, \|$) if $s(J_\lambda) = \bigcirc, \bullet, \vee, \wedge, \|$ and $\gamma_2 > 0$ (or $\gamma_2 < 0$), where γ_2 is as in Definition 1.2.

We denote by $\mathrm{TS}(\Phi, \Psi; \sigma)$ the set of TS models transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$.

21. The geometric criterion. We formulate the geometric criterion precisely. First we have the following. Since the proof is long and tedionus, we omit it.

THEOREM 8. *There exists a canonical commutative diagram*

$$\begin{tikzcd} t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) \arrow[r, "\bar{\tau}"] \arrow[dr, "\alpha"'] & \mathrm{TS}(\Phi, \Psi; \sigma) \arrow[d, "\bar{\alpha}"] \\ & \mathrm{am}(\Phi, \Psi; \sigma) . \end{tikzcd}$$

When a TS model $\mathscr{M}$ contains an infinite number of TS pieces, the construction of a foliation $\mathscr{G}$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$ corresponding to $\mathscr{M}$ has troubles concerning the differentiability of $\mathscr{G}$. In order to get a better formulation, we need the following.

DEFINITION 21.1. A TS model $\mathscr{M}$ is called *finite* if it contains at most a finite number of TS pieces.

DEFINITION 21.2. A TS model $\mathscr{M} = \{[\mathscr{T}(v)]\}_{v \in V(\Phi)}$ is called *irreducible* if the following conditions are satisfied.

(1) For each $v \in V(\Phi)$, a representative $\mathscr{T}(v)$ contains no regular TS piece $P = (\Delta, \nu, s\colon \mathscr{J} \to \mathscr{S}, \omega\colon \mathscr{K} \to \{1, -1\})$ such that $\nu = \mathrm{IX}$ and ω is not constant.

(2) Let $\mathscr{P}$ be the set of TS pieces in fixed representatives $\{\mathscr{T}(v)\}_{v \in V(\Phi)}$. On $\mathscr{P}$, we consider an equivalence relation $\sim$ determined by

$$P \sim P' \text{ if there is } s \in S(\Phi) \text{ with } \partial(s) = (v) - (v')$$

such that P (or P') belongs to $\mathscr{T}(v)$ (or $\mathscr{T}(v')$) and $\phi[s]$ in Definition 20.7 maps $\pi(^*J)$ to $\pi'(^*J')$ for some $J \in \mathscr{J}(|P|)$ and $J' \in \mathscr{J}(|P'|)$, where π (or π') is the projection to $L(C[v](s))$ (or $L(C[v'](s))$).

Then there is no equivalence class $\mathscr{C}$ with respect to $\sim$ such that if $P \subset \mathscr{C}$ then the symbols attached to P are $\vee$, $\wedge$ or $\|$.

We denote by $\mathrm{ts}(\Phi, \Psi; \sigma)$ the set of finite irreducible TS models transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$.

Now we have the following.

THEOREM 8*. (1) *There exists a canonical commutative diagram*

$$\begin{tikzcd} t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) \arrow[r, "\tau^*"] \arrow[dr, "\alpha"'] & \mathrm{ts}(\Phi, \Psi; \sigma) \arrow[r] \arrow[d, "\alpha^*"] & \mathrm{TS}(\Phi, \Phi; \sigma) \arrow[dl, "\bar{\alpha}"] \\ & \mathrm{am}(\Phi, \Psi; \sigma) & \end{tikzcd}$$

(2) *τ^* is surjective.*

The following follows directly from Theorem 8*.

THEOREM 8** (The geometric criterion). *$t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) \neq \emptyset$ if and only if* $\mathrm{ts}(\Phi, \Psi; \sigma) \neq \emptyset$.

PROOF OF THEOREM 8*. Let $\alpha^* = \bar{\alpha} \mid \mathrm{ts}(\Phi, \Psi; \sigma)$. We define τ^* as follows. Let $\mathscr{G} \in t_1^*(\mathscr{F}(\Phi, \Psi; \sigma))$ and $\mathscr{M} = \bar{\tau}(\mathscr{G})$. Using the theorem of Kopell [7] as in Nishimori [9], we can show that if $\mathscr{M}$ is irreducible then $\mathscr{M}$ is finite, since $\mathscr{M} \in \mathrm{Image}\,\bar{\tau}$. In this case, let $\tau^*(\mathscr{G}) = \mathscr{M}$. Suppose that $\mathscr{M}$ is not irreducible. Then $\mathscr{G}$ contains foliated I-bundles corresponding to regular TS pieces of type IX or to equivalence classes of $\mathscr{P}$ in Definition 21.2 (2). Collapsing such foliated I-bundles along fibers, we obtain a foliation $\mathscr{G}'$ such that $\bar{\tau}(\mathscr{G}')$ is irreducible. Then $\bar{\tau}(\mathscr{G}')$ is finite as above. Let $\tau^*(\mathscr{G}) = \bar{\tau}(\mathscr{G}')$. Since $\alpha(\mathscr{G}') = \alpha(\mathscr{G})$, we have $\alpha^* \circ \tau^* = \alpha$.

