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CORONA THEOREM WITH BOUNDS FOR FINITELY SHEETED DISKS

(Dedicated to Professor Yukio Kusunoki on the occasion of his 60^{th} birthday)

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The primary purpose of this paper is to show that the corona theorem with bounds is valid for finitely sheeted disks possibly with infinitely many branch points where the bounds are dominated by universal constants depending only on their sheet numbers. As a consequence the corona theorem with bounds is valid for finite Riemann surfaces where the bounds are dominated by universal constants depending only on their Euler characteristics.

We start by fixing terminology before stating our main result precisely. An n-tuple $\{f_j\}$ of functions $f_1, \cdots, f_n$ in $H^\infty(R)$ of the family of bounded holomorphic functions on a Riemann surface R is referred to as a *corona datum* of length n in $\boldsymbol{N}$, the set of positive integers, and of lower bound δ in the interval $(0, 1)$, or simply of *index* (n, δ), if the following two conditions are satisfied: $\max_{1\leqq j\leqq n}(\sup_R |f_j|) \leqq 1$ and $\inf_R(\sum_{j=1}^n |f_j|) \geqq \delta$. An n-tuple $\{g_j\}$ of functions $g_1, \cdots, g_n$ in $H^\infty(R)$ is said to be a *corona solution* of the datum $\{f_j\}$ if $\sum_{j=1}^n f_j g_j = 1$. The quantity $C(R; n, \delta)$ in $(0, \infty]$ given by

$$(1) \qquad C(R; n, \delta) = \sup_{\{f_j\}}(\inf_{\{g_j\}}(\max_{1\leqq j\leqq n}(\sup_R |g_j|)))$$

will be referred to as the *Gamelin constant* of R of index (n, δ) in $\boldsymbol{N}\times(0, 1)$ where the first supremum is taken with respect to corona data $\{f_j\}$ of index (n, δ) on R and the next infimum is taken with respect to corona solutions $\{g_j\}$ of each fixed datum $\{f_j\}$ under the usual convention that $\inf_{\{g_j\}} = \infty$ if there exist no corona solutions $\{g_j\}$ of the datum $\{f_j\}$. Since the quantity was first systematically considered for plane regions by Gamelin [6], we attach the name to the quantity for the convenience of references. We should mention that the quantity was also considered for plane regions by Behrens [2].

We say that the *corona theorem* is valid on a Riemann surface R if

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there always exist corona solutions of any given corona datum on R. If $C(R; n, \delta) < \infty$ for any index (n, δ), then the corona theorem is valid, and in this case, we say that the *corona theorem with bounds* is valid on R. Thus the corona theorem with bounds implies the simple corona theorem. The present study is motivated by the following

QUESTION. *Does the validity of the simple corona theorem on a Riemann surface R automatically imply that of the corona theorem with bounds on R?*

It seems to be very difficult to give a complet answer to the question in general and in the present paper we will only discuss it for a certain special class of Riemann surfaces. Consider two Riemann surfaces $\tilde{R}$ and R and an analytic mapping π of $\tilde{R}$ onto R. We say that $(\tilde{R}, R, \pi)$, or simply $\tilde{R}$, is a *covering surface* of R. We also say that $\tilde{R}$ is represented as a covering surface $(\tilde{R}, R, \pi)$. The surface R and the map π are referred to as the base surface and the covering map of the covering surface $(\tilde{R}, R, \pi)$. We say that the covering surface $(\tilde{R}, R, \pi)$ is *unbounded* (or unlimited) if for any curve C on R with its initial point z and any $\tilde{z}$ in $\pi^{-1}(z)$ there exists a curve $\tilde{C}$ on $\tilde{R}$ with $\tilde{z}$ its initial point such that $\pi(\tilde{C}) = C$. Let z_0 be in R and $\tilde{z}_0$ in $\pi^{-1}(z_0)$. We can always find local parameters ζ and $\tilde{\zeta}$ about z_0 and $\tilde{z}_0$ respectively such that the local expression of the covering map $z = \pi(\tilde{z})$ takes the form $\zeta = \tilde{\zeta}^m$. Here the positive integer m does not depend on the choice of local parameters ζ and $\tilde{\zeta}$. If $m > 1$, then $\tilde{z}_0$ is referred to as a branch point of order $m-1$. A branch point is isolated in $\tilde{R}$ and hence the set of branch points in $\tilde{R}$ may be finite or countably infinite. For each z in R we let $\#(\pi^{-1}(z)) = \infty$ if the set $\pi^{-1}(z)$ is infinite and $\#(\pi^{-1}(z)) = n$ if the set $\pi^{-1}(z)$ consists of a finite n number of points where a branch point of order $m-1$ is counted as m points. When $(\tilde{R}, R, \pi)$ is unbounded, we have

$$\#(\pi^{-1}(z)) = \sup_{\zeta \in R} \#(\pi^{-1}(\zeta)) = n \in \boldsymbol{N} \cup \{\infty\}$$

for any z in R. If $n \in \boldsymbol{N}$, then we say that $(\tilde{R}, R, \pi)$ is *n-sheeted* or more roughly *finitely sheeted* without referring to the specific n. We stress that if we say $(\tilde{R}, R, \pi)$ is finitely sheeted, then we apriori assume its unboundedness. If, in particular, a Riemann surface R is represented as a finitely sheeted covering surface (R, Δ, π) of the unit disk Δ: $|z| < 1$ in the complex plane $\boldsymbol{C}$, then we say that R is a *finitely sheeted disk* or an *m-sheeted disk* specifying the sheet number m. We will denote by $\mathscr{C}(m)$ $(m \in \boldsymbol{N})$ the class of Riemann surfaces R which are represented as m-sheeted disks (R, Δ, π) so that the class of finitely sheeted disks is

$\bigcup_{m\in N}\mathscr{E}(m)$. The starting point of our discussion is the following:

THE COVERING CORONA THEOREM ([11]). *The corona theorem is valid for any finitely sheeted covering surface $\tilde{R}$ if and only if the corona theorem is valid for the base surface R.*

This can be used to enlarge the class of surfaces for which the corona theorem is valid. For example, since the corona theorem is valid for Δ by the fundamental result of Carleson [4], the same is true of any R in $\bigcup_{m\in N}\mathscr{E}(m)$. The case for $m=2$ of this is found in [9] which seems to be the first nontrivial example of Riemann surfaces of infinite genus for which the corona theorem is valid. We are interested in what happens when the simple corona theorem is replaced by the corona theorem with bounds in the above theorem. Suppose that $C(\tilde{R}; n, \delta) < \infty$. Any corona datum $\{f_j\}$ of index (n, δ) on R gives rise to a corona datum $\{\tilde{f}_j\}$ of index (n, δ) on $\tilde{R}$ determined by $\tilde{f}_j = f_j \circ \pi$ $(j=1, \cdots, n)$. For any number t greater than $C(\tilde{R}; n, \delta)$ there exists a corona solution $\{\tilde{g}_j\}$ of the datum $\{\tilde{f}_j\}$ such that $\sup_{\tilde{R}}|\tilde{g}_j| < t$ $(j=1, \cdots, n)$. Observe that the n-tuple $\{g_j\}$ of functions in $H^\infty(R)$ given by $g_j(z) = (1/m)\sum_{w\in\pi^{-1}(z)}\tilde{g}_j(w)$ is a corona solution of the datum $\{f_j\}$ with $\sup_R|g_j| < t$ $(j=1, \cdots, n)$ so that $C(R; n, \delta)$ is dominated by t: If $(\tilde{R}, R, \pi)$ is finite covering surface, then we have

$$C(R; n, \delta) \leqq C(\tilde{R}; n, \delta) \tag{2}$$

for any (n, δ) in $N\times(0, 1)$. This means that if the corona theorem with bounds is valid for a covering surface $\tilde{R}$, then the same is true for its base surface R. The essential problem is whether the converse of this is valid or not. We are able to answer only partly to this question as follows which is the main result of this paper:

