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THE NON-EXISTENCE OF KILLING FIELDS

Dedicated to Professor Shigeo Sasaki on his seventieth birthday

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1. Introduction. We shall be in the C^∞-category. The manifolds under consideration are connected and orientable. In this note we prove the following theorem:

THEOREM. *Let M be a complete foliated Riemannian manifold with a foliation $\mathscr{F}$, and the Riemannian metric be a bundle-like metric with respect to $\mathscr{F}$. If all leaves of $\mathscr{F}$ are minimal and the Ricci operator ρ_∇ of $\mathscr{F}$ is non-positive everywhere and negative for at least one point of M, then every transverse Killing field of $\mathscr{F}$ with finite global norm is trivial.*

Examples of foliated Riemannian manifolds with bundle-like metrics and minimal leaves are shown in [2] and [6]. We remark that the assumption on the Ricci operator of $\mathscr{F}$ can also be interpreted as the quasi-negativity of the Ricci operator of $\mathscr{F}$ in the sense of Wu [10], [11].

The above theorem seems to be the ultimate generalization of the vanishing theorem of Killing vector fields started by Bochner. So far, we have already obtained the following results:

(i) If M in the above theorem is compact, then every transverse Killing field of $\mathscr{F}$ has finite global norm. The theorem in this case was obtained by Kamber and Tondeur [3].

(ii) If the foliation $\mathscr{F}$ on a complete Riemannian manifold M is the point foliation, then the Ricci operator ρ_∇ of $\mathscr{F}$ is the usual Ricci curvature operator and a transverse Killing field of $\mathscr{F}$ is a usual Killing vector field on M. The theorem in this case was obtained by Yorozu [8], [9].

(iii) The case of the point foliation $\mathscr{F}$ on a compact Riemannian manifold is the well-known theorem of Bochner [1].

Our discussions are essentially based on [3].

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2. Preliminaries. This section is devoted to the review of [3]. Let M be an n-dimensional complete foliated Riemmanian manifold with a foliation $\mathscr{F}$, a Riemannian metric g_M and the Levi-Civita connection ∇^M with respect to g_M. We assume that the foliation $\mathscr{F}$ is of codimension q $(0 \leqq q \leqq n)$ and the metric g_M is a bundle-like metric with respect to $\mathscr{F}$ in the sense of Reinhart [5]. The foliation $\mathscr{F}$ is given by an integrable subbundle E of the tangent bundle TM over M. Let Q denote the normal bundle TM/E. The metric g_M defines a splitting σ of the exact sequence

$$0 \to E \to TM \underset{\pi}{\overset{\sigma}{\leftrightarrows}} Q \to 0$$

with $\sigma(Q) = E^{\perp}$ (the orthogonal complement bundle of E). Thus g_M induces a metric g_Q on Q: $g_Q(\nu, \mu) = g_M(\sigma(\nu), \sigma(\mu))$ for all $\nu, \mu \in \Gamma(Q)$, where $\Gamma(\cdot)$ denotes the space of all sections of a bundle. For any connection D in Q, the torsion T_D of D is given by

$$T_D(X, Y) = D_X\pi(Y) - D_Y\pi(X) - \pi([X, Y])$$

for all $X, Y \in \Gamma(TM)$ and the curvature R_D of D is given by

$$R_D(X, Y)\nu = D_X D_Y \nu - D_Y D_X \nu - D_{[X,Y]}\nu$$

for all $X, Y \in \Gamma(TM)$ and all $\nu \in \Gamma(Q)$ (cf. [2], [3]). Now we define a connection ∇ in Q by

$$\nabla_X \nu = \pi([X, Y_\nu])$$

for all $X \in \Gamma(E)$ and all $\nu \in \Gamma(Q)$ with $Y_\nu = \sigma(\nu) \in \Gamma(\sigma(Q))$,

$$\nabla_X \nu = \pi(\nabla^M_X Y_\nu)$$

for all $X \in \Gamma(\sigma(Q))$ and all $\nu \in \Gamma(Q)$ with $Y_\nu = \sigma(\nu) \in \Gamma(\sigma(Q))$.

PROPOSITION 1 (cf. [2]). *The connection ∇ in Q is torsion-free and metrical with respect to g_Q, that is,*

$$T_\nabla = 0 \quad \text{and} \quad \nabla_X g_Q = 0$$

for all $X \in \Gamma(TM)$.