(2) Let $\mathscr{M} = \{[\mathscr{T}(v)]\}_{v \in V(\Phi)} \in \mathrm{ts}(\Phi, \Psi; \sigma)$. For each regular TS piece P of $\mathscr{T}(v)$, we take an appropriate component of the same type as P in Theorem 3 if $\mathscr{T}(v)$ is regular. When $\mathscr{T}(v)$ is singular, take C^+ in Definition 20.5 (1) or C in Definition 20.5 (2) and denote it by C. We construct a foliation $\hat{\mathscr{G}}$ of $S^1 \times S^1$ such that

(i) $\hat{\mathscr{G}} \mid (^*J_\lambda) \times S^1$ is a Reeb component such that the connected components of $\partial(^*J_\lambda) \times S^1$ have expanding holonomy in the same (or opposite) direction as the orientation of S^1 if $s(J_\lambda) = \bigcirc$ (or $\bullet$),

(ii) $\hat{\mathscr{G}} \mid (^*J_\lambda) \times S^1$ is a slope component such that the connected components of $\partial(^*J_\lambda) \times S^1$ have expanding holonomy in the same (or opposite) direction as the orientation of $\partial(^*J_\lambda) \times S^1$ as the boundary of $^*J_\lambda \times S^1$ if $s(J_\lambda) = \vee$ (or $\wedge$),

(iii) $\hat{\mathscr{G}} \mid (^*J_\lambda) \times S^1$ consists of leaves $\{x\} \times S^1$ for $x \in {}^*J_\lambda$ if $s(J_\lambda) = \|$, where $\mathscr{T}(v) = (\{J_\lambda\}_{\lambda \in \Lambda}, \{\iota_\lambda\}_{\lambda \in \Lambda}, \{s(J_\lambda)\}_{\lambda \in \Lambda}, (a, b; r))$. Now take a foliation $\mathscr{G}(v)$ of $E[v]$ with $\mathscr{G}(v) \mid C = \hat{\mathscr{G}}$ such that $\mathscr{G}(v)$ is C^∞ isomorphic to $\hat{\mathscr{G}} \times I$.

Since $\mathscr{M}$ satisfies the condition in Definition 20.7 and $\mathscr{M}$ is finite, we have a C^∞ foliation $\mathscr{G}$ of $M(\Phi, \Psi)$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$ with $\tau^*(\mathscr{G}) = \mathscr{M}$. We omit the details. This completes the proof of Theorem 8*.

The proof of Theorem 8* implies the following.

THEOREM 9. (1) *For each $\mathscr{M} \in \mathrm{TS}(\Phi, \Psi; \sigma)$, there is cannonically a C^0 foliation of $M(\Phi, \Psi)$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$.*

(2) *If $\mathrm{TS}(\Phi, \Psi; \sigma) \neq \emptyset$, there is a C^∞ 2-plane field of $M(\Phi, \Psi)$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$.*

PROOF. (1) is clear. As for (2) there is a transverse C^0 foliation $\mathscr{G}$ by (1). It suffices to take a C^∞ approximation of $T\mathscr{G}$.

22. **Some applications of the geometric criterion.** We treat $\mathscr{F}(\Phi, \Psi; \sigma)$ considered already in § 19. For such $\mathscr{F}(\Phi, \Psi; \sigma)$, we obtained a necessary and sufficient condition under which $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$ by Propositions 19.1, 19.2, 19.5 and Remark 19.6. First consider $\mathscr{F}(\Phi, \Psi; \sigma)$ such that Φ is a graph in Figure 19.1. We show that for such $\mathscr{F}(\Phi, \Psi; \sigma)$ the arithmetic criterion is complete. Precisely we have the following.

THEOREM 10. *Let $\mathscr{F}(\Phi, \Psi; \sigma)$ be as in* §1. *Suppose that*

(a) $V(\Phi) = \{v\}$ *and* $\# S(\Phi) > 1$,

or

(b) $V(\Phi) = \{v_0, \cdots, v_\mu\}$, $\mu > 2$,

$S(\Phi) = \{s_1, \cdots, s_\mu\}$ *and* $\partial(s_j) = (v_0) - (v_j)$

for all j.

