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THE SINGULARITIES OF THE CLOSURES OF NILPOTENT ORBITS IN CERTAIN SYMMETRIC PAIRS

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0. Introduction. Let $\mathfrak{g}$ be a complex reductive Lie algebra and θ a non-trivial involution of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ, i.e., $\mathfrak{k}=\{X\in\mathfrak{g}; \theta(X)=X\}$, $\mathfrak{p}=\{X\in\mathfrak{g}; \theta(X)=-X\}$. Clearly $\mathfrak{k}$ is then a subalgebra of $\mathfrak{g}$, $[\mathfrak{k}, \mathfrak{p}]\subset\mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}]\subset\mathfrak{k}$. We call the pair $(\mathfrak{g}, \mathfrak{k})$ a symmetric pair and $\mathfrak{p}$ the vector space associated to $(\mathfrak{g}, \mathfrak{k})$.

Let G be the adjoint group of $\mathfrak{g}$ and K_θ the subgroup of the elements in G which commute with θ. Then K_θ acts on $\mathfrak{p}$ by the adjoint action. Kostant and Rallis [KR] obtained several results on the orbit structure of $\mathfrak{p}$ under the action of K_θ. On the other hand, Kraft and Procesi [KP1], [KP2], [KP3] studied the singularities in the closures of nilpotent orbits of classical Lie algebras and gave a sufficient condition for an orbit closure to be normal. The purpose of this paper is to generalize some results of Kraft and Procesi to the following symmetric pairs

$$(\mathfrak{g}, \mathfrak{k})=\begin{cases}(\mathfrak{gl}(n, \boldsymbol{C}), \mathfrak{o}(n, \boldsymbol{C})) & (\varepsilon=1)\\ (\mathfrak{gl}(2m, \boldsymbol{C}), \mathfrak{sp}(m, \boldsymbol{C})) & (\varepsilon=-1)\,.\end{cases}$$

For the simplicity of expression, we attach $\varepsilon=1$ to $(\mathfrak{gl}(n, \boldsymbol{C}), \mathfrak{o}(n, \boldsymbol{C}))$ and $\varepsilon=-1$ to $(\mathfrak{gl}(2m, \boldsymbol{C}), (\mathfrak{sp}(m, \boldsymbol{C}))$.

In §1, we investigate the closure relation of nilpotent K_θ-orbits in $\mathfrak{p}$. Let $P(n)$ be the set of partitions of n. We frequently identify an element of $P(n)$ with a Young diagram of size n. Put

$$P_\varepsilon(n)=\begin{cases}P(n) & (\varepsilon=1)\\ P(m)^2:=\{(a_1, a_1, a_2, a_2, \cdots)\in P(n)\} & (\varepsilon=-1, n=2m)\,.\end{cases}$$

In this paper, we call an element of $P_\varepsilon(n)$ an ε-diagram. (Note that the ε-diagrams here do not coincide with the ε-diagrams in the sense of Kraft and Procesi [KP3].) It is known (Sekiguchi [S]) that there is a one-to-one correspondence between the set of nilpotent K_θ-orbits in $\mathfrak{p}$ and $P_\varepsilon(n)$ in each case. For an ε-diagram $\lambda\in P_\varepsilon(n)$, we denote by $C_{\varepsilon,\lambda}$ the corresponding nilpotent orbits in $\mathfrak{p}$. To describe the closure relation, we define a certain partial ordering $\leqq$ in $P_\varepsilon(n)$ (for the definition, see (1.4)). Then the closure relation is given as follows:

THEOREM 1. *For two ε-diagrams* λ *and* μ *in* $P_\varepsilon(n)$, *we have* $\bar{C}_{\varepsilon,\lambda}\supset C_{\varepsilon,\mu}$ *if and only if* $\lambda \geqq \mu$, *where* $\bar{C}_{\varepsilon,\lambda}$ *is the Zariski closure of* $C_{\varepsilon,\lambda}$.

In §2, we study the singularities of the closures of nilpotent orbits. If σ and η are ε-diagrams and $\sigma \leqq \eta$, we call $\sigma \leqq \eta$ an ε-degeneration. The main result of §2, which is an analogue of Proposition 3.1 of [KP2] and Theorem 12.3 of [KP3], is the following:

THEOREM 2. *Let* $\sigma=(\sigma_1, \sigma_2, \cdots, \sigma_p)\leqq(\eta_1, \eta_2, \cdots, \eta_q)$ *be an ε-degeneration. Suppose that for two integers r and s, the first r rows and the first s columns of* η *and* σ *coincide and that* $(\eta_1, \eta_2, \cdots, \eta_r)$ *is an ε-diagram. Denote by* η' *and* σ' *the diagrams we obtain by erasing these coinciding rows and columns of* η *and* σ, *respectively. Then* $\sigma' \leqq \eta'$ *is an ε-degeneration and*

$$\mathrm{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma}) = \mathrm{Sing}(\bar{C}_{\varepsilon,\eta'}, C_{\varepsilon,\sigma'}) .$$

(*For the definition of* $\mathrm{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma})$, *see* (2.1).)

In §3, we consider the normality of the closures of nilpotent orbits. This problem was first treated in Kostant [K]. He proved that the nilpotent variety of a complex semi-simple Lie algebra, which is the closure of the regular nilpotent orbit, is normal. Kraft and Procesi [KP1] showed that any closures of nilpotent orbits in the Lie algebras of type A are normal. Moreover, they gave a sufficient condition for the closure of a nilpotent orbit in simple Lie algebras of types B, C and D to be normal. The proof of Kostant for the nilpotent variety is mainly based on the fact that the nilpotent variety is a complete intersection in the Lie algebra. But the closure of a irregular nilpotent orbit is not a complete intersection in general. So Kraft and Procesi showed that the closure of some nilpotent orbit C in classical Lie algebras is normal by constructing a certain variety which is a complete intersection from which the closure $\bar{C}$ can be obtained as its quotient. We prove the following results by using the method of Kraft and Procesi [KP3].

THEOREM 4. *For the symmetric pair* $(\mathfrak{gl}(2m, \boldsymbol{C}))$, $\mathfrak{sp}(m, \boldsymbol{C}))$, *any closures of nilpotent orbits in the associated vector space are normal.*

This property does not hold for the symmetric pair $(\mathfrak{gl}(n, \boldsymbol{C}), \mathfrak{o}(n, \boldsymbol{C}))$. The reason will be given in (2.4).

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NOTATION. We denote by $\boldsymbol{C}$ the set of complex numbers. For a vector space V, we denote by $\mathfrak{gl}(V)$ the Lie algebra consisting of all endomorphisms of V. We denote by $GL(V)$ the group consisting of all invertible endomorphisms of V. We denote the adjoint representation of an algebraic group (resp. Lie algebra) by Ad (resp. ad). We always consider the Zariski topology unless we specify otherwise. Let X be an algebraic variety and Y be a subset of X. We denote by $\bar{Y}$ the (Zariski) closure of Y. We sometimes denote the closure by Y^- instead of $\bar{Y}$. If $f: S \to T$ is a map and S_1 is a subset of S, we denote by $f|S_1: S_1 \to T$ the restriction of f to S_1.