THE MAIN THEOREM. *The corona theorem with bounds is valid for any finitely sheeted disk where the bounds are dominated by a universal constant depending only on the sheet number of the disk.*

In other words, if the base surface is the unit disk Δ for which what the fundamental work of Carleson [4] states is not the mere corona theorem but the corona theorem with bounds (see also an extremely simple proof by Gamelin [8]), then the above question is in the affirmative. It is convenient to consider the quantity $c(m; n, \delta)$ given by

$$c(m; n, \delta) = \sup_{R\in\mathscr{E}(m)} C(R; n, \delta) \tag{3}$$

for $(m; n, \delta)\in N\times N\times(0, 1)$. Then our main theorem is equivalent to the assertion that $c(m; n, \delta) < \infty$. Hence the bound $C(R; n, \delta)$ for R in $\mathscr{E}(m)$

is dominated by the constant $c(m; n, \delta)$ which depends only on the sheet number m of R once (n, δ) is fixed. The proof of $c(m; n, \delta) < \infty$ will be given in no. 13 of Section 4. We will also clarify the dependence of $c(m; n, \delta)$ on m in no. 14 in Section 4 as follows:

$$\begin{cases} c(m; n, \delta) \leqq c(m+1; n, \delta) < \infty \quad (m \in \boldsymbol{N}) , \\ \lim_{m\to\infty} c(m; n, \delta) = \infty . \end{cases} \tag{4}$$

Many Riemann surfaces of infinite genus belong to the class $\mathscr{C}(m)$ but of course not every Riemann surface belongs to $\mathscr{C}(m)$. An important point with the class $\mathscr{C}(m)$ is that every finite Riemann surface belongs to $\mathscr{C}(m)$. Here a Riemann surface R is said to be *finite* if R is a subsurface of a closed Riemann surface so that it is of finite genus and the relative boundary ∂R of R consists of a finite number of disjoint nondegenerate continua. For a finite Riemann surface R we denote by $g = g(R)$ and $c = c(R)$ the *genus* of R and the *number of components* of ∂R, respectively. The *Euler characteristic* $\chi = \chi(R)$ of R is then given by $\chi(R) = -(2g(R) + c(R) - 2)$. We denote by $\mathscr{F}(g, c)$ $((g, c) \in \boldsymbol{Z}^+ \times \boldsymbol{N}, \boldsymbol{Z}^+ = \boldsymbol{N} \cup \{0\})$ the class of finite Riemann surface R such that $g(R) = g$ and $c(R) = c$. By the Ahlfors theorem [1] we have

$$\mathscr{F}(g, c) \subset \bigcup_{c \leqq m \leqq 2g+c} \mathscr{C}(m) \tag{5}$$

for every (g, c) in $\boldsymbol{Z}^+ \times \boldsymbol{N}$ so that a finite Riemann surface R of the Euler characteristic $\chi = \chi(R)$ is an at most $(2 - \chi)$-sheeted disk. Similar to (3) we define

$$f(g, c; n, \delta) = \sup_{R \in \mathscr{F}(g,c)} C(R; n, \delta) \tag{6}$$

By using (4) and (5) with $\chi = -(2g + c - 2)$ we have

$$f(g, c; n, \delta) \leqq c(2 - \chi; n, \delta) . \tag{7}$$

The above observation may be restated in the following

COROLLARY TO THE MAIN THEOREM. *The corona theorem with bounds is valid for any finite Riemann surface where the bounds are dominated by a universal constant depending only on the Euler characteritic of the surface.*

More precisely the bound $C(R; n, \delta)$ for a finite Riemann surface R is dominated by the constant $c(2 - \chi(R); n, \delta)$ depending only on the Euler characteristic $\chi(R)$ of R once (n, δ) is fixed. It is difficult to determine to whom the validity of the simple corona theorem or the corona theorem with bounds for finite Riemann surfaces owes. Instead we cite here some of papers related to the question: Stout [13], Forelli [5], Gamelin [6], and [10], among others. What seems to be new here with our corollary is the part that the bounds depend only on the Euler characteristic which generalizes a result in the above cited paper of Gamelin that asserts the sole dependance of the bounds for finitely connected plane regions on their connectivity. Similar to (4) we also study in nos. 9–12 in Section 3 the dependance of $f(g, c; n, \delta)$ on (g, c) and establish the following:

$$\begin{cases} f(g, c; n, \delta) \leqq f(g', c'; n, \delta) \quad (g \leqq g', c \leqq c')\,, \\ \lim_{g\to\infty} f(g, c; n, \delta) = \infty\,. \end{cases} \tag{8}$$

Whether $\lim_{c\to\infty} f(g, c; n, \delta)$ is finite or not is an important open question and nothing is known even for the case $g = 0$ except for that $\lim_{c\to\infty} f(0, c; n, \delta) < \infty$ is equivalent to the validity of the corona theorem for every plane regions (cf. [6]).

The paper is divided into four sections. In Section 1 consisting of nos. 1–4 we will study elementary properties concerning Gamelin constants. A Carleman type approximation method will be one of fundamental tools in this paper. For the sake of convenience and also completeness we derive it from the Mergelyan-Bishop theorem in Section 2 consisting of nos. 5–6. In Section 3 consisting of nos. 9–12 we will discuss the Gamelin constants of finite Riemann surfaces. Proofs of our main assertions in this paper will be given in nos. 13 and 14 of Section 4.

§ 1. Elementary properties of Gamelin constants.