Then the following conditions are equivalent.

(1) $t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) \neq \varnothing$.

(2) $\mathrm{ts}(\Phi, \Psi; \sigma) \neq \varnothing$.

(3) $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$.

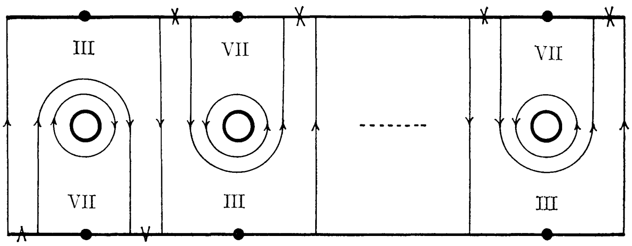
PROOF. Note that $(1) \Leftrightarrow (2)$ and $(2) \Rightarrow (3)$ are already known for general $\mathscr{F}(\Phi, \Psi; \sigma)$'s by Theorems 8^{**} and 1^*. Therefore it is sufficient to prove that (3) implies (2). Suppose that $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$.

Case (a). By Proposition 19.1, there is a longitude preserving side $s \in S(\Phi)$ with $\xi(s) = 1$. Furthermore we have an arithmetic model $(a, b; r)$: $\Gamma[\Phi] = \Gamma[v] \to (N \times \mathbf{Z})^* \times 2\mathbf{Z}$ such that

(i) $(a(C_1), b(C_1); r(C_1)) = (a(C_2), b(C_2); r(C_2)) = (1, 0; 2 - \mu)$,

(ii) $r(C) = 0$ for $C \in \Gamma[v] - \{C_1, C_2\}$,

where $C_1 = C[v](s^+)$ and $C_2 = C[v](s^-)$. Let $\Gamma[v] - \{C_1, C_2\} = \{C_3, \cdots, C_\mu\}$, where $\mu = 2 \cdot \# S(\Phi)$. Now we find a regular TS diagram $\mathscr{T}$ of $(\hat{E}(\mu); \sigma)$ indicated by Figure 22.1.



The right and left vertical segments are to be glued

FIGURE 22.1

Then it is easy to check that $[\mathscr{T}] \in \mathrm{ts}(\Phi, \Psi; \sigma)$ and $\alpha^*([\mathscr{T}]) = (a, b; r)$.

Case (b). By Proposition 19.2, there is $s_{j^*} \in S(\Phi)$ with $\xi(s_{j^*}) = 1$, and $\xi(s) = -1$ for all $s \in S(\Phi) - \{s_{j^*}\}$. We may suppose that $s_{j^*} = s_1$. Furthermore we have an arithmetic model $(a, b; r)\colon \Gamma[\Phi] \to (N \times \mathbf{Z})^* \times 2\mathbf{Z}$ such that

(i) $(a(C), b(C)) = (1, 0)$ for all $C \in \Gamma[\Phi]$,

(ii) $r(C_j') = 2$ and $r(C_j) = 2\xi(s_j)$ for $j > 0$,

where $C_j = C[v_0](s_j)$ and $C_j' = C[v_j](s_j)$. Now we find regular TS diagrams $\mathscr{T}_0, \cdots, \mathscr{T}_\mu$ indicated by Figure 22.2 in the case $\mu = 4$.

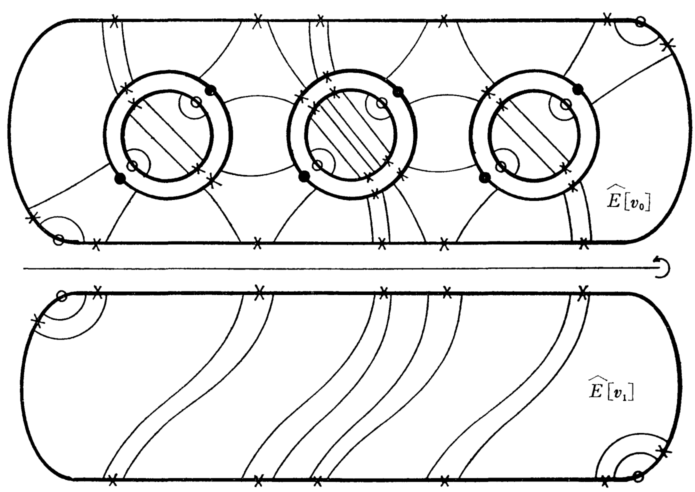


FIGURE 22.2

Then it is easy to check that $\mathscr{M} = \{[\mathscr{T}_i]\}_{i=1}^{\mu}$ is a finite TS model and $\alpha^*(\mathscr{M}) = (a, b; r)$. We have $\mathscr{M}' \in \mathrm{ts}(\Phi, \Psi; \sigma)$ from $\mathscr{M}$ by reduction as in the proof of Theorem 8*. This completes the proof of Theorem 10.