1. Closure relation.

(1.1) Preliminaries. Let ε be $+1$ or -1. A finite dimensional vector space V over $\boldsymbol{C}$ endowed with a non-degenerate bilinear form $(\ ,\ )$ such that $(u, v) = \varepsilon(v, u)$ for all $u, v \in V$ is called a quadratic space of type ε. Let V be a quadratic space of type ε of dimension n. For $X \in \mathfrak{gl}(V)$, we define the adjoint $X^* \in \mathfrak{gl}(V)$ of X by $(Xu, v) = (u, X^*v)$ for all $u, v \in V$. Then $X \mapsto -X^*$ gives an involution of the Lie algebra $\mathfrak{gl}(V)$. Define $\mathfrak{g}(V)$, $\mathfrak{p}(V)$ and $G(V)$ by

$$\mathfrak{g}(V) = \{X \in \mathfrak{gl}(V); X = -X^*\}, \quad \mathfrak{p}(V) = \{X \in \mathfrak{gl}(V); X = X^*\}$$
$$G(V) = \{g \in GL(V); g^* = g^{-1}\}.$$

Then $G(V)$ is a subgroup of $GL(V)$ with Lie algebra $\mathfrak{g}(V)$ and acts on $\mathfrak{p}(V)$ by the adjoint action. In this way, we have a symmetric pair $(\mathfrak{gl}(V), \mathfrak{g}(V))$ which is isomorphic to $(\mathfrak{gl}(n, \boldsymbol{C}), \mathfrak{o}(n, \boldsymbol{C}))$ if $\varepsilon = 1$ and $(\mathfrak{gl}(n, \boldsymbol{C}), \mathfrak{sp}(n/2, \boldsymbol{C}))$ if $\varepsilon = -1$. Note that $\mathrm{Ad}(G(V))$ coincides K_θ in the notation of [KR]. From now on, we consider $G(V)$-orbits in $\mathfrak{p}(V)$.

(1.2) Classification of nilpotent orbits. In order to study the geometric structure of the closures of nilpotent $G(V)$-orbits in $\mathfrak{p}(V)$, we first describe the classification of nilpotent orbits in $\mathfrak{p}(V)$.

Let $P(n)$ be the set of partitions of n. We frequently identify a partition in $P(n)$ with a Young diagram of size n. For a partition $\lambda \in P(n)$, we denote by C_λ the nilpotent orbit whose Jordan normal form has type

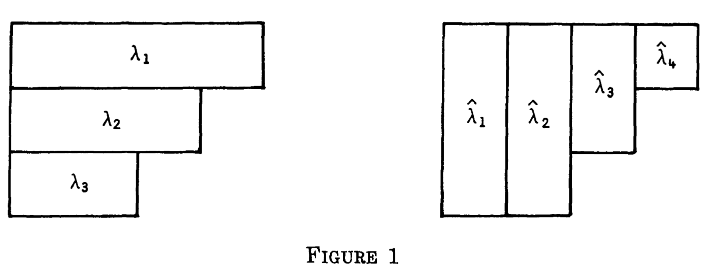

FIGURE 1

λ and put $C_{\varepsilon,\lambda} = \mathfrak{p}(V) \cap C_\lambda$. We denote by λ_i (resp. $\hat{\lambda}_j$) the length of the i-th row (resp. j-th column) of the Young diagram λ as in Figure 1.

DEFINITION. Let $\lambda \in P(n)$ with $\hat{\lambda}_1 = r$. Let β be a permutation of $\{1, 2, \cdots, r\}$ and α a map of $\{1, 2, \cdots, r\}$ into $\boldsymbol{C}^*$ (the multiplicative group of non-zero complex numbers) such that $\beta^2 = \mathrm{id}$, $\lambda_i = \lambda_{\beta(i)}$, and $\alpha(\beta(i)) = \varepsilon\alpha(i)$ for all $1 \leqq i \leqq r$. We call such a triple (λ, α, β) an ε-datum. If a Young diagram $\lambda \in P(n)$ is a member of an ε-datum (λ, α, β), we call λ an ε-diagram. We denote by $P_\varepsilon(n)$ the set of ε-diagrams in $P(n)$.

REMARK 1. It is easy to see that

$$P_\varepsilon(n) = \begin{cases} P(n) & (\varepsilon = 1) \\ P(m)^2 & (\varepsilon = -1) \end{cases}$$

where $P(m)^2 = \{(a_1, a_1, a_2, a_2, \cdots) \in P(n)\}$ with $m = n/2$.

The following result is given in [S].

PROPOSITION 1. *For a partition $\lambda \in P(n)$, we have $C_{\varepsilon,\lambda} \neq \varnothing$ if and only if $\lambda \in P_\varepsilon(n)$. Moreover, $C_{\varepsilon,\lambda}$ consists of a single $G(V)$-orbit. Thus there is a one-to-one correspondence between the set of nilpotent $G(V)$-orbits in $\mathfrak{p}(V)$ and $P_\varepsilon(n)$.*

(1.3) Good bases. Let λ be an ε-diagram in $P_\varepsilon(n)$ with $\hat{\lambda}_1 = r$ and (λ, α, β) and ε-datum. Then we have:

LEMMA 1. *There exists a nilpotent element $z \in \mathfrak{p}(V)$ and vectors $v_i \in V$ $(i = 1, 2, \cdots, r)$ such that $z^a v_i$ $(1 \leqq i \leqq r, 0 \leqq a \leqq \lambda_i - 1)$ form a basis of V and*

$$(z^a v_i, z^b v_j) = \begin{cases} \alpha(i) & (j = \beta(i) \text{ and } a + b + 1 = \lambda_i) \\ 0 & \text{otherwise}. \end{cases}$$

PROOF. Let $X \in \mathfrak{gl}(V)$ be a nilpotent element with a Young diagram λ and $\{X^a u_i; 1 \leqq i \leqq r, 0 \leqq a \leqq \lambda_i - 1\}$ a Jordan basis of X. Define a bilinear form ϕ on V by

$$\phi(X^a u_i, X^b u_j) := \begin{cases} \alpha(i) & (j = \beta(i) \text{ and } a + b + 1 = \lambda_i) \\ 0 & \text{otherwise}. \end{cases}$$

Then it is easy to see that ϕ is a non-degenerate bilinear form equivalent to (,). Therefore we can choose $g \in GL(V)$ so that $\phi(u, v) = (gu, gv)$ for $u, v \in V$. Since $((gXg^{-1})^a gu_i, (gXg^{-1})^b gu_j) = \phi(X^a u_i, X^b u_j)$, gXg^{-1} is a nilpotent element of $\mathfrak{p}(V)$ and gXg^{-1}, gu_i $(1 \leqq i \leqq r)$ satisfy the lemma.

q.e.d.