1. In the definition of the Gamelin constant $C(R; n, \delta)$ we made the restriction that δ is less than 1 in addition to its positivity. It is not essential but this convention will make a certain statement concerning $C(R; n, \delta)$ neater. It is clear from the definition that $C(R; 1, \delta) = 1/\delta$, and hence $C(R; n, \delta)$ has its essential meaning only for $n \geqq 2$. Suppose $C(R; n, \delta) < \infty$. By a normal family argument we can show the existence of a corona solution $\{g_j\}$ of a given corona datum $\{f_j\}$ of index (n, δ) such that $\max_{1\leqq j\leqq n}(\sup_R |g_j|) = \inf_{\{h_j\}}(\max_{1\leqq j\leqq n}(\sup_R |h_j|))$ where $\{h_j\}$ runs over corona solutions of $\{f_j\}$. Such a $\{g_j\}$ will be referred to as a *minimal* solution of $\{f_j\}$. The infimum and the minimum of the set $\{C\}$ of real numbers C such that $\max_{1\leqq j\leqq n}(\sup_R |g_j|) \leqq C$ for all minimal solutions $\{g_j\}$ are coincident. By the above observation $C(R; n, \delta)$ can also be characterized as the smallest constant for which given corona datum $\{f_j\}$ of index (n, δ) there exists a corona solution $\{g_j\}$ of $\{f_j\}$ with $\max_{1\leqq j\leqq n}(\sup_R |g_j|) \leqq C(R; n, \delta)$. Here we understand that $C(R; n, \delta) = \infty$

if there are no corona solutions. We have

$$(9) \qquad 1/\delta \leqq C(R; n, \delta) \leqq C(R; n', \delta') \quad (n \leqq n', \delta \geqq \delta') .$$

In fact there is a corona datum $\{f_j\}$ of index (n, δ) such that $\inf_R \sum_{j=1}^n |f_j| = \delta$ otherwise we only have to replace $\{f_j\}$ by $\{\alpha f_j\}$ with $\alpha = \delta/(\inf_R \sum_{j=1}^n |f_j|)$. Taking a minimal solution $\{g_j\}$ of $\{f_j\}$ we deduce $1 = |\sum_{j=1}^n f_j g_j| \leqq \sum_{j=1}^n |f_j||g_j| \leqq C(R; n, \delta) \sum_{j=1}^n |f_j|$. Taking the infimum of both sides we obtain $1 \leqq C(R; n, \delta)\delta$. A datum $\{f_j\}$ of length n can be viewed as one of length n' $(>n)$ by adding $n'-n$ constantly zero functions. A datum of lower bound δ is of course of lower bound δ' $(<\delta)$. Hence (9) is valid.

The dependance of $C(R; n, \delta)$ on R is very complicated. A closed set K in R is said to be *H^∞-removable* if $H^\infty(R-K) = H^\infty(R)$ i.e. any f in $H^\infty(R-K)$ can be continued to R so as to be in $H^\infty(R)$. In this case $C(R-K; n, \delta) = C(R; n, \delta)$. Finite sets are trivially H^∞-removable. Needless to say the Gamelin constant is conformally invariant, i.e. $C(\varphi(R); n, \delta) = C(R; n, \delta)$ if φ is a conformal mapping of R onto $\varphi(R)$.

2. For any open Riemann surface R there always exists an *exhaustion* $\{R_k\}_{k\in N}$ of R characterized by the following: each R_k is a relatively compact subregion of R; $\bar{R}_k \subset R_{k+1}$ $(k \in \boldsymbol{N})$; $\cup_{k\in N} R_k = R$. We can moreover assume that each R_k is a regular subregion and in this case $\{R_k\}$ is said to be a *regular exhaustion*. We denoted by $\mathscr{C}(m)$ the family of m-sheeted disks R, i.e. open Riemann surfaces R which are represented as m-sheeted covering surface (R, Δ, π) of the unit disk Δ: $|z| < 1$ in the complex plane $\boldsymbol{C}$. We also consider a subfamily $\mathscr{C}_0(m)$ of $\mathscr{C}(m)$ consisting of finite Riemann surfaces R with analytic borders such that R are represented as m-sheeted covering surfaces (R, Δ, π) of Δ whose sets of branch points are finite.

If $R \in \mathscr{C}(m)$, then there always exists a *$\mathscr{C}_0(m)$-exhaustion* $\{R_k\}_{k\in N}$ of R characterized by the following: $\{R_k\}_{k\in N}$ is a regular exhaustion of R; each R_k belongs to $\mathscr{C}_0(m)$. To show the existence of a $\mathscr{C}_0(m)$-exhaustion $\{R_k\}$ of R in $\mathscr{C}(m)$, let (R, Δ, π) be the representation of R as an m-sheeted covering surface of Δ and $\{r_k\}_{k\in N}$ be a strictly increasing sequence in $(0, 1)$ such that there exist no branch points in $\pi^{-1}(\cup_{k\in N}\{|z| = r_k\})$. Then $R_k = \pi^{-1}(\{|z| < r_k\})$ $(k \in \boldsymbol{N})$ give a required exhaustion since each R_k is represented as an m-sheeted covering surface $(R_k, \Delta, r_k^{-1}\pi)$ of Δ with the finite set of branch points.

3. The Gamelin constant $C(R; n, \delta)$ is *lower semicontinuous* in the following sense: for any t less than $C(R; n, \delta)$ and for any exhaustion $\{R_k\}_{k\in N}$ of R there exists an $N \in \boldsymbol{N}$ such that $C(R_k; n, \delta) > t$ $(k \geqq N)$. In

connection with this it is convenient to introduce the following notion. A corona datum $\{f_j\}$ of index (n, δ) on R is said to be *t-effective* $(t < C(R; n, \delta))$ on R if $\max_{1\leq j\leq n}(\sup_R|g_j|) > t$ where $\{g_j\}$ is a minimal solution of $\{f_j\}$. We denote by $E(R; n, \delta; t)$ the class of t-effective corona data $\{f_j\}$ of index (n, δ) on R. The t-effectiveness is *monotone* in the following sense: $\{f_j\}$ is a corona datum of index (n, δ) on R and R' is a subregion of R, then the t-effectiveness of $\{f_j\}$ on R' implies that on R. The above lower semicontinuity of $C(R; n, \delta)$ in R follows from the *lower semicontinuity of the t-effectiveness in surfaces* R in the following sense: for any exhaustion $\{R_k\}_{k\in N}$ of R there exists an $N\in \boldsymbol{N}$ such that $\{f_j\}\in E(R_k; n, \delta; t)$ $(k\geqq N)$. We prove this by contradiction. If $\{f_j\}\notin E(R_k; n, \delta; t)$ for every $k\in \boldsymbol{N}$, then the minimal solution $\{g_{kj}\}$ of $\{f_j\}$ on R_k must satisfy $\max_{1\leq j\leq n}(\sup_{R_k}|g_{kj}|) \leqq t$. Since $\{g_{kj}\}_k$ $(j = 1, \cdots, n)$ forms a normal family we can choose a subsequence $\{\nu(k)\}$ of $\{k\}$ such that $\{g_{\nu(k)j}\}_k$ converges to a $g_j\in H^\infty(R)$ $(j = 1, \cdots, n)$. Then $\{g_j\}$ is a corona solution of $\{f_j\}$ on R and $\max_{1\leq i\leq n}(\sup_R|g_j|) \leqq t$, contradicting $\{f_j\}\in E(R; n, \delta; t)$. □

We have another kind of lower semicontinuity of the t-effectiveness: *the t-effectiveness is lower semicontinuous in data* $\{f_j\}$. Namely, if $\{f_j\}\in E(R; n, \delta; t)$ and $\{\{f_{kj}\}_{j=1}^n\}_{k\in N}$ is a sequence of corona data $\{f_{kj}\}_{j=1}^n$ of index (n, δ) on R such that $\{f_{kj}\}_{k\in N}$ converges to f_j uniformly on each compact subset of R $(j = 1, \cdots, n)$, then there exists an $N\in \boldsymbol{N}$ such that $\{f_{kj}\}_j\in E(R; n, \delta; t)$ $(k \geqq N)$. The proof is again straightforward as above by using the normal family argument.