Consider $\mathscr{F}(\Phi, \Psi; \sigma)$ such that Φ is a graph in Figure 22.3, where

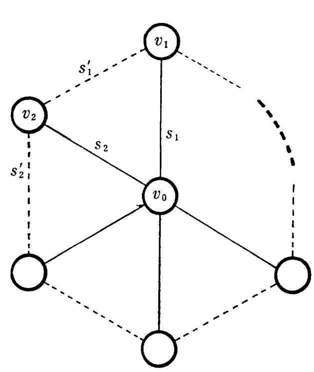


FIGURE 22.3

longitude preserving (or twisting) sides are represented by solid (or dotted) lines.

The arithmetic criterion is complete in this case, too, and we have the following.

THEOREM 11. *Let $\mathscr{F}(\Phi, \Psi; \sigma)$ be as in* §1. *Suppose that*

(1) $V(\Phi) = \{v_0, \cdots, v_\mu\}$,

$h(v_0) = \mu > 1$, $h(v_j) = 3$ *for* $j > 0$,

(2) $S(\Phi) = \{s_1, \cdots, s_\mu, s'_1, \cdots, s'_\mu\}$,

$\partial(s_j) = (v_0) - (v_j)$, $\partial(s'_j) = (v_j) - (v_{j+1})$,

s_j *(or* s'_j*) is longitude preserving (or twisting). Then* $t_1^*(\mathscr{F}(\Phi, \Psi; \sigma)) \neq \varnothing$ *if and only if* $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$.

PROOF. As above it suffices to prove that $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$ implies $\mathrm{ts}(\Phi, \Psi; \sigma) \neq \varnothing$. Suppose that $\mathrm{am}(\Phi, \Psi; \sigma) \neq \varnothing$. By Proposition 19.5, we have

$$\sum_{j=0}^{\mu} \xi(v_j, v_0)(4 - 2h(v_j)) = 2(2 - \mu + \sum_{j=1}^{\mu} \xi(s_j)) = 0 .$$

Therefore $\xi(s_{j^*}) = -1$ for some j^* and $\xi(s_j) = 1$ for all $j \in \{1, \cdots, \mu\} - \{j^*\}$. We may suppose that $j^* = 1$. Now we find regular TS diagrams $\mathscr{T}_0', \cdots, \mathscr{T}_\mu'$, where $\mathscr{T}_0'$ equals $\mathscr{T}_0$ in Case (b) in the proof of Theorem 9, and $\mathscr{T}_1', \cdots, \mathscr{T}_\mu'$ are indicated by Figure 22.4.

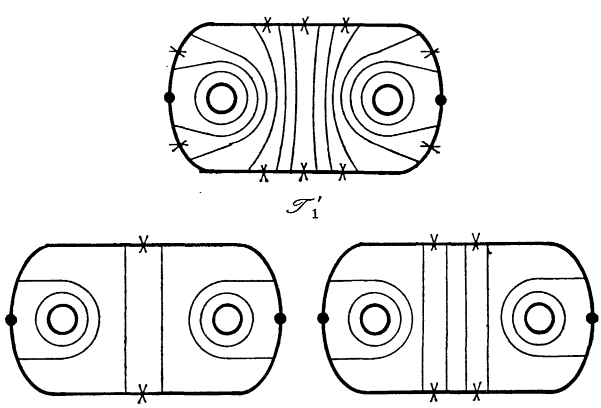


FIGURE 22.4

Then we see that $\mathrm{ts}(\Phi, \Psi; \sigma) \neq \varnothing$ as above. This completes the proof of Theorem 11.

REMARK 22.2. Let $\mathscr{F}(\Phi, \Psi; \sigma)$ satisfy the condition of Theorem 9

(b) or Theorem 10. Then $t_1^0(\mathscr{F}(\Phi, \Psi; \sigma)) \neq \emptyset$ if and only if $\mathrm{am}(\Phi, \Psi; \sigma) \neq \emptyset$, since $t_1^0(\mathscr{F}(\Phi, \Psi; \sigma)) = t^*(\mathscr{F}(\Phi, \Psi; \sigma))$ in the case Theorem 9 (b). Consider the case of Theorem 10. In order to obtain transversely orientable foliation $\mathscr{G}$ transverse to $\mathscr{F}(\Phi, \Psi; \sigma)$, it suffices to insert, for each $j > 0$, a regular TS piece $P = (\Delta, \nu, s: \mathscr{J} \to \mathscr{S}, \omega: \mathscr{K} \to \{-1, 1\})$, such that $\nu = \mathrm{IX}$ and ω is constant, between the TS pieces of type VI and VIII in $\mathscr{T}_j'$ if necessary.