Choose $z \in \mathfrak{p}(V)$ and v_i $(1 \leqq i \leqq r)$ as in Lemma 1. In order to know the closure relation of nilpotent orbits, we will construct good bases of $\mathfrak{g}(V)$ and $\mathfrak{p}(V)$ for z. Put $\Psi = \{(i, a);\ 1 \leqq i \leqq r,\ 0 \leqq a \leqq \lambda_i - 1\}$ and $v(i, a) = z^a v_i$. Then $\{v(\psi); \psi \in \Psi\}$ is a basis of V. Let $\{u(\psi); \psi \in \Psi\}$ be its dual basis. This means $u(\psi)(v(\psi')) = \delta_{\psi,\psi'}$ (the Kronecker delta) for $\psi, \psi' \in \Psi$. For $\psi, \psi' \in \Psi$, we define $\xi(\psi, \psi') \in \mathfrak{gl}(V)^*$ by $\xi(\psi, \psi')(X) := u(\psi)(Xv(\psi'))$ for $X \in \mathfrak{gl}(V)$. Then $\{\xi(\psi, \psi'); \psi, \psi' \in \Psi\}$ is a basis of $\mathfrak{gl}(V)^*$. Let $\{e(\psi, \psi'); \psi, \psi' \in \Psi\}$ be the dual basis of $\{\xi(\psi, \psi'); \psi, \psi' \in \Psi\}$. Then we have

$$e(\psi, \psi')v(\psi'') = \delta_{\psi',\psi''}v(\psi) \quad \text{for} \quad \psi, \psi', \psi'' \in \Psi$$
$$[e(i, a; j, b), z] = e(i, a; j, b-1) - e(i, a+1; j, b)\ ,$$

where $e(i, a; j, b) = 0$ if (i, a) or (j, b) is not contained in Ψ.

For $\psi, \psi' \in \Psi$, we define $\nu(\psi, \psi') \in \mathfrak{g}(V)^*$ and $\eta(\psi, \psi') \in \mathfrak{p}(V)$ by

$$\nu(\psi, \psi')(X) := (v(\psi), Xv(\psi')) \qquad (X \in \mathfrak{g}(V))$$
$$\eta(\psi, \psi')(X) := (v(\psi), Xv(\psi')) \qquad (X \in \mathfrak{p}(V))\ .$$

Let $\psi = (i, a)$ and $\psi' = (j, b)$ be two elements of Ψ. We write $\psi <^* \psi'$ if $i < j$ or if $i = j$ and $a < b + (1-\varepsilon)/2$ while we write $\psi < \psi'$ if $i < j$ or if $i = j$ and $a \leqq b - (1-\varepsilon)/2$. Since $\nu(\psi, \psi') = -\varepsilon\nu(\psi', \psi)$ for $\psi, \psi' \in \Psi$, $\nu(\psi, \psi')$ with $\psi <^* \psi'$ form a basis of $\mathfrak{g}(V)^*$. Similarly, $\eta(\psi, \psi')$ with $\psi < \psi'$ form a basis of $\mathfrak{p}(V)^*$, since $\eta(\psi, \psi') = \varepsilon\eta(\psi', \psi)$ for $\psi, \psi' \in \Psi$. Let $\{x(\psi, \psi'); \psi <^* \psi'\}$ be the dual basis of $\{\nu(\psi, \psi'); \psi <^* \psi'\}$ in $\mathfrak{g}(V)$ and $\{y(\psi, \psi'); \psi < \psi'\}$ the dual basis of $\{\eta(\psi, \psi'); \psi < \psi'\}$ in $\mathfrak{p}(V)$. Note that $(v(i, a), v) = \alpha(i)u(\beta(i), \lambda_i - a - 1)(v)$ for all $v \in V$. Then the following two lemmas can be easily proved.

LEMMA 2. (i) $\alpha(i)^{-1}\nu(i, a; j, b) = \xi(\beta(i), \lambda_i - a - 1; j, b)\,|\,\mathfrak{g}(V)$ *for* (i, a), $(j, b) \in \Psi$.

(ii) *If* $i < j$ *or if* $i = j$ *and* $a < b$, *then* $x(i, a; j, b) = \alpha(i)^{-1}e(\beta(i), \lambda_i - a - 1; j, b) - \varepsilon\alpha(j)^{-1}e(\beta(j), \lambda_j - b - 1; i, a)$.

(iii) *If* $\varepsilon = -1$, *then* $x(i, a; i, a) = \alpha(i)^{-1}e(\beta(i), \lambda_i - a - 1; i, a)$.

LEMMA 3. (i) $\alpha(i)^{-1}\eta(i, a; j, b) = \xi(\beta(i), \lambda_i - a - 1; j, b)\,|\,\mathfrak{p}(V)$ *for* (i, a), $(j, b) \in \Psi$.

(ii) *If* $i < j$ *or if* $i = j$ and $a < b$, *then* $y(i, a; j, b) = \alpha(i)^{-1}e(\beta(i), \lambda_i - a - 1; j, b) + \varepsilon\alpha(j)^{-1}e(\beta(j), \lambda_j - b - 1; i, a)$.

(iii) *If* $\varepsilon = 1$, *then* $y(i, a; i, a) = \alpha(i)^{-1}e(\beta(i), \lambda_i - a - 1; i, a)$.

The following lemma follows from Lemmas 2, 3.

LEMMA 4. (i) *For* $(i, a), (j, b) \in \Psi$ *with* $(i, a) <^* (j, b)$,

$$[x(i, a; j, b), z] = \begin{cases} 2y(i, a; i, a) - y(i, a-1; i, a+1) & (i = j \text{ and } a = b-1) \\ y(i, a; j, b-1) - y(i, a-1; j, b) & \textit{otherwise}, \end{cases}$$

where we put $y(\psi, \psi') = 0$ *if* $y(\psi, \psi')$ *is not yet defined.*

(ii) *For* $(i, a), (j, b) \in \Psi$ *with* $(i, a) < (j, b)$,

$$[y(i, a; j, b), z] = \begin{cases} 2x(i, a; i, a) - x(i, a-1; i, a+1) & (i = j \text{ and } a = b-1) \\ x(i, a; j, b-1) - x(i, a-1; j, b) & \textit{otherwise}, \end{cases}$$

where we put $x(\psi, \psi') = 0$ *if* $x(\psi, \psi')$ *is not yet defined.*

REMARK 2. Let $\mathfrak{g}(V)_{i,j}$ and $\mathfrak{p}(V)_{i,j}$ be the vector subspaces defined by

$$\mathfrak{g}(V)_{i,j} = \sum_{a,b} Cx(i, a; j, b) \quad \text{and} \quad \mathfrak{p}(V)_{i,j} = \sum_{a,b} Cy(i, a; j, b) .$$

Then we have

$$[\mathfrak{g}(V)_{i,j}, z] \subset \mathfrak{p}(V)_{i,j} , \quad [\mathfrak{p}(V)_{i,j}, z] \subset \mathfrak{g}(V)_{i,j} ,$$
$$\mathfrak{g}(V) = \bigoplus_{i \leqq j} \mathfrak{g}(V)_{i,j} \quad \text{and} \quad \mathfrak{p}(V) = \bigoplus_{i \leqq j} \mathfrak{p}(V)_{i,j} .$$

(1.4) Closure relation. Given two partitions $\lambda, \mu \in P(n)$, write $\lambda \geqq \mu$ if

$$\sum_{i=1}^{j} \lambda_i \geqq \sum_{i=1}^{j} \mu_i$$

for all j. This is equivalent to

$$\sum_{k>j} \hat{\lambda}_k \geqq \sum_{k>j} \hat{\mu}_k$$

for all j (cf., [KP3, Proposition 2.5]). For simplicity, we call such $\lambda \geqq \mu$ a degeneration. In particular, if $\lambda, \mu \in P_\varepsilon(n)$ we call $\lambda \geqq \mu$ an ε-degeneration.