4. We now state a technical fact about $C(R; n, \delta)$ which will be in an essential use later. It is derived from the so to speak *joint lower semicontinuity* of the t-effectiveness in both of surfaces and data: for any t less than $C(R; n, \delta)$ and any regular exhaustion $\{R_k\}_{k\in N}$ of R there exists an R_k and a system $\{f_1, \cdots, f_n\}$ of functions f_j holomorphic on $\bar{R}_k$ with the following three properties:

$$\max_{1\leq j\leq n}(\sup_{R_k}|f_j|) < 1\ , \tag{10}$$

$$\inf_{R_k}\left(\sum_{j=1}^n |f_j|\right) > \delta\ , \tag{11}$$

$$\{f_j\}\in E(R_k; n, \delta; t)\ . \tag{12}$$

Since $C(R; n, \delta) > t$ we can find a corona datum $\{f_j\}$ of index (n, δ) on R such that $\{f_j\}$ is t-effective, i.e. $\{f_j\}\in E(R; n, \delta; t)$. By the lower semicontinuity of the t-effectiveness in R, $\{f_j\}\in E(R_k; n, \delta; t)$ for some R_k. Needless to say f_j $(j = 1, \cdots, n)$ are holomorphic on $\bar{R}_k$. Suppose $f_j \equiv c_j$ on $\bar{R}_k$ $(j = 1, \cdots, p)$ with c_j a constant of modulus 1 and $|f_j| < 1$ on R_k

$(j = p+1, \cdots, n)$. Then $\{\eta f_1, \cdots, \eta f_p, f_{p+1}, \cdots, f_n\}$ is again a corona datum of index (n, δ) on R_k if the constant η is taken enough close to 1 in $(0, 1)$. Here the assumption $\delta \in (0, 1)$ is essential for the validity of the above assertion. Moreover $\{\eta f_1, \cdots, \eta f_p, f_{p+1}, \cdots, f_n\}$ is t-effective on R_k if η is chosen further close to 1 in $(0, 1)$ by the lower semicontinuity of t-effectiveness in data. Thus we have chosen an R_k and a system $\{f_1, \cdots, f_n\}$ of functions f_j holomorphic on $\bar{R}_k$ with $\max_{1 \leqq j \leqq n}(\sup_{R_k}|f_j|) < 1$ and $\{f_j\} \in E(R_k; n, \delta; t)$. Finally by taking a constant $\eta > 1$ enough close to 1 we see that $\{\eta f_1, \cdots, \eta f_n\}$ is the required one on R_k. $\square$

§ 2. An approximation lemma.

5. The discussion in this section will be developed based upon the following Bishop generalization [3] of the Mergelyan approximation theorem to the Riemann surface setting: Let K be a compact subset of an open Riemann surface R such that there exist no relatively compact components of $R - K$ and f a continuous function on K holomorphic in the interior of K. Then there exists a holomorphic function f_ε on R for an arbitrary positive number ε such that $\sup_K |f_\varepsilon - f| < \varepsilon$.

6. Consider a sequence $\{R_m\}_{m \in N}$ of finite regular Riemann surfaces R_m. We construct an ambient Riemann surface S containing $\cup_{m \in N} R_m$ as follows. First let S_m be a finite regular surface obtained from R_m by attaching an annulus to each boundary component of $\bar{R}_m$. Then S is obtained by connecting, for each $m \geqq 1$, at most k_m of the boundary components of S_m to at most k_m of those of S_{m+1} by means of k_m rectangular strips $s_{m\nu}$ $(1 \leqq \nu \leqq k_m)$. More precisely let $s_{m\nu}$ be represented as $\{a_{m\nu} \leqq \operatorname{Re} z \leqq b_{m\nu}, 0 < \operatorname{Im} z < 1\}$, and we denote by $\alpha_{m\nu}$ ($\beta_{m\nu}$, resp.) the part of $s_{m\nu}$ corresponding to $\{\operatorname{Re} z = a_{m\nu}, 0 < \operatorname{Im} z < 1\}$ ($\{\operatorname{Re} z = b_{m\nu}, 0 < \operatorname{Im} z < 1\}$, resp.). What we mean by connecting S_m to S_{m+1} by means of $s_{m\nu}$ $(1 \leqq \nu \leqq k_m)$ is that parts of ∂S_m (∂S_{m+1}, resp.) are identified with $\alpha_{m\nu}$ ($\beta_{m\nu}$, resp.) $(1 \leqq \nu \leqq k_m)$ so that

$$S = \bigcup_{m \in N}\left(S_m \cup \left(\bigcup_{\nu=1}^{k_m} s_{m\nu}\right)\right).$$

Here we assume that $\bar{S}_m \cap \bar{S}_{m'} = \varnothing$ $(m \neq m')$ and $\bar{s}_{m\nu} \cap \bar{s}_{m'\nu'} = \varnothing$ $((m, \nu) \neq (m', \nu'))$. We set $s_m = \cup_{\nu=1}^{k_m} s_{m\nu}$. Let $t_{m\nu}$ be a simple analytic arc that starts from ∂R_m, passes directly through the rectangle $s_{m\nu}$ without touching $R_m \cup R_{m+1}$, and terminates at ∂R_{m+1}. We assume that $t_{m\nu}$ intersects with ∂R_m, ∂S_m, ∂R_{m+1}, ∂S_{m+1} only once each. We also assume that $t_{m\nu} \cap t_{m'\nu'} = \varnothing$ $((m, \nu) \neq (m', \nu'))$. We set $T_m = \cup_{\nu=1}^{k_m} t_{m\nu}$. It is also assumed that $T_m \cap (\cup_{\nu \in N} R_\nu) = \varnothing$. Set

$$E = (\bigcup_{m \in N} \bar{R}_m) \cup (\bigcup_{m \in N} T_m)$$

which is a connected closed subset of S. We also consider

$$E_n = \left(\bigcup_{m=n}^{\infty} \bar{R}_m\right) \cup \left(\bigcup_{m=n}^{\infty} T_m\right) \quad (n = 1, 2, \cdots)$$

which are also connected closed subsets of S with $E_1 = E$. Then we have the following (cf. e.g. Scheinberg [12])

APPROXIMATION LEMMA. *For any continuous function f on E holomorphic in the interior $\bigcup_{m \in N} R_m$ of E and any sequence $\{\varepsilon_n\}$ of positive numbers ε_n there exists a holomorphic function F on S such that* $\sup_{E_n} |f - F| < \varepsilon_n$ $(n = 1, 2, \cdots)$.