We remark that $\nabla_X g_Q$ is defined by

$$(\nabla_X g_Q)(\nu, \mu) = X(g_Q(\nu, \mu)) - g_Q(\nabla_X \nu, \mu) - g_Q(\nu, \nabla_X \mu)$$

for all $X \in \Gamma(TM)$ and $\nu, \mu \in \Gamma(Q)$. We have that $i(X)R_\nabla = 0$ for all $X \in \Gamma(E)$, where i denotes the interior product (cf. [2]). We also have the following:

PROPOSITION 2 (cf. [2], [3]). *For all $\nu, \mu \in \Gamma(Q)$, the operator $R_\nabla(\nu, \mu)$:*

$\Gamma(Q) \to \Gamma(Q)$ is a well-defined endomorphism.

We introduce at a point $x \in M$ an orthonormal basis $e_{p+1}, \cdots, e_{p+q} = e_n$ of Q_x with $p = n - q$. Then the Ricci operator $\rho_\nabla: \Gamma(Q) \to \Gamma(Q)$ of $\mathscr{F}$ is defined by

$$(\rho_\nabla \nu)_x = \sum_{\alpha=p+1}^{n} R_\nabla(\nu, e_\alpha)e_\alpha$$

for all $\nu \in \Gamma(Q)$.

DEFINITION. The Ricci operator ρ_∇ of $\mathscr{F}$ is *non-positive* (resp. *negative*) *at a point* $x \in M$ if

$$g_Q(\rho_\nabla \nu, \nu)_x \leqq 0 \quad (\text{resp. } < 0)$$

for all $\nu \in \Gamma(Q)$ satisfying $\nu(x) \neq 0$.

Let $V(\mathscr{F})$ denote the space of all vector fields Y on M satisfying $[Y, Z] \in \Gamma(E)$ for all $Z \in \Gamma(E)$, where [ ] denotes the bracket operator. We define $\theta(Y): \Gamma(Q) \to \Gamma(Q)$ for $Y \in V(\mathscr{F})$ by

$$\theta(Y)\nu = \pi([Y, Y_\nu])$$

for all $\nu \in \Gamma(Q)$ and $Y_\nu \in \Gamma(TM)$ with $\pi(Y_\nu) = \nu$. The right hand side of the above equality is independent of the choice of the representative Y_ν of ν. For $Y \in V(\mathscr{F})$, $\theta(Y)g_Q$ is defined by

$$(\theta(Y)g_Q)(\nu, \mu) = Y(g_Q(\nu, \mu)) - g_Q(\theta(Y)\nu, \mu) - g_Q(\nu, \theta(Y)\mu)$$

for all $\nu, \mu \in \Gamma(Q)$.

DEFINITION (cf. [3]). If $Y \in V(\mathscr{F})$ satisfies $\theta(Y)g_Q = 0$, then $\pi(Y)$ is called a *transverse Killing field of* $\mathscr{F}$.

Let $\Omega^r(M, Q)$ be the space of all Q-valued r-forms on M. We define the exterior differential $d_\nabla: \Omega^r(M, Q) \to \Omega^{r+1}(M, Q)$ $(r \geqq 0)$ by

$$\begin{aligned} d_\nabla \eta(X_1, \cdots, X_{r+1}) = & \sum_{i=1}^{r+1} (-1)^{i+1} \nabla_{X_i} \eta(X_1, \cdots, \hat{X}_i, \cdots, X_{r+1}) \\ & + \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{r+1}), \end{aligned}$$

and define $d_\nabla^*: \Omega^r(M, Q) \to \Omega^{r-1}(M, Q)$ by

$$d_\nabla^* = (-1)^{rn+n+1} * d_\nabla *$$

where $*$ denotes the Hodge star operator. The Laplacian Δ is defined by $\Delta = d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$. Let $\Omega_o^r(M, Q)$ be the subspace of all Q-valued r-forms on M with compact support. For all $\eta, \tilde{\eta} \in \Omega^r(M, Q)$ with η or $\tilde{\eta}$ in $\Omega_o^r(M, Q)$, we define

$$\langle\langle \eta, \tilde{\eta} \rangle\rangle = \int_M g_Q(\eta \wedge * \tilde{\eta}) \quad (< +\infty)$$

and

$$\|\eta\|^2 = \langle\langle \eta, \eta \rangle\rangle .$$

For example, $g_Q(\eta \wedge * \tilde{\eta}) = g_Q(\nu, \tilde{\nu}) \xi \wedge * \tilde{\xi}$ if one of $\eta = \xi \otimes \nu$, $\tilde{\eta} = \tilde{\xi} \otimes \tilde{\nu} \in \Omega^1(M, Q)$ has compact support. If $\eta \in \Omega_o^r(M, Q)$ or $\tilde{\eta} \in \Omega_o^{r+1}(M, Q)$, we have

$$\langle\langle d_\nabla \eta, \tilde{\eta} \rangle\rangle = \langle\langle \eta, d_\Delta^* \tilde{\eta} \rangle\rangle .$$

The space $\Gamma(Q)$ is viewed as the space $\Omega^0(M, Q)$. Let $L_2^0(M, Q)$ be the completion of $\Omega_o^0(M, Q)$ with respect to the inner product $\langle\langle\ ,\ \rangle\rangle$.