LEMMA 5. ([H, Proposition 3.9]) *Let* $\lambda > \mu$ *be an adjacent degeneration in* $P(n)$ *(i.e., there is no partition* $\nu \in P(n)$ *such that* $\lambda > \nu > \mu$*) and* $\mu = (\mu_1, \mu_2, \cdots, \mu_t)$. *Then* λ *has one of the following forms*;

(I) $\lambda = (\mu_1, \cdots, \mu_{i-1}, \mu_i + 1, \mu_{i+1} - 1, \mu_{i+2}, \cdots, \mu_t)$ *for some* $1 \leqq i \leqq t-1$.

(II) $\lambda = (\mu_1, \cdots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \cdots, \mu_{j-1}, \mu_j - 1, \mu_{j+1}, \cdots, \mu_t)$ *with* $\mu_{j+1} < \mu_j = \mu_{j-1} = \cdots = \mu_i < \mu_{i-1}$ *for some* $1 \leqq i < j \leqq t$.

REMARK 3. Let λ and μ be as above. Suppose that $\lambda_1 = \mu_1, \cdots \lambda_p = \mu_p$, $\lambda_{p+1} \neq \mu_{p+1}$ and $\hat{\lambda}_1 = \hat{\mu}_1, \cdots, \hat{\lambda}_q = \hat{\mu}_q$, $\hat{\lambda}_{q+1} \neq \hat{\mu}_{q+1}$. Let λ' and μ' be the Young diagrams we obtain from λ and μ by erasing the first p rows and first q columns. Then λ' and μ' have the forms as in Figure 2;

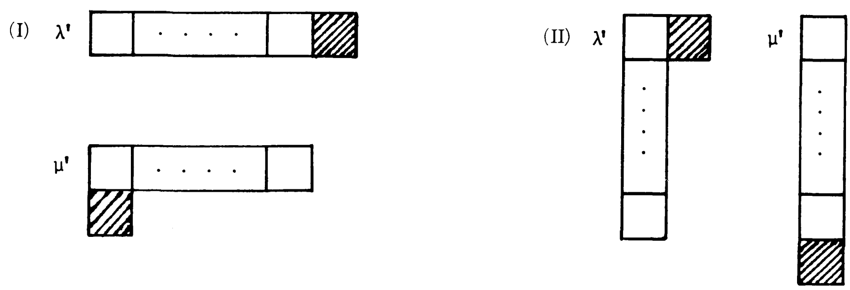

FIGURE 2

Suppose that $\varepsilon = -1$ and $\dim V = n = 2m$. For a partition $\mu = (\mu_1, \cdots, \mu_t) \in P(m)$, we write $\mu^2 = (\mu_1, \mu_1, \cdots, \mu_t, \mu_t) \in P(m)^2$. For two partitions λ and μ of $P(m)$, we have $\lambda^2 \geqq \mu^2$ if and only if $\lambda \geqq \mu$. Moreover, $\lambda^2 > \mu^2$ is adjacent in $P_{-1}(n) = P(m)^2$ if and only if $\lambda > \mu$ is adjacent in $P(m)$.

Now we give the closure relation for nilpotent orbits in $\mathfrak{p}(V)$.

THEOREM 1. *For two ε-diagrams λ and μ in $P_\varepsilon(n)$, we have $\bar{C}_{\varepsilon,\lambda} \supset C_{\varepsilon,\mu}$ if and only if $\lambda \geqq \mu$, where $n = \dim V$ and $\bar{C}_{\varepsilon,\lambda}$ is the Zariski closure of $C_{\varepsilon,\lambda}$.*

PROOF. The "only if" part is rather easily seen as follows. Suppose that $\bar{C}_{\varepsilon,\lambda} \supset C_{\varepsilon,\mu}$, $z \in C_{\varepsilon,\mu}$ and $X \in C_{\varepsilon,\lambda}$. Since $z \in \bar{C}_{\varepsilon,\lambda} = (\mathrm{Ad}(G(V))X)^-$, we have $z^i \in (\mathrm{Ad}(G(V))X^i)^-$. Therefore all minors of z^i of degree $\mathrm{rank}(X^i) + 1$ are 0 and hence $\mathrm{rank}(z^i) \leqq \mathrm{rank}(X^i)$. Since

$$\mathrm{rank}(X^i) = \sum_{j>i} \hat{\lambda}_j \quad \text{and} \quad \mathrm{rank}(z^i) = \sum_{j>i} \hat{\mu}_j$$

(cf., [KP3, (1.1)]), we have $\lambda \geqq \mu$.

We now prove the "if" part. Suppose that $\lambda > \mu$. We may assume that λ and μ are adjacent in $P_\varepsilon(n)$. Let (μ, α, β) be an ε-datum which contains μ. We note the following fact.

LEMMA 6. *Let λ, μ and (μ, α, β) be as above. In order to show that $\bar{C}_{\varepsilon,\lambda} \supset C_{\varepsilon,\mu}$, it is sufficient to show this in the following cases:*

(i) $\mu = (p, q)$, $\lambda = (p+1, q-1)$, $\beta = \mathrm{id}$, $\alpha(1) = \alpha(2) = 1$ $(\varepsilon = 1, p \geqq q \geqq 2)$.

(ii) $\mu = (p, p, q, q)$, $\lambda = (p+1, p+1, q-1, q-1)$, $\beta(1) = 2$, $\beta(3) = 4$, $\alpha(1) = \alpha(3) = 1$, $\alpha(2) = \alpha(4) = -1$ $(\varepsilon = -1, p \geqq q \geqq 2)$.

PROOF. Let $\mu = (\mu_1, \mu_2, \cdots, \mu_r)$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$, where $r = \hat{\mu}_1$ and $k = \hat{\lambda}_1$. Note that $r \geqq k$. Put $\{i_1, \cdots, i_s\} = \{i; \mu_i \neq \lambda_i\}$ and $\{j_1, \cdots, j_t\} = \{i; \mu_i = \lambda_i\}$ with $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$. Since $\lambda > \mu$ is adjacent, we have $s = 2$ if $\varepsilon = 1$ and $s = 4$ if $\varepsilon = -1$ (cf., Lemma 5).

