

GENERALIZED LIPSCHITZ CLASS OF FUNCTIONS AND THEIR FOURIER TRANSFORMS

Dedicated to Professor Tamotsu Tsuchikura on his sixtieth birthday

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(Received November 11, 1982)

1. This paper deals with a strong form of absolute convergence of the Walsh Fourier series of functions in a Lipschitz space, and with some analogue for the Fourier transform of functions on the k -dimensional space R_k .

Suppose that f is integrable on $(-\pi, \pi)$ and periodic with period 2π . Denote its Fourier series by

$$f \sim \sum c_n e^{inz}.$$

We use the following notations:

$$W = \left\{ (w_n): w_n > 0, w_n = w_{-n}, w_{|n|} \downarrow, \|w_n\|_1 = \sum_{n=-\infty}^{\infty} |w_n| < \infty \right\}.$$

$${}_a\|c_n\|_{p,w} = \left\{ \sum_{n=-\infty}^{\infty} [|c_n| (w_n)^{1/a-1/p}]^a \right\}^{1/a}.$$

$${}_a\|c_n\|_p = \inf_{(w_n) \in W} \{ \|w_n\|_1^{1/p-1/a} \|c_n\|_{p,w} \}.$$

$${}_aA_{p,j,\alpha}(f) = \left\{ \int_0^1 \left[\int_{-\pi}^{\pi} |t^{-\alpha} \Delta_t^j f(x)|^a dx \right]^{p/a} t^{-1} dt \right\}^{1/p}.$$

Note that $\|c_n\|_p \leq {}_a\|c_n\|_p$, if $0 < p \leq a \leq \infty$. In the previous paper [4], we proved the following results:

THEOREM 1.1. *Let us suppose that $1 < a \leq 2$, $1/a + 1/a' = 1$, $0 < p \leq a'$, $\alpha < j$, and $j = a$ positive integer. Then we have that*

$${}_{a'}\|c_n\|_p \|n|^{\alpha+1/a'-1/p}\|_p \leq K {}_aA_{p,j,\alpha}(f).$$

The inequality holds also for the case $a = 1$ and $0 < p < \infty = a'$.

(The letter K means a positive constant which may be different from one occurrence to another.)

THEOREM 1.2. *Let us suppose that $1 \leq a \leq 2$, $0 < p \leq a$, $1/a + 1/a' = 1$, $0 < \alpha < j$, and $j = a$ positive integer. Then we have*

$${}_{a'}A_{p,j,\alpha}(f) \leq K_a \|c_n\| n^{|\alpha-1/p+1/a|} \|_p.$$

REMARK. (i) In the previous paper, we assumed the condition $0 < \alpha < j$ in Theorem 1.1. However, by the careful examination of the original proof, we see that a condition $\alpha < j$ is enough to support the assertion of Theorem 1.1. (ii) The case $a = 1$ or $a' = p$ in Theorem 1.1 and the case $a = 1$ or $a = p$ in Theorem 1.2 were not discussed in the previous paper. However those extremal cases are rather easily proved. (See the next section of the paper or [6].)

One of the corollaries of Theorem 1.1 is

THEOREM 1.3. Suppose that $1 \leq a \leq 2$ and $0 < p \leq a' < \infty$ or $0 < p < \infty = a'$. Let us put

$$\omega_a(f, \delta) = \sup_{0 < h \leq \delta} \left[\int_{-\pi}^{\pi} |f(x+h) - f(x)|^a dx \right]^{1/a}.$$

Then we have that

$$(1.1) \quad \sum_{n=1}^{\infty} n^{\gamma-p/a'} [\omega_a(f, 1/n)]^p < \infty$$

implies

$$(1.2) \quad {}_{a'}\|c_n\| n^{|\gamma/p|} \|_p < \infty.$$

Since (1.2) implies

$$(1.3) \quad \sum_{n=-\infty}^{\infty} |c_n|^p |n|^{\gamma} < \infty,$$

our Theorem 1.3 is a generalization of J. R. McLaughlin's result [7].

For the proof of Theorem 1.3, put $\alpha = \gamma/p + 1/p - 1/a'$ in Theorem 1.1, then we have

$$\begin{aligned} [{}_aA_{p,j,\alpha}(f)]^p &\leq K \sum_{n=1}^{\infty} [\omega_a(f, 1/n)]^p \int_{1/(n+1)}^{1/n} t^{-\alpha p-1} dt \\ &\leq K \sum_{n=1}^{\infty} n^{\alpha p-1} [\omega_a(f, 1/n)]^p, \end{aligned}$$

where $\alpha p - 1 = \gamma - p/a'$.

McLaughlin stated that all known sufficiency conditions for absolute convergence of the trigonometric Fourier series given in terms of $\text{Lip } \alpha$, $\text{Lip } (\alpha, p)$, $\text{Var } p$, $\omega_p(f, \delta)$, etc. follow from (1.1). McLaughlin also proved the same conclusion for the Walsh Fourier series.