23. A construction of regular TS diagrams of $(\hat{E}(h); \sigma)$ with given $(a, b; r)$ in the case $h > 2$. The purpose of this section is to make preparations for the proof of Theorem 2. We prove the following.

THEOREM 12. *Let $\mathscr{F}(h; \sigma)$ be as in §1 and suppose that $h > 2$. Let $(a, b; r): \Gamma(h) \to (N \times \mathbf{Z})^* \times 2\mathbf{Z}$ be a map such that*

(i) *if $r(C) = 0$ for $C \in \Gamma(h)$, then $(a(C), b(C)) = (1, 0)$.*

(ii) $\sum_{C \in \Gamma(h)} a(C)r(C) = 4 - 2h$.

Then there is a regular TS diagram $\mathscr{T} = (\hat{\mathscr{T}}, \{\phi_t\}_{t \in I}, (a', b'; r'))$ of $(\hat{E}(h); \sigma)$ with $(a', b'; r') = (a, b; r)$.

REMARK 23.1. When $h = 1$ or 2, we obtain results similar to Theorem 12 more easily.

PROOF OF THEOREM 12. Denote by Γ^+ (or Γ^-, Γ^0) the set of $C \in \Gamma(h)$ with $r(C) > 0$ (or <0, $=0$), and let $\Gamma^+ = \{C_1^+, \cdots, C_{k(+)}^+\}$, $\Gamma^- = \{C_1^-, \cdots, C_{k(-)}^-\}$, and $\Gamma^0 = \{C_1^0, \cdots, C_{k(0)}^0\}$. For each $C \in \Gamma(h)$, take a set $\Pi(C)$ of $|r(C)|$ points of $\hat{C}$. Let $\Pi^+ = \bigcup \{\Pi(C) | C \in \Gamma^+\}$ and $\Pi^- = \bigcup \{\Pi(C) | C \in \Gamma^-\}$. Number the elements of Π^+ in such a way that

$$\pi^+\Big(\sum_{j=1}^{k-1} r(C_j^+) + 1\Big), \cdots, \pi^+\Big(\sum_{j=1}^{k} r(C_j^+)\Big) \in \Pi^+$$

are on $\hat{C}_k^+$ in the order opposite to the orientation of $\hat{C}_k^+$ for $k = 1, \cdots, k(+)$. Number the elements of Π^- in such a way that

$$\pi^-\Big(\sum_{j=1}^{k-1} |r(C_j^-)| + 1\Big), \cdots, \pi^-\Big(\sum_{j=1}^{k} |r(C_j^-)|\Big) \in \Pi^-$$

are on $\hat{C}_k^-$ in the same order as the orientation of $\hat{C}_k^-$ for $k = 1, \cdots, k(-)$. Take an orientation preserving imbedding $\iota: \hat{E}(h) \to \mathbf{R}^2$ such that

(1) $\iota(\hat{C}_{k(-)}^-) = C_N(N, 0)$ for large N,

(2) $\iota(\hat{C}_j^-) = C_1(9j, -9)$ for $j = 1, \cdots, k(-) - 1$,

(3) $\iota(\hat{C}_j^+) = C_1(9j, 9)$ for $j = 1, \cdots, k(+)$,

(4) $\iota(\hat{C}_j^0) = C_1(9(j + k(+)), 9)$ for $j = 1, \cdots, k(0)$,

where $C_\rho(x, y)$ is the circle of radius ρ with center (x, y). Identifying

$\hat{E}(h)$ and $\iota(\hat{E}(h))$, we regard $\hat{E}(h)$ as a subspace of $\boldsymbol{R}^2$.

Since $\sum_{C\in\Gamma} r(C) = \sum_{C\in\Gamma} a(C)r(C) = 4 - 2h < 0$, we have $k(-) > 0$. Since $h - 1 \leqq |4 - 2h|$ for $h \geqq 3$ and $k(0) \leqq h-1$, it follows that $k(0) + r^+ \leqq r^-$, where $r^+ = \sharp \Pi^+$ and $r^- = \sharp \Pi^-$. Sliding the points of Π^+ and Π^- if necessary, we may take disjoint line segments $L(1), \cdots, L(r^+)$ such that $\partial L(j) = \{\pi^+(j), \pi^-(j)\}$. Furthermore we may take disjoint line segments $L(r^+ + 1), \cdots, L(r^+ + k(0))$ in such a way that an endpoint of $L(r^+ + j)$ equals $\pi^-(r^+ + j)$ and the other one belongs to $\hat{C}_j^0$. In addition to these, take line segments $K(1), \cdots, K(\kappa_1)$ satisfying the following conditions (1)–(3).