7. To prove the above lemma we may assume that $\{\varepsilon_n\}$ is decreasing. We put $X_n = (\bigcup_{m=1}^{n} S_m) \cup (\bigcup_{m=1}^{n-1} s_m)$ which is a subsurface of S and $\{X_n\}$ exhausts S. We also put

$$Y_n = \left(\bigcup_{m=1}^{n} \bar{R}_m\right) \cup \left(\bigcup_{m=1}^{n-1} T_m\right) \cup (\bar{S}_n \cap T_n) .$$

First let $f_1 = f | Y_1$ and X_1' be a slightly larger surface than X_1 such that X_1 is normal in X_1', i.e. $\bar{X}_1 \subset X_1'$ and $X_1' - X_1$ has no compact components. Then by the Bishop theorem there exists an F_1 holomorphic on X_1' and hence on $\bar{X}_1$ such that $\sup_{Y_1} |f_1 - F_1| < \varepsilon_1/4$. We then define a continuous function g_1 on T_1 such that $g_1 = F_1$ on $T_1 \cap \bar{X}_1$, $g_1 = f$ at $T_1 \cap \partial R_2$, and $\sup_{T_1} |g_1 - f| < \varepsilon_1/4$. Using g_1 we define a continuous function f_2 on $X_1 \cup Y_2$ such that $f_2 = F_1$ on $\bar{X}_1$, $f_2 = g_1$ in T_1, and $f_2 = f$ on $\bar{R}_2 \cup (S_2 \cap T_2)$. Let X_2' be a slightly larger surface than X_2 such that X_2 is normal in X_2'. By the Bishop theorem there exists a function F_2 holomorphic on X_2' and hence on $\bar{X}_2$ such that

$$\sup_{X_1 \cup Y_2} |f_2 - F_2| < \varepsilon_2/4^2 .$$

By repeating the same process we construct sequences $\{f_n\}_{n \in N}$, $\{g_n\}_{n \in N}$, and $\{F_n\}_{n \in N}$ of functions as follows. Suppose $f_1, \cdots, f_n$ and $F_1, \cdots, F_n$ have been constructed. Define g_n on T_n such that $g_n = F_n$ on $T_n \cap \bar{X}_n$, $g_n = f$ at $T_n \cap \partial R_{n+1}$, and $\sup_{T_n} |g_n - f| < \varepsilon_n/4^n$. Using g_n we define a continuous function f_{n+1} on $X_n \cup Y_{n+1}$ such that $f_{n+1} = F_n$ on $\bar{X}_n$, $f_{n+1} = g_n$ on T_n, and $f_{n+1} = f$ on $\bar{R}_{n+1} \cup (\bar{S}_{n+1} \cap T_{n+1})$. Then define a function F_{n+1} holomorphic on $\bar{X}_{n+1}$ as before such that

$$\sup_{X_n \cup Y_{n+1}} |f_{n+1} - F_{n+1}| < \varepsilon_{n+1}/4^{n+1} .$$

By the construction $\{F_n\}$ converges to a holomorphic function F on S and $\sup_{E_n} |f - F| \leqq \varepsilon_n/3 \cdot 4^{n-1} \leqq \varepsilon_n$. $\square$

8. We insert here a simple remark which will be used later. Let $A_\nu = (\alpha_{\nu 1}, \cdots, \alpha_{\nu n})$ $(\nu = 1, 2)$ be two points in C^n satisfying $\max_{1 \leqq j \leqq n} |\alpha_{\nu j}| \leqq a$ and $\sum_{j=1}^n |\alpha_{\nu j}| \geqq b$ $(\nu = 1, 2)$. We maintain that there exists a continuous curve $Z = Z(t) = (z_1(t), \cdots, z_n(t))$ $(0 \leqq t \leqq 1)$ in C^n such that $Z(0) = A_1$, $Z(1) = A_2$, $\max_{1 \leqq j \leqq n}(\sup_{0 \leqq t \leqq 1} |z_j(t)|) \leqq a$, and $\inf_{0 \leqq t \leqq 1}(\sum_{j=1}^n |z_j(t)|) \geqq b$. Actually we can construct such a curve explicitly as follows. Let $\arg \alpha_{\nu j} = \theta_{\nu j}$ $(\nu = 1, 2; j = 1, \cdots, n)$ under the convention $\arg 0 = 0$ and

$$z_j(t) = \begin{cases} (1 - 3t)\alpha_{1j} + 3at \exp(i\theta_{1j}) & (0 \leqq t \leqq 1/3)\ , \\ a \exp(i((2 - 3t)\theta_{1j} + (3t - 1)\theta_{2j})) & (1/3 \leqq t \leqq 2/3)\ , \\ (3t - 2)\alpha_{2j} + 3a(1 - t)\exp(i\theta_{2j}) & (2/3 \leqq t \leqq 1) \end{cases}$$

for $j = 1, \cdots, n$. Then it is easily checked that $Z = Z(t) = (z_1(t), \cdots, z_n(t))$ is the required curve. □

§ 3. Finite Riemann surfaces.

9. We now give a proof for (8). We denoted by $\mathscr{F}(g, c)$ the class of finite Riemann surfaces R of genus $g \geqq 0$ and of $c(\geqq 1)$ boundary components, and by $f(g, c; n, \delta)$ the supremum of $C(R; n, \delta)$ as R runs over $\mathscr{F}(g, c)$. We will prove that $f(g, c; n, \delta) \leqq f(g', c'; n, \delta)$ $(g \leqq g', c \leqq c')$. We first prove that $f(g, c; n, \delta) \leqq f(g, c-1; n, \delta)$. Take an arbitrary positive number t less than $f(g, c; n, \delta)$ so that there is an R in $\mathscr{F}(g, c)$ with $C(R; n, \delta) > t$. Fix an arbitrary point p in R and set $R' = R - \{p\}$. Observe that $C(R'; n, \delta) = C(R; n, \delta) > t$. Let $\{R_k\}_{k \in N}$ be a regular exhaustion of R' where we can assume that $R_k \in \mathscr{F}(g, c + 1)$ $(k \in N)$. By the lower semicontinuity of $C(R; n, \delta)$ (cf. no. 3) we can find an R_k such that $C(R_k; n, \delta) > t$. Hence $f(g, c + 1; n, \delta) > t$, proving $f(g, c; n, \delta) \leqq f(g, c + 1; n, \delta)$.