The first purpose of the present paper is to prove Walsh Fourier series analogue of Theorem 1.1 and Theorem 1.2, and their generalizations. The second purpose is to discuss the same for the case of Fourier

transform in the k -dimensional Euclidean space.

2. In this section, we suppose that f is integrable on $(0, 1)$ and periodic with period 1. The Walsh Fourier series of f will be denoted by

$$f \sim \sum_{n=0}^{\infty} c_n \psi_n(x)$$

where $\{\psi_n(x)\}$ is the system of Walsh functions.

We should modify the notation ${}_a A_{p,j,\alpha}(f)$ in the following way: We denote the dyadic addition by a symbol $+$ and put

$${}_a A_{p,\alpha}(f) = \left\{ \int_0^1 \left[\int_0^1 |t^{-\alpha} \Delta_t f(x)|^a dx \right]^{p/a} t^{-1} dt \right\}^{1/p},$$

where

$$\Delta_t f(x) = f(x + t) - f(x).$$

(In our case, any higher order difference has no meaning.)

We have the following theorems for the Walsh Fourier series:

THEOREM 2.1. *The result of Theorem 1.1 also holds for $j = 1$ and any α .*

THEOREM 2.2. *The result of Theorem 1.2 also holds for $\alpha > 0$.*

THEOREM 2.3. *The result of Theorem 1.3 also holds for the Walsh Fourier series.*

The main part of the proof of the above theorems is to prove the following Lemmas 2.1 and 2.2 corresponding to Theorems 2.1 and 2.2, respectively.

LEMMA 2.1. *Let us suppose that $0 < p \leq a$. Then we have*

$${}_a \|c_n n^{\alpha-1/p+1/a}\|_p \leq K {}_a \tilde{A}_{p,\alpha}(c_n),$$

where

$${}_a \tilde{A}_{p,\alpha}(c_n) = \left\{ \int_0^1 |t^{-\alpha} Y_a(t, c_n)|^p t^{-1} dt \right\}^{1/p}$$

and

$$Y_a(t, c_n) = \left\{ \sum_{n=0}^{\infty} |c_n|^a |\psi_n(t) - 1|^a \right\}^{1/a}.$$

LEMMA 2.2. *Let us suppose that $0 < p \leq a$ and $0 < \alpha$. Then, we have*

$${}_a \tilde{A}_{p,\alpha}(c_n) \leq K {}_a \|c_n n^{\alpha-1/p+1/a}\|_p.$$

The proofs of these lemmas are essentially the same as in the case of trigonometric Fourier series shown in [4], in which the definition of

$Y_a(t, c_n)$ is given by

$$Y_{a,j}(t, c_n) = \left\{ \sum_{n=-\infty}^{\infty} |c_n|^a |\sin nt/2|^{aj} \right\}^{1/a}.$$

Therefore we shall give the full statements of the proofs only for the points where the proofs are different from the case of trigonometric Fourier series. For details, compare the previous paper [4].

PROOF OF LEMMA 2.1. Let us discuss first the case $0 < p < a < \infty$. We have

$$\begin{aligned} (2.1) \quad [{}_a\tilde{A}_{p,a}(c_k)]^p &= \sum_{n=1}^{\infty} |c_n|^a \int_0^1 t^{-1-\alpha p} [Y_a(t, c_k)]^{p-a} |\psi_n(t) - 1|^a dt \\ &= \sum_{n=1}^{\infty} |c_n|^a M_n, \quad \text{say.} \end{aligned}$$

We define a decreasing sequence $\{w_n\}$ by

$$w_n = \int_0^{1/2^m} t^{-\alpha p} [Y_a(t, c_k)]^p dt, \quad \text{for } 2^m \leq n < 2^{m+1}.$$

Then, we have

$$\begin{aligned} \sum_{n=1}^{\infty} w_n &= \sum_{m=0}^{\infty} \sum_{n=2^m}^{2^{m+1}-1} \int_0^{1/2^m} t^{-\alpha p} [Y_a(t, c_k)]^p dt \\ &= \sum_{m=0}^{\infty} 2^m \sum_{s=m}^{\infty} \int_{1/2^{s+1}}^{1/2^s} t^{-\alpha p} [Y_a(t, c_k)]^p dt \\ &= \sum_{s=0}^{\infty} \sum_{m=0}^s 2^m \int_{1/2^{s+1}}^{1/2^s} t^{-\alpha p} [Y_a(t, c_k)]^p dt \\ &\leq K \int_0^1 t^{-1-\alpha p} [Y_a(t, c_k)]^p dt, \end{aligned}$$

that is, we have $\{w_n\} \in W$ and

$$(2.2) \quad \|w_n\|_1 \leq K [{}_aA_{p,a}(c_n)]^p.$$

Then, by Hölder's inequality, we have

$$[w_n]^{1/Q} [M_n]^{1/P} \geq \int_0^{1/2^m} t^{-\alpha p - p/a} |\psi_n(t) - 1|^p dt,$$