If $\varepsilon = 1$, we may assume that $\beta = \text{id}$. If $\varepsilon = -1$, we may assume that $\mu_{i_1} = \mu_{i_2} \geqq \mu_{i_3} = \mu_{i_4}$ and $\beta(i_1) = i_2$, $\beta(i_3) = i_4$. Choose a nilpotent element $z \in C_{\varepsilon,\mu}$ and a Jordan basis $\{z^a v_i; 1 \leqq i \leqq r,\ 0 \leqq a \leqq \mu_i - 1\}$ of z such that

$$(z^a v_i, z^b v_j) = \begin{cases} \alpha(i) & (j = \beta(i) \text{ and } a + b + 1 = \mu_i) \\ 0 & \text{otherwise}. \end{cases}$$

Put $v(i, a) = z^a v_i$,

$$V_1 = \bigoplus_{a=1}^{t} \big(\sum_{b \geqq 0} Cv(j_a, b)\big) \quad \text{and} \quad V' = \bigoplus_{a=1}^{s} \big(\sum_{b \geqq 0} Cv(i_a, b)\big).$$

Then we have the orthogonal decomposition $V = V' \oplus V_1$ with respect to $(\ ,\ )$. Hence V' and V_1 are quadratic spaces of type ε with respect to the restrictions of $(\ ,\ )$. Put $\mu' = (\mu_{i_1}, \cdots, \mu_{i_s})$, $\lambda' = (\lambda_{i_1}, \cdots, \lambda_{i_s})$ and $\nu = (\mu_{j_1}, \cdots, \mu_{j_t})$. Since V' and V_1 are z-stable, z is decomposed as $z = (z', z_1)$ where $z' \in C_{\varepsilon,\mu'}(\subset \mathfrak{p}(V'))$ and $z_1 \in C_{\varepsilon,\nu}(\subset \mathfrak{p}(V_1))$. Take $X' \in C_{\varepsilon,\lambda'}$ and put $X = (X', z_1)$. Then clearly $X \in C_{\varepsilon,\lambda}$. If we can show that $z' \in (\text{Ad}(G(V'))X')^- = \bar{C}_{\varepsilon,\lambda'}$, we get

$$z \in (\text{Ad}(G(V')X)^- = (\bar{C}_{\varepsilon,\lambda'}, \{z_1\}) \subset \bar{C}_{\varepsilon,\lambda}.$$

Thus we may assume that $\mu = (p, q)$, $\lambda = (p+1, q-1)$ if $\varepsilon = 1$ while $\mu = (p, p, q, q)$, $\lambda = (p+1, p+1, q-1, q-1)$ if $\varepsilon = -1$ with $p \geqq q \geqq 1$. If $q = 1$, $C_{\varepsilon,\lambda}$ is the principal nilpotent orbit in the sense of [KR] and so we have $\bar{C}_{\varepsilon,\lambda} \supset C_{\varepsilon,\mu}$. Therefore we may assume that $q \geqq 2$. The remaining assertions for α and β are easily checked. Thus Lemma 6 has been proved.

Now we assume that λ, μ and (μ, α, β) are as in Lemma 6. We first consider the case $\varepsilon = -1$. Put

$$\begin{aligned} y = & -y(2, p-1; 3, 0) - y(3, p-1; 4, 0) \\ & + y(1, p-1; 3, 0) + y(1, p-1; 4, 0), \quad z(t) = z + ty, \end{aligned}$$

$$v_1(t) = v_1, \quad v_2(t) = v_3, \quad v_3(t) = zv_3, \quad v_4(t) = z^{p-q-1}v_1 - tv_3 + tv_4 \qquad (t \in C).$$

Then we know that $z(t)$ is a nilpotent element of $\mathfrak{p}(V)$ and $\{z(t)^a v_i(t); 1 \leqq i \leqq 4,\ 0 \leqq a \leqq \lambda_i - 1\}$ is a Jordan basis of $z(t)$ for each $t \in C^*$. Hence $z(t) \in C_{\varepsilon,\lambda}$ if $t \neq 0$ and $z(0) = z \in C_{\varepsilon,\mu}$. This implies that $C_{\varepsilon,\mu} \subset \bar{C}_{\varepsilon,\lambda}$.

Next we consider the case $\varepsilon = 1$. What we want to construct is a morphism $z\colon C \to \mathfrak{p}(V)$ such that $z(t) \in C_{\varepsilon,\lambda}$ if $t \neq 0$ and $z(0) = z$. For this purpose, put

$$\begin{aligned} y(a, b, c) = & -\sum_{i=1}^{p} a_i y(1, i-1;\ 1, p-1) \\ & - \sum_{i=1}^{q} b_i y(2, i-1;\ 2, q-1) - \sum_{i=1}^{q} c_i y(1, p-1; 2, i-1), \end{aligned}$$

where $a=(a_1, a_2, \cdots, a_p) \in C^p$, $b=(b_1; b_2, \cdots, b_q) \in C^q$ and $c=(c_1, c_2, \cdots, c_q) \in C^q$. If we express z and $y(a, b, c)$ by matrices with respect to the basis $u_1 = v(1, p-1)$, $u_2 = v(1, p-2)$, $\cdots$, $u_p = v(1, 0)$, $u_{p+1} = v(2, q-1)$, $u_{p+2} = v(2, q-2)$, $\cdots$, $u_{p+q} = v(2, 0)$ of V, we have

$$z = \left[\begin{array}{c|c} J_p & O \\ \hline O & J_q \end{array}\right] \quad \text{and} \quad y(a, b, c) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{-A} & \begin{matrix} O \\ -c' \end{matrix} \\ \hline -{}^t c & O & -B \end{array}\right]$$

where

$$J_p = \underbrace{\begin{bmatrix} 0 & 1 & & & O \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & O & & \ddots & 1 \\ & & & & 0 \end{bmatrix}}_{p} \Bigg\} p \qquad A = \begin{bmatrix} a_1 & & & \\ a_2 & & O & \\ \vdots & & & \\ a_p & \cdots\cdots & a_2 & a_1 \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 & & & \\ b_2 & & O & \\ \vdots & & & \\ b_q & \cdots\cdots & b_2 & b_1 \end{bmatrix} \quad \text{and} \quad c' = (c_q, \cdots, c_2, c_1) .$$

Put $z(a, b, c) = z + y(a, b, c)$. Let T be a variable and $M_{p+q}(C[T])$ the ring of matrices with coefficients in $C[T]$. For two matrices $X(T)$ and $Y(T)$ in $M_{p+q}(C[T])$, we write $X(T) \sim Y(T)$ if there are two invertible matrices $U(T)$ and $V(T)$ in $M_{p+q}(C[T])$ such that $X(T) = U(T)Y(T)V(T)$. We denote by I_n the unit matrix of degree n. Then multiplying

$$\left[\begin{array}{c|c} O & 1 \\ \hline I_{p+q+1} & O \end{array}\right]$$

to $TI_{p+q} - z(a, b, c)$ from the right and erasing components other than the first $p-1$ diagonal components, we have

$$TI_{p+q} - z(a, b, c) \sim \left[\begin{array}{c|c|c} -I_{p-1} & O & O \\ \hline O & c' & h(T) \\ \hline O & B(T) & {}^t c \end{array}\right]$$