10. We next prove that $f(g, c; n, \delta) \leqq f(g + 1, c; n, \delta)$. Let t be an arbitrary positive number less than $f(g, c; n, \delta)$ so that there is an R' in $\mathscr{F}(g, c)$ with $C(R'; n, \delta) > t$. By no. 4, we can find a regular subregion R of R' such that $R \in \mathscr{F}(g, c)$ and a system $\{f_1, \cdots, f_n\}$ of functions f_j holomorphic on $\bar{R}$ such that (10), (11), and (12) are satisfied. Let S' be a finite regular surface obtained from R by attaching an annulus to each boundary component of $\bar{R}$. Let B' be obtained from a torus by removing a closed parametric disk and B by removing slightly larger closed concentric parametric disk so that $B' - B$ is an annulus. Finally let S be obtained by connecting one arbitrary fixed boundary component of S' to the boundary component of B' by means of a rectangular strip s as in no. 6. Let γ be a simple analytic arc that starts from an arbitrary fixed point a of ∂R, passes directly through the rectangle s without touching

$R \cup B$, and terminates at ∂B. We define functions $\varphi_j \in C(\bar{R} \cup \gamma \cup \bar{B})$ by $\varphi_j | \bar{R} = f_j$ and $\varphi_j | \gamma \cup \bar{B} = \varphi_j(a)$ $(j = 1, \cdots, n)$. By the Bishop theorem there exist holomorphic functions F_j on S approximating f_j on $\bar{R} \cup \gamma \cup \bar{B}$ $(j = 1, \cdots, n)$ such that (10), (11), and (12) are satisfied for $\{F_j\}$ on R. Take a sufficiently small neighborhood W of $\bar{R} \cup \gamma \cup \bar{B}$ such that $W \in \mathscr{F}(g + 1, c)$ and $\{F_j\} \in E(W; n, \delta; t)$. Hence $C(W; n, \delta) > t$ and then $f(g + 1, c; n, \delta) > t$, proving $f(g, c; n, \delta) \leqq f(g + 1, c; n, \delta)$. We have thus proved the inequality in (8).

11. We will now show that there exists a finite regular surface $R = R(\delta, \eta, c)$ such that $C(R; 2, \delta) > \eta$ and the number $c(R)$ of boundary components of R is c for any (δ, η, c) in $(0, 1) \times (0, \infty) \times \boldsymbol{N}$. We have no explicit information about the genus $g(R)$ of R but possibly very large for large η.

We only have to show the existence of $R(\delta, \eta, 1)$. In fact, if $c > 1$, then remove $c - 1$ points $p_1, \cdots, p_{c-1}$ from $R(\delta, \eta, 1)$ and let W be the resulting surface. Observe that $C(W; 2, \delta) = C(R(\delta, \eta, 1); 2, \delta) > \eta$. Take a regular exhaustion $\{W_k\}$ of W where we can assume $c(W_k) = c$. By the lower semicontinuity of $C(R; 2, \delta)$ in R there exists an W_k with $C(W_k; 2, \delta) > \eta$. We can then take an $R(\delta, \eta, c)$ as W_k.

To construct an $R(\delta, \eta, 1)$ we take a finite regular surface $W = W(\delta, \eta)$ such that $C(W; 2, \delta) > \eta$ (Gamelin [7, pp. 47-49]). If $c(W) = 1$, then W is the required one. Suppose that $c(W) > 1$. As in no. 10 by replacing W with its regular subregion if necessary, we can assume the existence of two holomorphic functions f_1 and f_2 on $\bar{W}$ such that $\max_{1 \leqq j \leqq 2}(\sup_W |f_j|) < 1$, $\inf_W(|f_1| + |f_2|) > \delta$, and $\{f_1, f_2\}$ is η-effective on W. First let S' be a finite regular surface obtained from W by attaching an annulus to each boundary component of $\bar{W}$. Then we construct an S by connecting two boundary component of S' by means of a rectangular strip s. Let γ be a simple analytic arc that start from a point in ∂W, passes directly through the rectangle s without touching S', and terminates at a point in ∂W. We define two functions $\varphi_j \in C(\bar{W} \cup \gamma)$ $(j = 1, 2)$ such that $\varphi_j = f_j$ on $\bar{W}$, $\max_{1 \leqq j \leqq 2}(\sup_{\bar{W} \cup \gamma} |\varphi_j|) < 1$, and $\inf_\gamma(|\varphi_1| + |\varphi_2|) > \delta$. Here we have used the result in no. 8. By the Bishop theorem there exist holomorphic functions F_1 and F_2 on S approximating φ_1 and φ_2 on $\bar{W} \cup \gamma$ respectively such that $\max_{1 \leqq j \leqq 2}(\sup_{\bar{W} \cup \gamma} |F_j|) < 1$, $\inf_{\bar{W} \cup \gamma}(|F_1| + |F_2|) > \delta$, and $\{F_1, F_2\}$ is η-effective on W. Take a sufficiently small regular neighborhood W_1 of $\bar{W} \cup \gamma$ such that $\{F_1, F_2\} \subset E(W_1; 2, \delta; \eta)$. Then $c(W_1) = c(W) - 1$ and $C(W_1; 2, \delta) > \eta$. Repeating this process we can reduce $c(W)$ to 1. $\square$

12. We complete the proof for (8) by showing $\lim_{g \to \infty} f(g, c; n, \delta) = \infty$ for any $(c, n, \delta) \in \boldsymbol{N} \times \boldsymbol{N} \times (0, 1)$. Let $R = R(\delta, k, c)$ be the surface formed

for $(\delta, k, c) \in (0, 1) \times \boldsymbol{N} \times \boldsymbol{N}$ in no. 11. By (9) we have $C(R; n, \delta) \geqq C(R; 2, \delta) \geqq k$ and $R \in \mathscr{F}(g(\delta, k, c), c)$. Therefore $f(g(\delta, k, c), c; n, \delta) \geqq k$. Since $f(g, c; n, \delta)$ is increasing in g, we have $\lim_{g\to\infty} f(g, c; n, \delta) \geqq k$ which implies $\lim_{g\to\infty} f(g, c; n, \delta) = \infty$ by making $k \to \infty$.

§ 4. Proof of the main theorem.

13. We are now giving a proof of our main result in this paper. Our main theorem is equivalent to the finiteness of the quantity $c(m; n, \delta)$ for any $(m, n, \delta) \in \boldsymbol{N} \times \boldsymbol{N} \times (0, 1)$ which is the supremum of $C(R; n, \delta)$ when R runs over $\mathscr{C}(m)$. Contrariwise we suppose $c(m; n, \delta) = \infty$ for some $(m, n, \delta) \in \boldsymbol{N} \times \boldsymbol{N} \times (0, 1)$ so that there exists a sequence $\{R_k\}_{k \in \boldsymbol{N}}$ in $\mathscr{C}(m)$ such that $C(R_k; n, \delta) > k$ $(k \in \boldsymbol{N})$. Taking a $\mathscr{C}_0(m)$-exhaustion of R_k (cf. no. 2) and using the result in no. 4 we can assume for each $k \in \boldsymbol{N}$ that $R_k \in \mathscr{C}_0(m)$ and there exists a system $\{f_{k1}, \cdots, f_{kn}\}$ of functions f_{kj} holomorphic on $\bar{R}_k = R_k \cup \partial R_k$ with the following three properties:

$$\max_{1 \leqq j \leqq n}(\sup_{R_k} |f_{kj}|) = 1 - l_k \quad (k \in \boldsymbol{N}) , \tag{13}$$

$$\inf_{R_k}\Bigl(\sum_{j=1}^{n} |f_{kj}|\Bigr) = \delta + d_k \quad (k \in \boldsymbol{N}) , \tag{14}$$

$$\{f_{k1}, \cdots, f_{kn}\} \in E(R_k; n, \delta; k) \quad (k \in \boldsymbol{N}) , \tag{15}$$

where $\{l_k\}_{k \in \boldsymbol{N}}$ is a sequence in $(0, 1)$ and $\{d_k\}_{k \in \boldsymbol{N}}$ is a sequence of positive numbers. Since $R_k \in \mathscr{C}_0(m)$, R_k has a representation (R_k, Δ, π_k) as an m-sheeted covering surface of Δ with a finite number of branch points.