for $2^m \leq n < 2^{m+1}$, where we have put $P = a/p$ and $1/P + 1/Q = 1$. Since $\psi_n(t) = 1$ ($0 \leq t < 1/2^{m+1}$), and $= -1$ ($1/2^{m+1} \leq t < 1/2^m$), the above integral is

$$K \int_{1/2^{m+1}}^{1/2^m} t^{-\alpha p - p/a} dt \geq K n^{\alpha p + p/a - 1},$$

that is, $M_n \geq K n^{\alpha(a-1/p+1/a)} [w_n]^{1-a/p}$. By using the last estimate, (2.1) and (2.2), we have the conclusion.

Let us show Lemma 2.1 for the case $0 < p = a < \infty$. We have to prove that $\|c_n n^a\|_p^p \leq K[\tilde{A}_{p,j,\alpha}(c_n)]^p$. However, the right hand side of the above inequality is

$$\begin{aligned} \sum_{n=1}^{\infty} |c_n|^p \int_0^1 t^{-\alpha p-1} |\psi_n(t) - 1|^p dt &\geq K \sum_{m=0}^{\infty} \sum_{n=2^m}^{2^{m+1}-1} |c_n|^p \int_{1/2^{m+1}}^{1/2^m} t^{-\alpha p-1} dt \\ &\geq K \sum_{n=1}^{\infty} |c_n|^p n^{\alpha p}, \end{aligned}$$

which is what is required.

Let us show Lemma 2.1 for the case $0 < p < \infty = a$. Set

$$Y_{\infty}(t, c_k) = \sup_k (|c_k| |\psi_k(t) - 1|)$$

and

$$w_n = \int_0^{1/2^m} t^{-\alpha p} [Y_{\infty}(t, c_k)]^p dt, \quad (2^m \leq n < 2^{m+1}).$$

Then, in the same way as before, we have

$$(2.3) \quad \|w_n\|_1 \leq K[\tilde{A}_{p,\alpha}(c_n)]^p.$$

We have to show that $\|w_n\|_1^{1/p} \leq \|c_n n^{\alpha-1/p}\|_{p,w} \leq K \tilde{A}_{p,\alpha}(c_n)$. Since (2.3) holds, the above inequality is equivalent to $\|c_n n^{\alpha-1/p}\|_{p,w} \leq K$, which must be read as $\|c_n n^{\alpha-1/p} (w_n)^{-1/p}\|_{\infty} \leq K$. Therefore, we have to show that

$$|c_n| n^{\alpha-1/p} \leq K \left\{ \int_0^{1/2^m} t^{-\alpha p} \sup_k (|c_k| |\psi_k(t) - 1|)^p dt \right\}^{1/p} \quad \text{for } 2^m \leq n < 2^{m+1},$$

that is,

$$n^{\alpha-1/p} \leq K \left\{ \int_0^{1/2^m} t^{-\alpha p} |\psi_n(t) - 1|^p dt \right\}^{1/p}.$$

However, the integral on the right hand side is

$$K \int_{1/2^{m+1}}^{1/2^m} t^{-\alpha p} dt \geq K n^{\alpha p-1},$$

and we have the conclusion.

PROOF OF LEMMA 2.2. The proof is essentially same as in the case of trigonometric Fourier series (cf. [4, Lemma 2]). The only difference is to estimate the following integral:

$$J = \int_0^1 t^{-2+a/p-a\alpha} [w^*(1/t)]^{1-a/p} |\psi_n(t) - 1|^a dt,$$

where $w^*(t)$ satisfies (i) $t^{\delta} w^*(t)$ is increasing, and (ii) $t^{\varepsilon} w^*(t)$ is decreasing for some δ and ε such that $0 < \varepsilon < 1 < \delta$. We need to show that $J \leq K n^{a\alpha-a/p+1} [w^*(n)]^{1-a/p}$. For this purpose, we divide the integral J into

two parts:

$$J = \int_0^{1/n} + \int_{1/n}^1 = J_1 + J_2, \quad \text{say.}$$

Suppose $2^m \leq n < 2^{m+1}$. Then, by the property (i), we have

$$\begin{aligned} J_1 &= K \int_{1/2^{m+1}}^{1/n} t^{-2+a/p-a\alpha} [w^*(1/t)]^{1-a/p} dt \\ &\leq K n^{\delta(1-a/p)} [w^*(n)]^{1-a/p} \int_{1/2^{m+1}}^{1/n} t^{-2+a/p-a\alpha+\delta(1-a/p)} dt \\ &\leq K n^{a\alpha-a/p+1} [w^*(n)]^{1-a/p}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} J_2 &\leq K \int_{1/n}^1 t^{-2+a/p-a\alpha} [w^*(1/t)]^{1-a/p} dt \\ &\leq K n^{\varepsilon(1-a/p)} [w^*(n)]^{1-a/p} \int_{1/n}^1 t^{-2+a/p-a\alpha+\varepsilon(1-a/p)} dt. \end{aligned}$$

Since $\alpha > 0$, we may choose ε in such a way that $0 < \varepsilon < 1$ and $-1 + a/p - a\alpha + \varepsilon(1 - a/p) < 0$. Therefore, we have $J_2 \leq K n^{a\alpha-a/p+1} [w^*(n)]^{1-a/p}$.