where $h(T) = T^p + \sum_{t=1}^{p} (\sum_{s=1}^{t-1} a_s a_{t-s} + 2a_t) T^{p-t} - a_p$ and $B(T) = TI_q + B - J_q$. Multiplying

$$\left[\begin{array}{c|c} O & I_q \\ \hline 1 & O \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} O & 1 \\ \hline I_q & O \end{array}\right]$$

from the left and right to

$$\left[\begin{array}{c:c} c' & h(T) \\ \hdashline B(T) & {}^t c \end{array}\right]$$

respectively, we get

$$\left[\begin{array}{c:c} c' & h(T) \\ \hdashline B(T) & {}^t c \end{array}\right] \sim \left[\begin{array}{c:c} -I_{q-1} & O \\ \hdashline O & \begin{matrix} P(T) & Q(T) \\ R(T) & S(T) \end{matrix} \end{array}\right],$$

where

$$P(T) = S(T) = \sum_{t=1}^{q} \left(c_t + \sum_{i=1}^{t-1} b_i c_{t-i}\right) T^{q-t},$$

$$Q(T) = T^q + \sum_{t=1}^{q} \left(2b_t + \sum_{i=1}^{t-1} b_i b_{t-i}\right) T^{q-t} - b_q, \quad \text{and}$$

$$R(T) = T^p + \sum_{t=1}^{p} \left(2a_t + \sum_{i=1}^{t-1} a_i a_{t-i}\right) T^{p-t} + \sum_{t=2}^{q} \left(\sum_{i=1}^{t-1} c_i c_{t-i}\right) T^{q-t} - a_p.$$

In order that $z(a, b, c)$ is nilpotent and its corresponding partition is $\lambda = (p+1, q-1)$, it is sufficient to show that the following condition $(*)$ holds

$$(*) \qquad \begin{aligned} & S(T) = P(T) = c_1 T^{q-1} \quad \text{with} \quad c_1 \neq 0, \; Q(T) = T^q + 2b_1 T^{q-1}, \\ & R(T) = T^p + \sum_{t=1}^{p-q+1} \left(2a_t + \sum_{i=1}^{t-1} a_i a_{t-i}\right) T^{p-t}, \quad \text{and} \\ & R(T)Q(T) - P(T)S(T) = T^{p+q}. \end{aligned}$$

This condition $(*)$ is satisfied if the following $(*)'$ holds:

$$(*)' \qquad \begin{aligned} & c_t + \sum_{i=1}^{t-1} b_i c_{t-i} = 0 \quad (2 \leqq t \leqq q), \quad 2b_t + \sum_{i=1}^{t-1} b_i b_{t-i} = 0 \quad (2 \leqq t \leqq q-1), \\ & b_q + \sum_{i=1}^{q-1} b_i b_{q-i} = 0, \; c_1 \neq 0, \\ & \sum_{i=1}^{t-1} c_i c_{t-i} + 2a_{p-q+t} + \sum_{i=1}^{p-q+t-1} a_i a_{p-q+t-i} = 0 \quad (2 \leqq t \leqq q-1), \\ & \sum_{i=1}^{q-1} c_i c_{q-i} + a_p + \sum_{i=1}^{p-1} a_i a_{p-i} = 0, \quad 2b_1 + 2a_1 = 0, \\ & 2a_{t+1} + \sum_{i=1}^{t} a_i a_{t+1-i} + 2b_1\left(2a_t + \sum_{i=1}^{t-1} a_i a_{t-i}\right) = 0 \quad (1 \leqq t \leqq p-q), \\ & c_1^2 = 2b_1\left(2a_{p-q+1} + \sum_{i=1}^{p-q} a_i a_{p-q+1-i}\right). \end{aligned}$$

Put A_s $(1 \leqq s \leqq p)$, B_s $(1 \leqq s \leqq q)$ and C_s $(1 \leqq s \leqq q)$ as follows;

$$A_1 = 1,\ A_{s+1} = -\Big(\sum_{i=1}^{s} A_iA_{s+1-i}\Big)/2 + 2A_s + \sum_{i=1}^{s-1} A_iA_{s-i} \qquad (1 \leqq s \leqq p-q)\,.$$

(Note that $A_{p-q+1} + (\sum_{i=1}^{p-q} A_iA_{p-q+1-i})/2 = 2^{p-q}$).

$$B_1 = -1\,,\quad B_s = -\Big(\sum_{i=1}^{s-1} B_iB_{s-i}\Big)/2 \qquad (2 \leqq s \leqq q-1)\,,\quad B_q = -\Big(\sum_{i=1}^{q-1} B_iB_{q-i}\Big)$$

$$C_1 = (-2^{p-q+2})^{1/2}\,,\quad C_s = -\Big(\sum_{i=1}^{s-1} B_iC_{s-i}\Big) \qquad (2 \leqq s \leqq q)\,.$$

$$A_{p-q+s} = -\Big(\sum_{i=1}^{p-q+s-1} A_iA_{p-q+s-i} + \sum_{i=1}^{s-1} C_iC_{s-i}\Big)/2 \qquad (2 \leqq s \leqq q-1)\,,$$

$$A_p = -\Big(\sum_{i=1}^{p-1} A_iA_{p-i} + \sum_{i=1}^{q-1} C_iC_{q-i}\Big)\,.$$

Define $a_i(t)$, $b_i(t)$, $c_i(t)$ by $a_i(t) = A_it^{2i}$ $(1 \leqq i \leqq p)$, $b_i(t) = B_it^{2i}$ $(1 \leqq i \leqq q)$ and $c_i(t) = C_it^{2i+p-q}$ $(1 \leqq i \leqq q)$ for $t \in \boldsymbol{C}$. Then they satisfy $(*)'$ if $t \neq 0$. Therefore if we define $z(t)$ by

$$z(t) = z(a_1(t), \cdots, a_p(t), b_1(t), \cdots, b_q(t), c_1(t), \cdots, c_q(t))\,,$$

we have $z(t) \in C_{\varepsilon,\lambda}$ $(t \neq 0)$ and $z(0) = z$. This implies $\bar{C}_{\varepsilon,\lambda} \supset C_{\varepsilon,\mu}$. Thus the proof of Theorem 1 is completed.

2. Singularities in the closures of nilpotent orbits.

(2.1) Smooth equivalence classes.

Definition ([KP3]). Consider two varieties X, Y and let $x \in X$, $y \in Y$. The singularity of X at x is called smoothly equivalent to the singularity of Y at y if there exist a variety Z, a point $z \in Z$ and two maps

$$\begin{array}{ccc} Z & \xrightarrow{\ \phi\ } & X \\ {\scriptstyle\psi}\big\downarrow & & \\ Y & & \end{array}$$

such that $\phi(z) = x$, $\psi(z) = y$, and ϕ and ψ are smooth at z. This clearly defines an equivalence relation among pointed variaties (X, x). We denote by $\mathrm{Sing}(X, x)$ the equivalence class to which (X, x) belongs.

Suppose that an algebraic group G acts on a variety X. Then $\mathrm{Sing}(X, x) = \mathrm{Sing}(X, x')$ if x and x' belong to the same orbit O. In this case we denote the equivalence class also by $\mathrm{Sing}(X, O)$.