(1) An endpoint of $k(j)$ belongs to $\hat{C}_{j'}^+$ and the other one belongs to $\hat{C}_{j''}^-$ for some j' and j''.

(2) The set $B_1 = \hat{C}_1^+ \cup \cdots \cup \hat{C}_{k(+)}^+ \cup \hat{C}_1^- \cup \cdots \cup \hat{C}_{k^*}^- \cup L(1) \cup \cdots \cup L(r^+) \cup K(1) \cup \cdots \cup K(\kappa_1)$ is connected, where $\hat{C}_{k^*}^-$ contains $\pi^-(r^+)$.

(3) For each $j \in \{1, \cdots, \kappa_1\}$, the set $B_1 - K(j)$ is not connected.

Finally take line segments $K(\kappa_1 + 1), \cdots, K(\kappa_2)$ satisfying the following conditions (4)–(6).

(4) An endpoint of $K(j)$ belongs to $\hat{C}_{j'}^-$ and the other one belongs to $\hat{C}_{j'+1}^-$ for some j'.

(5) The set $B_2 = B_1 \cup \hat{C}_{k^*+1}^- \cup \cdots \cup \hat{C}_{k(-)}^- \cup K(\kappa_1 + 1) \cup \cdots \cup K(\kappa_2)$ is connected.

(6) For each $j \in \{\kappa_1 + 1, \cdots, \kappa_2\}$, the set $B_2 - K(j)$ is not connected. (See Figure 23.1.)

Let H_0 be the connected component of

$$H = \hat{E}(h) - (L(1) \cup \cdots \cup L(r^+ + k(0)) \cup K(1) \cup \cdots \cup K(\kappa_2))$$

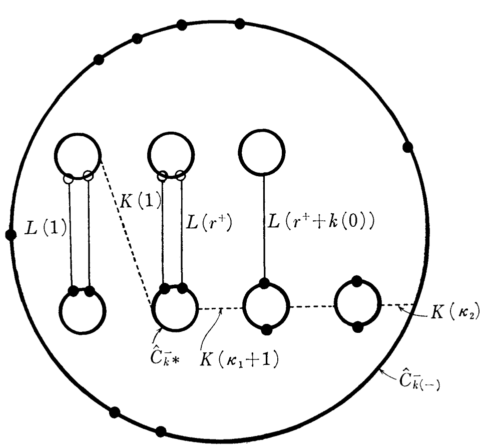


FIGURE 23.1

containing the point $(-N, 0) \in \boldsymbol{R}^2$. We denote by $\bar{H}_0$ the compact manifold with corner obtained from H_0 by attaching the boundary. Then $\bar{H}_0$ is homeomorphic to D^2. Each other connected component H_j of H is surrounded by $\hat{C}^+_{j'}$, $\hat{C}^-_{j''}$, $L(j^*)$ and $L(j^*+1)$ for some j', j'' and j^*. The closure of H_j is homeomorphic to D^2.

Take a non-singular vector field Z on a neighborhood of $\hat{E}(h) - \operatorname{Int} H_0 = B_2 \cup (H - H_0)$ satisfying the following conditions (1)-(4).

(1) Z is tangent to $\partial\hat{E}(h)$ at and only at $\Pi^+ \cup \Pi^- \cup \hat{C}^0_1 \cup \cdots \cup \hat{C}^0_{k(0)}$.

(2) The orbits of Z make concentric half circles (or confocal parabolas) near a point of Π^+ (or Π^-).

(3) The line segments $K(1), \cdots, K(\kappa_2)$ are orbits of Z.

(4) For each $j = 1, \cdots, r^+$ the orbits of Z make figures as in Figure 23.2 (a) in a closed neighborhood U_j of $L(j)$, and for each $j = 1, \cdots, k(0)$ they do so as in Figure 23.2 (b) in a closed neighborhood V_j of $L(r^+ + j) \cup \hat{C}^0_j$.