PROOF OF THEOREM 2.1. The Walsh Fourier series of the dyadic difference $\Delta_t f(x)$ is

$$\Delta_t f(x) \sim \sum_{n=1}^{\infty} c_n (\psi_n(t) - 1) \psi_n(x).$$

Therefore, by the Hausdorff-Young inequality, we have $Y_{a'}(t, c_n) \leq K \|\Delta_t f(\cdot)\|_{a'}$, and hence ${}_a\tilde{A}_{p,\alpha}(c_n) \leq K {}_aA_{p,\alpha}(f)$. Then, by Lemma 2.1, we have the conclusion.

PROOF OF THEOREM 2.2. By the Hausdorff-Young inequality, we have $\|\Delta_t f(\cdot)\|_{a'} \leq K Y_a(t, c_n)$. Therefore we have ${}_aA_{p,\alpha}(f) \leq K {}_a\tilde{A}_{p,\alpha}(c_n)$. By Lemma 2.2, we have the conclusion.

From Theorems 2.1 and 2.2, we have a contraction theorem for the Walsh Fourier series:

THEOREM 2.4. *Let us denote the Walsh Fourier series of f and g by*

$$f \sim \sum c_n \psi_n \quad \text{and} \quad g \sim \sum d_n \psi_n.$$

Suppose that $0 < p \leq 2$, $0 < \alpha < 1$,

$$(2.4) \quad \sum_{n=1}^{\infty} n^{\alpha p-1} \left(\sum_{k=n}^{\infty} |c_k|^2 \right)^{p/2} < \infty,$$

and g is a contraction of f , that is, for any x_1 and x_2 ,

$$|g(x_1) - g(x_2)| \leq K |f(x_1) - f(x_2)|.$$

Then

$${}_2\|d_n n^{1/2-1/p+\alpha}\|_p < \infty.$$

(The above result is a generalization of Watari's result [9].)

PROOF OF THEOREM 2.4. Since the finiteness of ${}_2\|c_n n^{1/2-1/p+\alpha}\|_p$, ${}_2\tilde{A}_{p,\alpha}(c_n)$ and ${}_2A_{p,\alpha}(f)$ are mutually equivalent, we have to show the finiteness of $[{}_2\tilde{A}_{p,\alpha}(c_n)]^p$. We have

$$\begin{aligned} [{}_2A_{p,\alpha}(c_n)]^p &= \sum_{m=0}^{\infty} \int_{1/2^{m+1}}^{1/2^m} t^{-1+\alpha p} \left[\sum_{n=2^m}^{\infty} |c_n|^2 |\psi_n(t) - 1|^2 \right]^{p/2} dt \\ &\leq K \sum_{m=0}^{\infty} (2^m)^{\alpha p} \left[\sum_{n=2^m}^{\infty} |c_n|^2 \right]^{p/2}. \end{aligned}$$

Since the finiteness of the last series derives from (2.4), we have the conclusion. (In the same way, we can generalize the result due to Okuyama [8].)

Finally we state some generalizations of Theorems 2.1 and 2.2: We use the following notations. Suppose that a function $I(t)$ is positive and increasing on $(0, 1)$, and satisfies $I(2h)/I(h) \leq K$, where K is a positive constant. We define ${}_aA_{p,I}(f)$ by

$${}_aA_{p,I}(f) = \left\{ \int_0^1 \left[\int_0^1 \left| \frac{\Delta_t f(x)}{I(t)} \right|^a dx \right]^{p/a} \frac{dt}{t} \right\}^{1/p}.$$

Then, we have the following theorems.

THEOREM 2.5. Let us suppose that $1 < a \leq 2$, $1/a + 1/a' = 1$ and $0 < p \leq a'$. Then, we have

$${}_a\|c_n n^{1/a'-1/p} [I(1/n)]^{-1}\|_p \leq K {}_aA_{p,I}(f).$$

The inequality holds also for the case $a = 1$ and $0 < p < \infty = a'$.

THEOREM 2.6. Let us suppose that $1 \leq a \leq 2$ and $0 < p \leq a$. Suppose that $I(t)$ satisfies the following conditions: There exist positive numbers ε and δ with $0 < \varepsilon < 1 < \delta$ such that

$$\int_0^t u^\beta [I(u)]^{-a} du \leq K t^{\beta+1} [I(t)]^{-a},$$

where $\beta = -2 + a/p + \delta(1 - a/p)$, and

$$\int_t^1 u^\gamma [I(u)]^{-a} du \leq K t^{\gamma+1} [I(t)]^{-a},$$

where $\gamma = -2 + a/p + \varepsilon(1 - a/p)$. Then, we have

$${}_aA_{p,I}(f) \leq K_a \|c_n n^{1/a-1/p} [I(1/n)]^{-1}\|_p.$$

(For the proof, cf. [6].)