Remark 4. Let (X, x) and (Y, y) be pointed varieties over $\boldsymbol{C}$. Suppose that $\dim_x X = \dim_y Y + r$ for some integer $r \geqq 0$. Then $\mathrm{Sing}(X, x) = \mathrm{Sing}(Y, y)$ if and only if some neighbourhoods (in the classical topology) of $x \in X$ and $(y, 0) \in Y \times C^r$ are analyticaly isomorphic. Therefore, various geometric properties of X at x depend only on the equivalence class $\mathrm{Sing}(X, x)$, for example; X is smooth, normal, seminormal, unibranched or has a Cohen-Macaulay or rational singularity (cf., [KP3, 12.2]).

The following theorem is the main result of this section.

THEOREM 2. *Let $\sigma \leqq \eta$ be an ε-degeneration. Suppose that for two integers r and s the first r rows and the first s columns of η and σ coincide and that $(\eta_1, \eta_2, \cdots \eta_r)$ is an ε-diagram. Denote by η' and σ' the diagrams we obtain by erasing these coincident rows and columns of η and σ, respectively. Then $\sigma' \leqq \eta'$ is an ε-degeneration and*

$$\mathrm{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma}) = \mathrm{Sing}(\bar{C}_{\varepsilon,\eta'}, C_{\varepsilon,\sigma'}) .$$

REMARK 5. In the setting of Theorem 2, we say that the ε-degeneration $\sigma \leqq \eta$ is obtained from the ε-degeneration $\sigma' \leqq \eta'$ by addition of rows and columns.

This is an analogue to the results of Kraft and Procesi for classical Lie algebras ([KP3, Proposition 3.1] and [KP3, Theorem 12.3]). The proof is similar to that for Theorem 12.3 of [KP3]. We will treat separately the two steps "cancelling columns" and "cancelling rows".

(2.2) Cancelling columns. Let U and V be two quadratic spaces of type ε and put $L(V, U) := \mathrm{Hom}(V, U)$. For $X \in L(V, U)$, we define the adjoint $X^* \in L(U, V)$ by $(Xv, u)_U = (v, X^*u)_V$ for all $u \in U$ and $v \in V$. Then $(X^*)^* = X$ for $X \in L(V, U)$. We define two morphisms

$$\begin{array}{ccc} L(V, U) & \xrightarrow{\ \pi\ } & \mathfrak{p}(U) \\ {\scriptstyle \rho}\big\downarrow & & \\ \mathfrak{p}(V) & & \end{array}$$

by $\rho(X) := X^*X$ and $\pi(X) := XX^*$ for $X \in L(V, U)$. The group $G(V) \times G(U)$ acts on $L(V, U)$ by $(g, h)X = gXh^{-1}$ and π and ρ are equivariant with respect to the adjoint actions of $G(U)$ and $G(V)$ on $\mathfrak{p}(U)$ and $\mathfrak{p}(V)$, respectively.

DEFINITION ([KP1]). Let X be an affine variety with an action of a reductive group G and Y an affine variety. A morphism $\pi: X \to Y$ is called the quotient map under G if, via π, the coordinate ring of Y is identified with the ring of G-invariant functions on X.

REMARK 6. If $\pi: X \to Y$ is a quotient map under G and X_1 is a G-invariant closed subset of X, then $\pi(X_1)$ is a closed subset of Y and the restriction $\pi | X_1: X_1 \to \pi(X_1)$ is also a quotient map under G (cf., [MF, Chap. 1, §2]). If X is normal, then so is Y.

Similar to the case of classical Lie algebras in [KP3], we have the following theorem which we can prove by using Theorem 5.6 (i) and Theorem 6.6 of [DP].

THEOREM 3. *Let U and V be two quadratic spaces of type ε of dimensions n and m, respectively. Suppose that $n \geqq m$. Then $\pi: L(V, U) \to \mathfrak{p}(U)$ is surjective and is the quotient map under $G(V)$. On the other hand, the image of ρ is the determinantal variety in $\mathfrak{p}(V)$ of the endmorphisms of rank $\leqq m$ and $\rho: L(V, U) \to \operatorname{Im} \rho$ is a quotient map under $G(U)$.*

Let D be a nilpotent element in $\mathfrak{p}(V)$ and η its ε-diagram. Put $U = \operatorname{Im} D$. As in [KP3, 4.1], we can define a bilinear form $(\ ,\ )_U$ on U by $(Du, Dv)_U = (u, Dv)_V$ for $u, v \in V$. Then U becomes a quadratic space of type ε. Let $X^{\cdot} := [D: V \to U] \in L(V, U)$ and let $[I: U \to V] \in L(U, V)$ be the inclusion. Then we have $(X^{\cdot})^* = I$, $D = IX^{\cdot} = (X^{\cdot})^* X^{\cdot}$ and $D' := D|U = X^{\cdot} I = X^{\cdot}(X^{\cdot})^*$. In particular, $D' \in \mathfrak{p}(U)$ and it follows from the construction that $D' \in C_{\varepsilon,\eta'}$, where η' is the ε-diagram we obtain from η by erasing the first column. Now we consider the previous two morphisms

$$\begin{array}{ccc} L(V,U) & \xrightarrow{\pi} & \mathfrak{p}(U) \\ \rho \big\downarrow & & \\ \mathfrak{p}(V) & & \end{array} \qquad \pi(X) = XX^*\,, \quad \rho(X) = X^*X$$

in this situation. Put $L'(V, U) = \{X \in L(V, U); X \text{ is surjective}\}$. Then we have the following three lemmas whose proofs are similar to the ones for [KP3, Lemmas 4.2 and 4.3 and Proposition 11.1].

LEMMA 7. *For any $Y \in L'(V, U)$, the stabilizer of Y in $G(U)$ is trivial and $\rho^{-1}(\rho(Y))$ is a single orbit under $G(U)$.*

LEMMA 8. *Let η' be an ε-diagram we obtain from η by erasing the first column and consider the following diagram*

$$\begin{array}{ccc} L(V,U) & \xrightarrow{\pi} & \mathfrak{p}(U) \supset C_{\varepsilon,\eta'} \\ \rho \big\downarrow & & \\ \mathfrak{p}(V) \supset C_{\varepsilon,\eta}\,. & & \end{array}$$

Put $N_{\varepsilon,\eta} = \pi^{-1}(\overline{C}_{\varepsilon,\eta'})$. Then

(i) $\rho(N_{\varepsilon,\eta}) = \bar{C}_{\varepsilon,\eta}$.

Let σ *be an* ε*-diagram such that* $\sigma \leqq \eta$ *and* $\hat{\sigma}_1 = \hat{\eta}_1$. *Then*

(ii) $\rho^{-1}(C_{\varepsilon,\sigma})$ *is a single orbit under* $G(U)\times G(V)$ *and is contained in* $N_{\varepsilon,\eta}\cap L'(V, U)$.

(iii) $\pi(\rho^{-1}(C_{\varepsilon,\sigma})) = C_{\varepsilon,\sigma'}$ *where* σ' *is an* ε*-diagram we obtain from* σ *by erasing the first column.*

LEMMA 9. (i) π *is smooth in* $L' := L'(V, U)$.