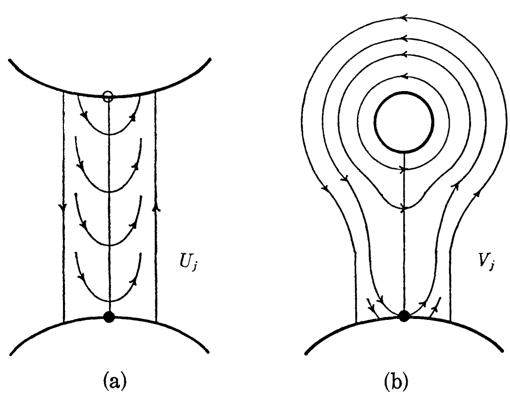


FIGURE 23.2

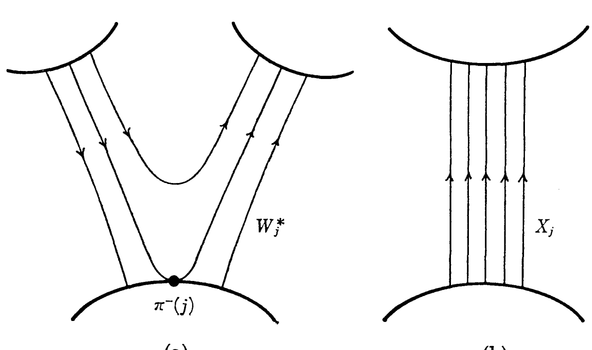


FIGURE 23.3

Since $\sum_{C \in \Gamma(h)} r(C) = 4 - 2h$, we can extend Z to a non-singular vector field Z^* on $\hat{E}(h)$. For each $j = r^+ + k(0) + 1, \cdots, r^-$, take a small closed interval $J(j)$ in $\partial\hat{E}(h)$ containing $\pi^-(j)$. Since $J(j) \subset H_0$, the saturation $J(j)^*$ of $J(j)$ with respect to Z^* is contained in H_0. Modifying Z^* if necessary, we may suppose that $J(j)^* \cap (\Pi^- - \{\pi^-(j)\}) = \emptyset$ for all j. For a small closed neighborhood W_j of $\pi^-(j)$, the saturation W_j^* of W_j is as in Figure 23.3 (a).

It is easy to see that for each connected component X_i of

$$\hat{E}(h) - \left(\bigcup_{j=1}^{r^+} U_j\right) \cup \left(\bigcup_{j=1}^{k(0)} V_j\right) \cup \left(\bigcup_{j=r^*}^{r^-} W_j\right),$$

where $r^* = r^+ + k(0) + 1$, the orbits of Z^* passing through X_j are as in Figure 23.3 (b).

Now we take a regular TS piece for each of U_j, V_j, W_j and X_j, as in Figure 23.4.

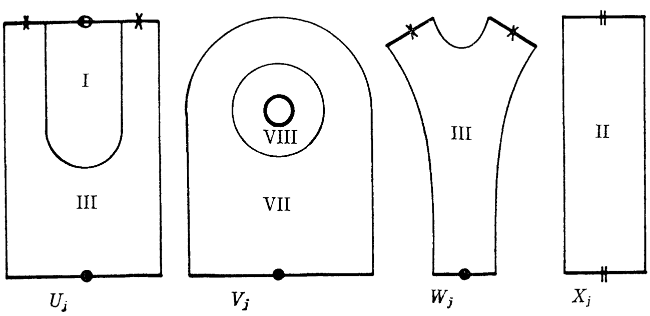


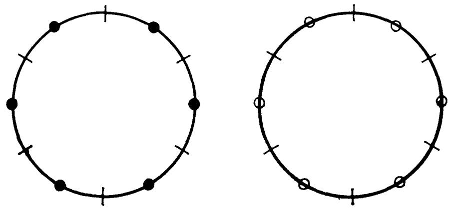
FIGURE 23.4

Then it is easy to construct a regular TS diagram containing the above TS pieces and satisfying the condition of Theorem 12. This completes the proof of Theorem 12.

24. The proof of Theorem 2. Theorem 2 follows from Theorem 9 (2) and the following.

THEOREM 13. *The map* $\bar{\alpha}$: $\mathrm{TS}(\Phi, \Psi; \sigma) \to \mathrm{am}(\Phi, \Psi; \sigma)$ *is surjective.*