3. The object of the present section is to give some analogue of the previous results for the case of the Fourier transform in R_k .

We shall use the following notations:

Functions f , g , I and w are defined on the k -dimensional Euclidean space R_k .

$$\Delta^j g(x) = \sum_{m=0}^j (-1)^{j+m} \binom{j}{m} g(x + mt), \quad \text{for } x, t \in R_k.$$

$I(x)$ is a radial function and $I(|x|) = I(r)$ is positive and increasing on $(0, \infty)$.

$$\tilde{Y}_{a,j}(t, g) = \tilde{Y}_{a,j}(t) = \|g(\cdot) |\sin(t, \cdot)|/2|^j\|_a$$

$$Y_{a,j}(t, g) = Y_{a,j}(t) = \|\Delta^j g(\cdot)\|_a$$

$${}_a\tilde{A}_{p,j,I}(g) = \left\{ \int_{R_k} [\tilde{Y}_{a,j}(t)/I(t)]^p |t|^{-k} dt \right\}^{1/p}$$

$${}_aA_{p,j,I}(g) = \left\{ \int_{R_k} [Y_{a,j}(t)/I(t)]^p |t|^{-k} dt \right\}^{1/p}.$$

Let W be the class of radial functions $w(x) \in L^1(R_k)$ such that $w(|x|) = w(r)$ are positive and decreasing on $(0, \infty)$. For each $w \in W$, we define

$${}_a\|g\|_{p,w} = \left\{ \int_{R_k} [|g(x)| [w(x)]^{1/a-1/p}]^a dx \right\}^{1/a},$$

$${}_a\|g\|_p = \inf_{w \in W} \{ {}_a\|g\|_{p,w} \|w\|_1^{1/p-1/a} \}$$

and

$${}_aL_p = \{g: {}_a\|g\|_p < \infty\}.$$

Note that ${}_aL_p \subseteq L^p$ if $0 < p \leq a \leq \infty$, and ${}_aL_a = L^a$.

We shall prove the following theorems:

THEOREM 3.1. Suppose that $1 \leq a \leq 2$ and $1/a + 1/a' = 1$, $0 < p \leq a'$ (if $1 < a \leq 2$) or $0 < p < \infty = a'$ (if $a = 1$), and that $f \in L^q(R_k)$ where $1 \leq q \leq 2$. Then, for the Fourier transform \hat{f} of f in $L^q(R_k)$, we have

$${}_a\|\hat{f}(x)|x|^{k(1/a'-1/p)}[I(1/|x|)]^{-1}\|_p \leq K_a A_{p,j,I}(f).$$

THEOREM 3.2. Let a , a' , p and f be as in Theorem 3.1. Then, we have

$${}_a\|f(x)|x|^{k(1/a'-1/p)}[I(1/|x|)]^{-1}\|_p \leq K_a A_{p,j,I}(\hat{f}).$$

THEOREM 3.3. Suppose that $1 \leq a \leq 2$, $1/a + 1/a' = 1$ and $0 < p \leq a$, and that $f \in L^q(R_k)$ where $1 \leq q \leq 2$. Furthermore, we suppose that there exist positive ε and δ such that

$$(i) \quad 0 < \varepsilon < 1 < \delta < \infty,$$

$$(ii) \quad \int_0^u r^\beta [I(r)]^{-a} dr \leq Ku^{\beta+1} [I(u)]^{-a} \text{ where } \beta = k(a/p - 1) + aj + \delta(1 -$$

$a/p) - 1$, and

(iii) $\int_u^\infty r^\gamma [I(r)]^{-a} dr \leq K u^{\gamma+1} [I(u)]^{-a}$ where $\gamma = k(a/p-1) + \varepsilon(1-a/p) - 1$.

Then, for the Fourier transform \hat{f} of f in $L^q(R_k)$, we have

$${}_a A_{p,j,I}(\hat{f}) \leq K_a \|f(x)|x|^{k(1/a-1/p)} [I(1/|x|)]^{-1}\|_p.$$

THEOREM 3.4. Let a , a' , p , f and I be as in Theorem 3.3. Then we have

$${}_a A_{p,j,I}(f) \leq K_a \|\hat{f}(x)|x|^{k(1/a-1/p)} [I(1/|x|)]^{-1}\|_p.$$

In particular, we have the following

THEOREM 3.5. Suppose that $0 < p \leq 2$, and that $I(t)$ satisfies the same condition (in $a = 2$) as in Theorem 3.3. Then, ${}_2 A_{p,j,I}(f)$ is finite, if and only if ${}_2 \|\hat{f}(x)|x|^{k(1/2-1/p)} [I(1/|x|)]^{-1}\|_p$ is finite.