DEFINITION. If $\nu \in \Gamma(Q)$ belongs to $L_2^0(M, Q) \cap \Omega^0(M, Q)$, then ν is called a *field of $\mathscr{F}$ with finite golbal norm.*

Now, we define $A_\nabla(Y): \Gamma(Q) \to \Gamma(Q)$ for $Y \in V(\mathscr{F})$ by

$$A_\nabla(Y)\nu = \theta(Y)\nu - \nabla_Y \nu$$

for all $\nu \in \Gamma(Q)$ (cf. [3]). By the torsion-freeness of ∇, we have

$$A_\nabla(Y)\nu = -\nabla_{Y_\nu} \pi(Y)$$

where $Y_\nu \in \Gamma(TM)$ with $\pi(Y_\nu) = \nu$. Thus we may define

$$A_\nabla(\nu): \Gamma(Q) \to \Gamma(Q)$$

for all $\nu \in \Gamma(Q)$ by $A_\nabla(\nu) = A_\nabla(Y)$ with $\pi(Y) = \nu$ and $A_\nabla(\nu) f = 0$ for any function f on M.

PROPOSITION 3. *Under the assumption that all leaves of $\mathscr{F}$ are minimal, a transverse Killing field $\nu \in \Gamma(Q)$ of $\mathscr{F}$ satisfies $\Delta \nu = \rho_\nabla \nu$.*

PROOF. Let $X_1, \cdots, X_{p+q}$ be an orthonormal local frame of TM on a neighborhood of $x \in M$ such that $X_1, \cdots, X_p \in \Gamma(E)$, $X_{p+1}, \cdots, X_{p+q} \in \Gamma(\sigma(Q))$, and let $(X_i)_x = e_i$ and $(X_\alpha)_x = e_\alpha$ ($i = 1, \cdots, p$; $\alpha = p+1, \cdots, p+q$). For a transverse Killing field ν of $\mathscr{F}$, we have $\nabla_{X_i} \nu = 0$, and $\nabla_Y \nu = 0$ with $Y = \sum_{i=1}^p (\nabla_{X_i}^M X_i)$, since the minimality of leaves of $\mathscr{F}$ implies that $Y_{\sigma(Q)} = 0$ where $(\ )_{\sigma(Q)}$ denotes the $\sigma(Q)$-component of $(\ )$. Thus we have

$$\begin{aligned}
(\Delta \nu)_x &= (d_\nabla^* d_\nabla \nu)_x \\
&= -\sum_{i=1}^p (\nabla_{e_i} \nabla_{X_i} \nu - \nabla_{Y_i} \nu) - \sum_{\alpha=p+1}^{p+q} (\nabla_{e_\alpha} \nabla_{X_\alpha} \nu - \nabla_{U_\alpha} \nu) \\
&= -\sum_{\alpha=p+1}^{p+q} (\nabla_{e_\alpha} \nabla_{X_\alpha} \nu - \nabla_{(U_\alpha)_{\sigma(Q)}} \nu)
\end{aligned}$$

with $Y_i = \nabla_{e_i}^M X_i$ and $U_\alpha = \nabla_{e_\alpha}^M X_\alpha$, and

$$(\nabla_{X_\alpha} A_\nabla(\nu))(\pi(X_\beta)) = \nabla_{X_\alpha}(A_\nabla(\nu)\pi(X_\beta)) - A_\nabla(\nu)(\nabla_{X_\alpha}\pi(X_\beta))$$
$$= -\nabla_{X_\alpha}\nabla_{X_\beta}\nu + \nabla_{W_{\sigma(Q)}}\nu$$

with $W = \nabla^M_{X_\alpha} X_\beta$. Hence, in a neighborhood of x, we have

$$\Delta\nu = \sum_{\alpha=p+1}^{p+q} (\nabla_{X_\alpha} A_\nabla(\nu))(\pi(X_\alpha)) .$$

On the other hand, we have

$$(\nabla_{X_\alpha} A_\nabla(\nu))(\pi(X_\beta)) = R_\nabla(\nu, \pi(X_\alpha))\pi(X_\beta)$$

(cf. [3, Proposition 3.17]). In the case of the point foliation, the above equality is well-known (cf. [4, Proposition 2.2]). Therefore we have $\Delta\nu = \rho_\nabla \nu$.