We denote by U_k the disk $\{|z - (3k - 2)| < 1\}$ and by φ_k the conformal mapping $z \mapsto z + (3k - 2)$ of Δ onto U_k $(k \in \boldsymbol{N})$. We consider the m-sheeted covering surface $(C_m, \boldsymbol{C}, \pi)$ of $\boldsymbol{C}$ such that the set of branch points is contained in $\pi^{-1}(\cup_{k \in \boldsymbol{N}} U_k)$ and each $\pi^{-1}(U_k)$ can be identified with R_k such that $\pi = \varphi_k \circ \pi_k$ $(k \in \boldsymbol{N})$. We denote by τ_k the line segment $[3k - 1, 3k]$ on the real axis in $\boldsymbol{C}$ $(k \in \boldsymbol{N})$. We choose sequences $\{\rho_k\}_{k \in \boldsymbol{N}}$ in $(0, 1/2)$ and $\{\sigma_k\}_{k \in \boldsymbol{N}}$ in $(0, 1)$. We denote by V_k the disk with its center $3k - 2$ and of radius $1 + \rho_k$ so that $\bar{U}_k \subset V_k$ $(k \in \boldsymbol{N})$. We set

$$W_k = \{|\mathrm{Im}\, z| < \sigma_k, 3k - 1 < |\mathrm{Re}\, z| < 3k\} \cap (\boldsymbol{C} - V_k \cup V_{k+1})$$

for each $k \in \boldsymbol{N}$ and then consider $D = (\cup_{k \in \boldsymbol{N}} V_k) \cup (\cup_{k \in \boldsymbol{N}} W_k)$ which is simply connected so that there exists a conformal mapping φ of D onto Δ. The subsequence $D_m = \pi^{-1}(D)$ then belongs to $\mathscr{C}(m)$ since it has a representation $(D_m, \Delta, \varphi \circ \pi)$ as an m-sheeted covering surface of Δ. The set $T_k = \pi^{-1}(\tau_k)$ consists of m line segments $t_{k1}, \cdots, t_{km}$ each of which starts from a point in ∂R_k and terminates at a point in ∂R_{k+1} $(k \in \boldsymbol{N})$. We set

$$E = \left(\bigcup_{k=1}^{\infty} \bar{R}_k\right) \cup \left(\bigcup_{k=1}^{\infty} T_k\right)$$

which is a connected closed subset of C_m. We also set

$$E_\nu = \left(\bigcup_{k=\nu}^{\infty} \bar{R}_k\right) \cup \left(\bigcup_{k=\nu}^{\infty} T_k\right) \quad (\nu \in \boldsymbol{N})$$

which are also connected closed subsets of C_m with $E_1 = E$. Observe that D_m is a surface containing E as its closed subset and moreover we can make D_m so small by choosing $\{\rho_k\}_{k\in N}$ and $\{\sigma_k\}_{k\in N}$ convergent to zero enough rapidly that D_m is contained in any preassigned neighborhood of E in C_m.

By the technical remark stated in no. 8 with (13) and (14) we can find continuous functions $F_1, \cdots, F_n$ on E such that $F_j|\bar{R}_k = f_{kj}$ ($k \in \boldsymbol{N}$, $j = 1, \cdots, n$) and that the following two properties are satisfied:

$$\max_{1\leqq j\leqq n}\Big(\sup_{\bar{R}_k \cup T_k} |F_j|\Big) \leqq 1 - \min(l_k, l_{k+1}) \quad (k \in \boldsymbol{N}) \ , \tag{16}$$

$$\inf_{\bar{R}_k \cup T_k}\left(\sum_{j=1}^{n} |F_j|\right) \geqq \delta + \min(d_k, d_{k+1}) \quad (k \in \boldsymbol{N}) \ . \tag{17}$$

We apply the approximation lemma in no. 6 to functions $F_1, \cdots, F_n$ and the closed set E on the surface C_m. Then, in view of (15), (16), and (17), we can find holomorphic functions $f_1, \cdots, f_n$ on C_m with the following three properties:

$$\max_{1\leqq j\leqq n}\Big(\sup_{E} |f_j|\Big) \leqq 1 \ , \tag{18}$$

$$\inf_{E}\left(\sum_{j=1}^{n} |f_j|\right) \geqq \delta \ , \tag{19}$$

$$\{f_1, \cdots, f_n\} \in E(R_k; n, \delta; k) \quad (k \in \boldsymbol{N}) \ . \tag{20}$$

Here we have also used the lower semicontinuity of the k-effectiveness in data (cf. no. 3). By choosing D_m so small that (18) and (19) are also valid if E is replaced by $\bar{D}_m$ we can assume that $\{f_1, \cdots, f_n\}$ is a corona datum with index (n, δ) on D_m. Since $D_m \in \mathscr{C}(m)$, the covering corona theorem assures that there exists a corona solution $\{g_1, \cdots, g_n\}$ to the datum $\{f_1, \cdots, f_n\}$ on D_m. Therefore

$$k \leqq \max_{1\leqq j\leqq n}\Big(\sup_{R_k} |g_j|\Big) \leqq \max_{1\leqq j\leqq n}\Big(\sup_{D_m} |g_j|\Big)$$

for every $k \in \boldsymbol{N}$. The first inequality follows from (20). This clearly contradicts $g_j \in H^\infty(D_m)$ $(j = 1, \cdots, n)$. □

14. We close the paper by establishing (4). From (7) and (8) it follows that $\limsup_{m\to\infty} c(m; n, \delta) = \infty$. Therefore we only have to prove the first inequality of (4) to complete the proof of (4).