REMARK: If we take $I(x) = |x|^\alpha$ ($\alpha < j$), then our theorems imply the Beurling and Herz result (cf. [1] and [3, in particular, Theorem 2]). In fact, the norm ${}_a A_{p,j,I}(f)$ in the case when $I(x) = |x|^\alpha$ is equivalent to the norm of Besov space B_{ap}^α .

The proof of these theorems is based upon the following lemmas.

LEMMA 3.1. Suppose that $0 < p \leq a < \infty$. Then, we have

$${}_a \|g(x)|x|^{k(1/a-1/p)} [I(1/|x|)]^{-1}\|_p \leq K_a \tilde{A}_{p,j,I}(g).$$

The inequality holds for the case $0 < p < \infty = a$.

LEMMA 3.2. Suppose that $0 < p \leq a < \infty$, and that $I(t)$ satisfies the conditions (i), (ii) and (iii) in the statement of Theorem 3.3. Then we have

$$\tilde{A}_{p,j,I}(g) \leq K_a \|g(x)|x|^{k(1/a-1/p)} [I(1/|x|)]^{-1}\|_p.$$

PROOF OF LEMMA 3.1. We discuss first the case $0 < p < a < \infty$. In this case, we can prove the lemma in the same way as in the proof of Lemma 3 of the previous paper [5]. Therefore we shall give a sketch.

Let us put

$$w(x) = \int_{|t| < 1/|x|} [I(t)]^{-p} [\tilde{Y}_{a,j}(t)]^p dt.$$

Since $\tilde{Y}(t)$ is radial (cf. [5, Lemma 1]), and since $I(t)$ and $w(x)$ are radial, we have

$$\|w\|_1 = K \int_0^\infty [I(r)]^{-p} [\tilde{Y}_{a,j}(r)]^p r^{-k} r^{k-1} dr,$$

that is,

$$(3.1) \quad \|w\|_1 = K [{}_a \tilde{A}_{p,j,I}(g)]^p < \infty, \quad \text{and} \quad w \in W.$$

By definition, we have

$$(3.2) \quad [{}_a A_{p,j,I}(g)]^p = \int_{R_x} |g(x)|^a G(x) dx,$$

where

$$G(x) = \int_{R_x} |t|^{-k} [I(t)]^{-p} [\tilde{Y}_{a,j}(t)]^{p-a} |\sin(t, x)/2|^{aj} dt.$$

Let us estimate the following integral:

$$V = \int_{|t| < 1/|x|} |t|^{-kp/a} [I(t)]^{-p} |\sin(t, x)/2|^{pj} dt.$$

Put $P = a/p > 1$ and $1/P + 1/Q = 1$. Then, by the Hölder inequality, $V \leq [G(x)]^{1/P} [w(x)]^{1/Q}$. Therefore, $G(x) \geq [w(x)]^{-P/Q} V^P$. On the other hand, we have, for $k \geq 2$,

$$\begin{aligned} V &= \int_{|y| \leq 1/|x|} |y|^{-kp/a} [I(y)]^{-p} |\sin(y_1|x|)/2|^{pj} dy \\ &\geq K [I(1/|x|)]^{-p} |x|^{-k+ka/p}. \end{aligned}$$

We see that the above inequality holds also for $k = 1$. Now we have

$$(3.3) \quad G(x) \geq [w(x)]^{1-a/p} |x|^{-ka/p+k} [I(1/|x|)]^{-a}.$$

Summing up (3.2) and (3.3), we have

$$[{}_a \tilde{A}_{p,j,I}(g)]^p \geq K_a \|g(x)|x|^{k(1/a-1/p)} [I(1/|x|)]^{-1}\|_{p,w}^a.$$

Using the equality (3.1), we have

$$\begin{aligned} {}_a \tilde{A}_{p,j,I}(g) &= [\tilde{A}]^{p(1/p-1/a)} [\tilde{A}]^{p/a} \\ &\geq K \|w\|_1^{1/p-1/a} \|g(x)|x|^{k(1/a-1/p)} [I(1/|x|)]^{-1}\|_{p,w}, \end{aligned}$$

by which we have the conclusion in the case $0 < p < a < \infty$.

For the case $0 < p = a < \infty$, the result of Lemma 3.1 is easily seen.