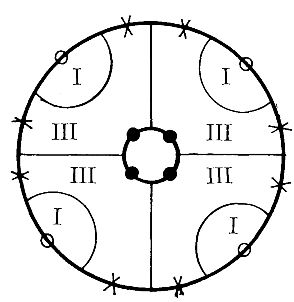
PROOF. Let $(a, b; r) \in \mathrm{am}(\Phi, \Psi; \sigma)$. For each $v \in V(\Phi)$ with $h(v) > 2$, we take the regular TS diagram $\mathscr{T}(v)$ constructed in §23. For $v \in V(\Phi)$ with $h(v) = 1$, we have $a(C) = 1$ and $r(C) = 2$, where $\{C\} = \Gamma[v]$. Take a regular TS diagram $\mathscr{T}(v)$ containing exactly two TS pieces of type I. Consider $v \in V(\Phi)$ with $h(v) = 2$. Let $\Gamma[v] = \{C, C'\}$. When $r(C) = 0$, we take a regular TS diagram $\mathscr{T}(v)$ containing exactly two TS pieces of type VIII. When $r(C) \neq 0$ and $a(C) = 0$, we may suppose that $\sigma(C) = 1$



$r(C) < 0$ $r(C) > 0$

FIGURE 24.1

and we take a singular TS diagram $\mathscr{T}(v)$ as in Figure 24.1. When $r(C) \neq 0$ and $a(C) > 0$, we take a regular TS diagram $\mathscr{T}(v)$ indicated by Figure 24.2.



$\#(\bigcirc) = a(C)|r(C)|$

FIGURE 24.2

Unfortunately $\{\mathscr{T}(v)\}_{v \in V(\Phi)}$ constructed above does not satisfy the compatibility condition described in Definition 20.7. For each $s \in S(\Phi)$ with $\partial(s) = (v) - (v')$, we see that $C = C[v](s)$ and $C' = C[v'](s)$ satisfy the compatibility condition on the symbols $\bigcirc$, $\bullet$ attached to $L(C)$ and $L(C')$, while we have trouble with the symbols $\vee$, $\wedge$, $\|$. We can overcome this trouble by using the following trick.

Suppose that $r(C) \neq 0$. First take a homeomorphism $\phi\colon L(C) \to L(C')$ satisfying the following conditions (1)-(3).

(1) For each J with $s(J) = \bigcirc$ or $\bullet$, the image $\phi(\operatorname{Int} J)$ intersects only one J' with $s(J') = \bigcirc$ or $\bullet$.

(2) For each J with $s(J) = \vee$, $\wedge$ or $\|$, the image $\phi(J)$ is contained in some J'.

(3) For each J' with $s(J') = \vee$, $\wedge$ or $\|$, there is J with $\phi(J) \supset J'$.

Inserting regular TS pieces of type II or singular TS pieces with symbols $\vee$, $\wedge$, $\|$ into $\mathscr{T}(v')$ for each J with $s(J) = \vee$, $\wedge$, $\|$, we can modify ϕ to ϕ_1 in such a way that ϕ_1 satisfies the conditions corresponding

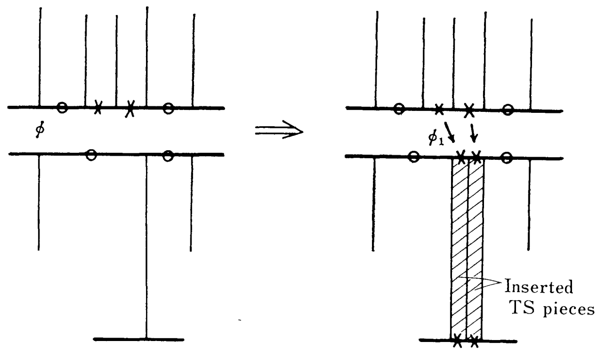


FIGURE 24.3

to (1), (2) and (3) above and ϕ_1 maps each J with $s(J) = \vee, \wedge, \|$ to some J' with the same symbol. (See Figure 24.3.)

Performing this for all $s \in S(\Phi)$, we have a family $\{\mathscr{T}(v)^{(1)}\}_{v \in V(\Phi)}$ of TS diagrams. Now make similar modifications in the opposite direction of s for all $s \in S(\Phi)$ and get $\{\mathscr{T}(v)^{(2)}\}_{v \in V(\Phi)}$. Then $L(C[v](s))$ has possibly new J's with $s(J) = \vee, \wedge$ or $\|$ for $s \in S(\Phi)$ with $\partial(s) = (v) - (v')$. Repeating the process of inserting TS pieces with symbols $\vee, \wedge, \|$ infinitely many times, we have a limit family $\{\mathscr{T}(v)^{(\infty)}\}_{v \in V(\Phi)}$ of TS diagrams. By construction, this limit family $\mathscr{M}$ is a TS model transverse to $\mathscr{T}(\Phi, \Psi; \sigma)$ with $\bar{\alpha}(\mathscr{M}) = (a, b; r)$. This completes the proof of Theorem 13.

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