3. Proof of Theorem. Let us pick and fix a point o of M. For $r > 0$, we set

$$B(r) = \{x \in M \mid \rho(x) < r\} ,$$

where $\rho(x)$ denotes the geodesic distance from o to x. There exists a family of Lipschitz continuous functions $\{w_r; r > 0\}$ on M satisfying the following properties:

$$\begin{aligned} &0 \leqq w_r(x) \leqq 1 && \text{for all} \quad x \in M \\ &\operatorname{supp} w_r \subset B(2r) && \\ &w_r(x) = 1 && \text{for all} \quad x \in B(r) \\ &\lim_{r\to\infty} w_r = 1 && \\ &|dw_r| \leqq C/r && \text{almost everywhere on} \quad M , \end{aligned}$$

where C is a positive constant independent of r (cf. [7], [8], [9]). Then we have the following:

LEMMA 1. *For all $\nu \in \Gamma(Q)$, there exists a positive constant A independent of r such that*

$$\| dw_r \otimes \nu \|^2_{B(2r)} \leqq (A/r^2) \| \nu \|^2_{B(2r)}$$

where $\| \nu \|^2_{B(2r)} = \langle\!\langle \nu, \nu \rangle\!\rangle_{B(2r)} = \int_{B(2r)} g_Q(\nu, \nu)^*1.$

We define $d_\nabla(w_r^2\nu)$ by $d_\nabla(w_r^2\nu) = 2w_r dw_r \otimes \nu + w_r^2 d_\nabla \nu$ almost everywhere on M. By the Schwarz inequality and Lemma 1, we have

$$\begin{aligned} \langle\!\langle \Delta\nu, w_r^2\nu \rangle\!\rangle_{B(2r)} &= \langle\!\langle d_\nabla\nu, 2w_r dw_r \otimes \nu + w_r^2 d_\nabla\nu \rangle\!\rangle_{B(2r)} \\ &= \| w_r d_\nabla \nu \|^2_{B(2r)} + 2\langle\!\langle w_r d_\nabla\nu, dw_r \otimes \nu \rangle\!\rangle_{B(2r)} \\ &\geqq \| w_r d_\nabla \nu \|^2_{B(2r)} - 2\| w_r d_\nabla \nu \|_{B(2r)} \| dw_r \otimes \nu \|_{B(2r)} \end{aligned}$$

$$\geqq \| w_r d_\nabla \nu \|^2_{B(2r)} - ((1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} + 2\| dw_r \otimes \nu \|^2_{B(2r)})$$
$$\geqq (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} - (2A/r^2)\| \nu \|^2_{B(2r)} .$$

Thus, by Proposition 3, we have:

LEMMA 2. *Suppose that all leaves of $\mathscr{F}$ are minimal. For a transverse Killing field ν of $\mathscr{F}$, the following holds*:

$$\langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} \geqq (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} - (2A/r^2)\| \nu \|^2_{B(2r)} .$$

Since the Ricci operator ρ_∇ of $\mathscr{F}$ is non-positive everywhere, we have that, for a transverse Killing field ν of $\mathscr{F}$ with finite global norm,

$$0 \geqq \limsup_{r\to\infty} \langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)}$$
$$\liminf_{r\to\infty} \| w_r d_\nabla \nu \|^2_{B(2r)} \geqq 0$$
$$\lim_{r\to\infty} (2A/r^2)\| \nu \|^2_{B(2r)} = 0 .$$

From these, we have

$$0 \geqq \liminf_{r\to\infty} \langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} \geqq \liminf_{r\to\infty} (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} \geqq 0 ,$$
$$0 \geqq \limsup_{r\to\infty} \langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} \geqq \limsup_{r\to\infty} (1/2)\| w_r d_\nabla \nu \|^2_{B(2r)} \geqq 0 .$$

Thus $d_\nabla \nu = 0$, i.e., $\nabla_Y \nu = 0$ for all $Y \in \Gamma(TM)$, and $\langle\langle \rho_\nabla \nu, w_r^2 \nu \rangle\rangle_{B(2r)} = 0$ for all $r > 0$. Since the Ricci operator ρ_∇ of $\mathscr{F}$ is negative for at least one point of M, say x_o, we have $\nu(x_o) = 0$. Since $\nabla_Y \nu = 0$, we see that ν vanishes identically. Therefore our theorem is proved.

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