Let us finally show the result in the case $0 < p < \infty = a$: Set $\tilde{Y}_{\infty,j}(t; g) = \sup_x [g(x) |\sin(t, x)/2|^j]$ and

$$w(x) = \int_{|t| < 1/|x|} [I(t)]^{-p} [\tilde{Y}_{\infty,j}(t; g)]^p dt,$$

then we have

$$(3.4) \quad \|w\|_1 \leq K [{}_a \tilde{A}_{p,j,I}(g)]^p.$$

We have to show that

$$\|w\|_1^{1/p} \|g(x)|x|^{-k/p} [I(1/|x|)]^{-1}\|_{p,w} \leq K {}_\infty \tilde{A}_{p,j,I}(g).$$

Due to the inequality (3.4), the above inequality is equivalent to $\|g(x)|x|^{-k/p} [I(1/|x|)]^{-1} [w(x)]^{-1/p}\|_\infty \leq K$. Therefore, we have to show that

$$|x|^{-k}[I(1/|x|)]^{-p} \leq K \int_{|t| < 1/|x|} [I(t)]^{-p} |\sin(t, x)/2|^{pj} dt.$$

However, the right hand side integral is greater than

$$[I(1/|x|)]^{-p} \int_{|t| < 1/|x|} |\sin(t, x)/2|^{pj} dt = K[I(1/|x|)]^{-p} |x|^{-k},$$

by which we have the conclusion (cf. [5, proof of Lemma 3]).

PROOF OF LEMMA 3.2. For the detail, see the proof of [5, Lemma 4].

Let us discuss the case $0 < p < a < \infty$. Suppose that there exists $w \in W$ such that

$$_a \|g(x)|x|^{k(1/a-1/p)}[I(1/|x|)]^{-1}\|_{p,w} < \infty.$$

For this w , we find w^* which has the following properties:

$$(3.5) \quad w^*(t) \text{ is radial and positive.}$$

$$(3.6) \quad w(t) \leq w^*(t).$$

$$(3.7) \quad r^\varepsilon w^*(r) \text{ is decreasing on } (0, \infty).$$

$$(3.8) \quad r^\delta w^*(r) \text{ is increasing on } (0, \infty).$$

$$(3.9) \quad \|w^*\|_1 = K\|w\|_1.$$

Let $P = a/p$, $1/P + 1/Q = 1$, $\alpha_1 = 2k/Q - k$ and $\alpha_2 = -2k/Q$. Then, by the Hölder inequality, we have

$$\begin{aligned} & [_a \tilde{A}_{p,j,I}(g)]^p \\ & \leq K \|w^*\|_1^{1/Q} \left\{ \int_{R_+} [\tilde{Y}(t)]^a [I(t)]^{-a} |t|^{\alpha_1 P} [w^*(1/|t|)]^{1-a/p} dt \right\}^{1/P} \\ & = K \|w^*\|_1^{1/Q} \left\{ \int_{R_+} |g(x)|^a dx \int_{R_+} |\sin(t, x)/2|^{aj} [I(t)]^{-a} |t|^{\alpha_1 P} [w^*(1/|t|)]^{1-a/p} dt \right\}^{1/P}. \end{aligned}$$

We denote by J the second integral on the right hand side. Then,

$$\begin{aligned} J & \leq K \int_{|t| < 1/|x|} |x|^{aj} |t|^{\alpha_1 P + aj} [I(t)]^{-a} [w^*(1/|t|)]^{1-a/p} dt \\ & \quad + K \int_{|t| > 1/|x|} |t|^{\alpha_1 P} [I(t)]^{-a} [w^*(1/|t|)]^{1-a/p} dt \\ & = K(J_1 + J_2), \quad \text{say.} \end{aligned}$$

Then, by using properties (ii) and (3.8), we have

$$\begin{aligned} J_1 & \leq |x|^{aj} \int_0^{1/|x|} r^\delta [I(r)]^{-a} [(1/r)^\delta w^*(1/r)]^{-(a/p-1)} dr \\ & \leq |x|^{aj-\delta(a/p-1)} [w^*(|x|)]^{1-a/p} \int_0^{1/|x|} r^\delta [I(r)]^{-a} dr \\ & \leq K |x|^{k(1-a/p)} [w^*(|x|)]^{1-a/p} [I(1/|x|)]^{-a}. \end{aligned}$$

Similarly, we have, by (iii) and (3.7),

$$\begin{aligned} J_2 &\leq |x|^{-\varepsilon(a/p-1)} [w^*(|x|)]^{1-a/p} \int_{1/|x|}^{\infty} r^{\varepsilon} [I(r)]^{-a} dr \\ &\leq K |x|^{k(1-a/p)} [w^*(|x|)]^{1-a/p} [I(1/|x|)]^{-a}. \end{aligned}$$

Summing up J_1 and J_2 and using (3.6), we have $J \leq K |x|^{k(1-a/p)} [I(1/|x|)]^{-a} \times [w(|x|)]^{1-a/p}$. So we have the conclusion in the case $0 < p < a < \infty$.

For the case $0 < p = a < \infty$, the result of Lemma 3.2 is easily seen from the properties (ii) and (iii) with $a = p$.