We start the proof for $c(m; n, \delta) \leqq c(m+1; n, \delta)$ for an arbitrary (m, n, δ) in $\boldsymbol{N}\times\boldsymbol{N}\times(0, 1)$. What we need to show is that $c(m+1; n, \delta) > t$ whenever $c(m; n, \delta) > t \geqq 1/\delta$. Hence we take an arbitrary but then fixed t in $[1/\delta, c(m; n, \delta))$ and will show that $c(m+1; n, \delta) > t$. There exists an $R \in \mathscr{C}(m)$ such that $C(R; n, \delta) > t$. By taking a $\mathscr{C}_0(m)$-exhaustion of R (cf. no. 2) and using the technical remark in no. 4, we can assume that $R \in \mathscr{C}_0(m)$ with $(R, \varDelta, \pi)$ as its representation as an m-sheeted covering surface of $\varDelta$ having a finite number of branch points and there exists a system $\{f_1, \cdots, f_n\}$ of functions f_j holomorphic on $\bar{R}$ with the following three properties:

$$\max_{1\leqq j\leqq n}(\sup_R |f_j|) = 1 - l , \tag{21}$$

$$\inf_R\Big(\sum_{j=1}^n |f_j|\Big) = \delta + d , \tag{22}$$

$$\{f_j\} \in E(R; n, \delta; t) , \tag{23}$$

where l is in $(0, 1)$ and d is a positive number. Choose a point p in R which is not a branch point of the covering surface $(R, \varDelta, \pi)$ and a closed parametric disk K in R about p with a finite radius, i.e. $K: |z| \leqq r$ with $z(p) = 0$, such that there is no branch point in K. Clearly (21)-(23) are also valid if R is replaced by $R - \{p\}$ and in particular $\{f_j\}$ is t-effective on $R - \{p\}$. By the lower semicontinuity of the t-effectiveness of $\{f_j\}$ (cf. no. 3) we see that $\{f_j\}$ is an t-effective corona datum of index (n, δ) on $R - K$ if r is chosen sufficiently close to zero. Hence we can and hereafter we will fix K in addition to R, p, and $\{f_j\}$ such that $\{f_j\}$ satisfies the following two conditions:

$$\{f_j\} \in E(R - K; n, \delta; t) , \tag{24}$$

$$\max_{1\leqq j\leqq n}(\max_K |f_j - f_j(p)|) < \frac{1}{3n}\max(1, d) . \tag{25}$$

Let σ be contained in the interior of K corresponding to a line segment $[0, s]$ in terms of the local parameter z and R_s be obtained from $R - \sigma$ and $\hat{C} - \sigma$, $\hat{C}$ being the extended complex plane, by connecting each other crosswise along the segment σ. We set $u_j = \operatorname{Re} f_j$ $(j = 1, \cdots, n)$ which are harmonic on $\bar{R}$. We denote by $U_j = U_{js}$ the solution of the Dirichlet problem on R_s with boundary values u_j on $\partial R_s = \partial R$ $(j = 1, \cdots, n)$ which are also harmonic on $\bar{R}_s = R_s \cup \partial R_s$. By the maximum principle it is easy to see that

$$\lim_{s\to 0}(\sup_{R-K} |U_j - u_j|) = 0 \quad (j = 1, \cdots, n) . \tag{26}$$

Let $C_1, \cdots C_q$ be the homology basis of R_s which can be taken in $R - K$.

Then $C_1, \cdots, C_q$ also form the homology basis of R. It is well known that the mapping $\psi \mapsto \left(\int_{C_1} \psi, \cdots, \int_{C_q} \psi\right)$ from the space $A(\bar{R})$ of Abelian differentials ψ of the first kind on $\bar{R}$ to C^q is surjective. Let ψ_k be in $A(\bar{R})$ such that

$$\left(\int_{C_1} \psi_k, \cdots, \int_{C_q} \psi_k\right) = (i\delta_{k1}, \cdots, i\delta_{kq}) \quad (k = 1, \cdots, q)$$

where δ_{kj} be the Kronecker delta. Then $\int_{C_j} \operatorname{Re} \psi_k = 0$ $(j = 1, \cdots, q)$ and thus $\operatorname{Re} \psi_k$ is exact, i.e. there exists a g_k harmonic on $\bar{R}$ with $\operatorname{Re} \psi_k = dg_k$ on $\bar{R}$ $(k = 1, \cdots, q)$. Therefore $\psi_k = dg_k + i*dg_k$ and

$$\int_{C_j} *dg_k = \delta_{kj} \quad (k, j = 1, \cdots, q) .$$

We denote by $G_k = G_{ks}$ the solution of the Dirichlet problem on R_s with boundary values g_k on $\partial R_s = \partial R$. Thus G_k is harmonic on $\bar{R}_s$ $(k = 1, \cdots, q)$. As in (26) we obtain

$$\lim_{s\to 0}(\sup_{R-K} |G_k - g_k|) = 0 \quad (k = 1, \cdots, q) . \tag{27}$$

Hence in particular we see that

$$\lim_{s\to 0} \int_{C_j} *dG_k = \int_{C_j} *dg_k = \delta_{kj} \quad (k, j = 1, \cdots, q) . \tag{28}$$

We set $H_h = H_{hs} = \sum_{k=1}^{q} c_{hk}\, G_k$ with $c_{hk} = c_{hks}$ satisfying

$$\sum_{k=1}^{q} c_{hk} \int_{C_j} *dG_k = \delta_{hj} \quad (h, j = 1, \cdots, q) .$$

In view of (28) we can define H_h for sufficiently small s and moreover c_{hk} $(h, k = 1, \cdots, q)$ are bounded as $s \to 0$. Hence H_h $(h = 1, \cdots, q)$ are bounded on $\overline{R - K}$ and a fortiori on $\bar{R}_s$ uniformly for $s \to 0$.

Now consider the differentials

$$dU_j + i*dU_j - \sum_{h=1}^{q} \left(\int_{C_h} *dU_j\right)(dH_h + i*dH_h) \quad (j = 1, \cdots, n)$$

on $\bar{R}_s$ which are easily seen to have no period along any of C_h $(h = 1, \cdots, q)$ and therefore exact. There exists a holomorphic functions $F_j = F_{js}$ on $\bar{R}_s$ such that

$$dF_j = dU_j + i*dU_j - \sum_{h=1}^{q} \left(\int_{C_h} *dU_j\right)(dH_h + i*dH_h) \quad (j = 1, \cdots, n) .$$

Fixing a point w in $R - K$ we normalize F_j as $\operatorname{Im} F_j(w) = \operatorname{Im} f_j(w)$ by adding a suitable constant to F_j if necessary. Observe that $\int_{C_h} *du_j = 0$

$(h = 1, \cdots, q)$ and thus by (26)

$$\lim_{s\to 0} \int_{C_h} *dU_j = \lim_{s\to 0} \int_{C_h} *d(U_j - u_j) = 0 \quad (j = 1, \cdots, n) .$$

Since we have

$$\operatorname{Re}(F_j - f_j) = (U_j - u_j) + \sum_{h=1}^{q} \left(\int_{C_h} *dU_j \right) H_h \to 0 \quad (s \to 0)$$

uniformly on $\overline{R - K}$, the normalization $\operatorname{Im}(F_j(w) - f_j(w)) = 0$ implies that

$$\lim_{s\to 0} \sup_{\overline{R-K}} |F_j - f_j| = 0 \quad (j = 1, \cdots, n) . \tag{26}$$

Let $W = W_s$ be the part of R_s over Δ so that W is obtained from $R - \sigma$ and $\Delta - \sigma$ by connecting each other crosswise along σ. Clearly W belongs to $\mathscr{C}(m + 1)$, By using the maximum principle, (21), (22), (24), (25), and (29), we can see that $\{F_j\}$ is a t-effective corona datum of index (n, δ) on W if s is sufficiently close to zero. In particular we have $C(W; n, \delta) > t$ and a fortiori $c(m + 1; n, \delta) > t$. $\square$

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