PROOF OF THEOREM 3.1. Suppose that ${}_a A_{p,j,I}(f)$ is finite. Then we have $\Delta_i^j f \in L^a(R_k)$, and there exists its Fourier transform in $L^{a'}(R_k)$, which must be equal to the Fourier transform of $\Delta_i^j f(x)$ in $L^{a'}(R_k)$, that is, $(\Delta_i^j f)^\wedge(x) = (e^{i(x,t)} - 1)^j \hat{f}(x)$. By the Hausdorff-Young inequality, we have $\|\hat{f}(\cdot) |\sin(\cdot, t)/2|^j\|_{a'} \leq K \|\Delta_i^j f(\cdot)\|_a$. It follows that ${}_a \tilde{A}_{p,j,I}(\hat{f}) \leq K {}_a A_{p,j,I}(f)$. Applying Lemma 3.1, we have the conclusion.

PROOF OF THEOREM 3.2. Suppose that ${}_a A_{p,j,I}(\hat{f}) < \infty$. Then we have $\|\Delta_i^j \hat{f}\|_a < \infty$. Therefore, there exists the Fourier (inverse) transform of $\Delta_i^j \hat{f}$, which may be expressed as

$$(3.10) \quad \text{l.i.m.}_{N \rightarrow \infty}^{(a')} \int_{NP} [\Delta_i^j \hat{f}(u)] e^{i(x,u)} du = f^*(x) (e^{-i(x,z)} - 1)^j, \quad \text{say}.$$

(For the inversion argument, see, for example, Herz [2]: Here we denote by P a convex polyhedron in R_k containing 0 in its interior and NP the homothetic dilation of P by the amount N .) On the other hand $\Delta_i^j \hat{f}(u)$ is the Fourier transform of $f(x) (e^{-i(x,z)} - 1)^j$ in $L^{a'}(R_k)$, and we have, by the inversion argument,

$$(3.11) \quad f(x) (e^{-i(x,z)} - 1)^j = \text{l.i.m.}_{N \rightarrow \infty}^{(a)} (2\pi)^{-n/2} \int_{NP} [\Delta_i^j \hat{f}(u)] e^{i(x,u)} du.$$

From the two equalities (3.10) and (3.11), we have $(2\pi)^{n/2} f(x) = f^*(x)$. Apply the Hausdorff-Young inequality, and we have $\|f(\cdot) (e^{-i(\cdot,z)} - 1)^j\|_{a'} \leq K \|\Delta_i^j \hat{f}\|_a$, that is, ${}_a \tilde{A}_{p,j,I}(f) \leq K {}_a A_{p,j,I}(\hat{f})$. By Lemma 3.1, we have the conclusion.

PROOF OF THEOREM 3.3. We may suppose that ${}_a \tilde{A}_{p,j,I}(f)$ is finite (cf. Lemma 3.2). Then, $f(\cdot) (e^{-i(\cdot,t)} - 1)^j \in L^a(R_k)$, and we have its Fourier transform in $L^{a'}(R_k)$ which must be equal to the Fourier transform of $f(\cdot) (e^{-i(\cdot,t)} - 1)^j$ in $L^{a'}(R_k)$, that is,

$$[f(\cdot) (e^{-i(\cdot,t)} - 1)^j]^\wedge(u) = \Delta_i^j \hat{f}(u).$$

By the Hausdorff-Young inequality, we have $\|\Delta_i^j \hat{f}(\cdot)\|_{a'} \leq K \|f(\cdot) |\sin(\cdot, t)|^j\|_a$.

$t)/2^j\|_a$ and, by Lemma 3.2, we have the conclusion.

PROOF OF THEOREM 3.4. Suppose that

$$_a\|\hat{f}(x)|x|^{k(1/a-1/p)}[I(1/|x|)]^{-1}\|_p < \infty.$$

Then, by Lemma 3.2, we have $_a\tilde{A}_{p,j,I}(\hat{f}) < \infty$, and $(e^{i(t,u)} - 1)^j \hat{f}(u) \in L^a(R_k)$. Therefore, there exists the limit

$$\text{l.i.m.}_{N \rightarrow \infty}^{(a')} \int_{NP} (e^{i(t,u)} - 1)^j \hat{f}(u) e^{i(x,u)} du = f^*(x, t), \quad \text{say}.$$

On the other hand, we have $(\Delta_t^j f)^\wedge(u) = (e^{i(t,u)} - 1)^j \hat{f}(u)$ in $L^{q'}(R_k)$, and, by the inversion argument, we have

$$\text{l.i.m.}_{N \rightarrow \infty}^{(q)} \int_{NP} (e^{i(t,u)} - 1)^j \hat{f}(u) e^{i(x,u)} du = (2\pi)^{n/2} \Delta_t^j f(x),$$

which must be equal to $f^*(x, t)$. Now apply the Hausdorff-Young inequality, and we have $\|\Delta_t^j f\|_{a'} \leq K \|(e^{i(t,\cdot)} - 1)^j \hat{f}(\cdot)\|_a$, that is, $_a\tilde{A}_{p,j,I}(f) \leq K _a\tilde{A}_{p,j,I}(\hat{f})$. Using Lemma 3.2, we have the conclusion.

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