(ii) $\rho(L') = \{A \in \mathfrak{p}(V); \operatorname{rank}(A) = m\}$ *and* $\rho|L'\colon L' \to \rho(L')$ *is locally trivial in the classical topology with typical fibre* $G(U)$.

We can prove the following part of Theorem 2 in the same way as [KP3, Proposition 13.5] by using the above three lemmas.

PROPOSITION 2. *Suppose that the* ε*-degeneration* $\sigma \leqq \eta$ *is obtained from the* ε*-degeneration* $\sigma' \leqq \eta'$ *by addition of columns. Then*

$$\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma}) = \operatorname{Sing}(\bar{C}_{\varepsilon,\eta'}, C_{\varepsilon,\sigma'}) .$$

(2.3) Cancelling rows. To prove the remaining part of Theorem 2, we need the following concept.

DEFINITION. Let X be a variety with an action of an algebraic group G. A cross section at a point $x \in X$ is defined to be a locally closed subvariety S of X such that $x \in S$ and the map $G\times S \to X$, $((g, s) \mapsto gs)$ is smooth at (e, x).

REMARK 7. Let V be a vector space with a linear G-action and X a closed G-invariant subvariety of V. Let N be a subspace of V complementary to the tangent space $T_x(Gx)$ for an $x \in X$. Put $S = (N + x)\cap X$. Then S is a cross section at x. If X is irreducible or equidimensional, then we have $\dim_x S = \operatorname{codim}(X, Gx)$ (cf., [KP3, 12.4]).

PROPOSITION 3. *Suppose that an* ε*-degeneration* $\sigma \leqq \eta$ *is obtained from an* ε*-degeneration* $\sigma' \leqq \eta'$ *by addition of rows. Then*

$$\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma}) = \operatorname{Sing}(\bar{C}_{\varepsilon,\eta'}, C_{\varepsilon,\sigma'}) .$$

PROOF. If $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_r)$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_t)$, then σ' and η' are written as $\sigma' = (\sigma_s, \cdots, \sigma_r)$ and $\eta' = (\eta_s, \cdots, \eta_t)$ with $(\sigma_1, \cdots, \sigma_s) = (\eta_1, \cdots, \eta_s)$ for some integer s. Put $\nu = (\sigma_1, \cdots, \sigma_s)$. Let (σ, α, β) be an ε-datum which contains σ. We may assume that $\beta = \mathrm{id}$ if $\varepsilon = 1$ and $\beta(2i + 1) = 2i + 2$, $\beta(2i + 2) = 2i + 1$ $(i \geqq 0)$ if $\varepsilon = -1$. Choose $E \in C_{\varepsilon,\sigma}$ and the Jordan basis $\{E^a v_i; 1 \leqq i \leqq r,\ 0 \leqq a \leqq \sigma_i - 1\}$ of E such that

$$(E^a v_i, E^b v_j) = \begin{cases} \alpha(i) & (\beta(i) = j \text{ and } a + b + 1 = \sigma_i) \\ 0 & \text{otherwise} . \end{cases}$$

Let W and V' be subspaces spanned by $\{E^a v_i; 1 \leqq i \leqq s, 0 \leqq a \leqq \sigma_i - 1\}$ and $\{E^a v_i; s < i < r, 0 \leqq a \leqq \sigma_i - 1\}$, respectively. Then we have an orthogonal decomposition $V = W \oplus V'$. With respect to the restrictions of $(\, , \,)$, W and V' become quadratic spaces of type ε. Since W and V' are E-stable, E is decomposed as $E = (F, E')$ where $F \in C_{\varepsilon,\nu}$ $(\subset \mathfrak{p}(W))$ and $E' \in C_{\varepsilon,\sigma'}$ $(\subset \mathfrak{p}(V'))$. Take $D' \in C_{\varepsilon,\eta'}$ $(\subset \mathfrak{p}(V'))$ and put $D = (F, D')$. Then $D \in C_{\varepsilon,\eta}$. Now we use the notations of Remark 2 in (1.3) for $z = E$ and $\lambda = \sigma$. It is easy to see that

$$\mathfrak{p}(W) = \bigoplus_{i \leqq j \leqq s} \mathfrak{p}(V)_{i,j}, \quad \mathfrak{p}(V') = \bigoplus_{s < i \leqq j} \mathfrak{p}(V)_{i,j}$$

$$\mathfrak{g}(W) = \bigoplus_{i \leqq j \leqq s} \mathfrak{g}(V)_{i,j} \quad \text{and} \quad \mathfrak{g}(V') = \bigoplus_{s < i \leqq j} \mathfrak{g}(V)_{i,j}.$$

Put

$$Y = \bigoplus_{i \leqq s < j} \mathfrak{p}(V)_{i,j} \quad \text{and} \quad X = \bigoplus_{i \leqq s < j} \mathfrak{g}(V)_{i,j}.$$

Then we have

$$\begin{aligned}
&\mathfrak{p}(V) = \mathfrak{p}(W) \oplus \mathfrak{p}(V') \oplus Y, \quad \mathfrak{g}(V) = \mathfrak{g}(W) \oplus \mathfrak{g}(V') \oplus X, \\
&[X, E] \subset X, \quad [Y, E] \subset X, \\
&[\mathfrak{g}(W), E] \subset \mathfrak{p}(W), \quad [\mathfrak{p}(W), E] \subset \mathfrak{g}(W), \\
&[\mathfrak{g}(V'), E] \subset \mathfrak{p}(V'), \quad [\mathfrak{p}(V'), E] \subset \mathfrak{g}(V'), \\
&\mathfrak{gl}(V) = \mathfrak{p}(W) \oplus \mathfrak{p}(V') \oplus \mathfrak{g}(W) \oplus \mathfrak{g}(V') \oplus Y \oplus X.
\end{aligned}$$

Take vector subspaces N_1, N_2, N_3, N_4 of $\mathfrak{gl}(V)$ such that

$$\begin{aligned}
&\mathfrak{p}(W) \oplus \mathfrak{p}(V') = [\mathfrak{g}(W) \oplus \mathfrak{g}(V'), E] \oplus N_1, \\
&\mathfrak{g}(W) \oplus \mathfrak{g}(V') = [\mathfrak{p}(W) \oplus \mathfrak{p}(V'), E] \oplus N_2, \\
&Y = [X, E] \oplus N_3, \quad X = [Y, E] \oplus N_4.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\mathfrak{gl}(V) = [\mathfrak{gl}(V), E] \oplus N, \quad \mathfrak{p}(V) = [\mathfrak{g}(V), E] \oplus N_0, \\
&\mathfrak{gl}(W) \oplus \mathfrak{gl}(V') = [\mathfrak{gl}(W) \oplus \mathfrak{gl}(V'), E] \oplus N', \\
&\mathfrak{p}(W) \oplus \mathfrak{p}(V') = [\mathfrak{g}(W) \oplus \mathfrak{g}(V'), E] \oplus N_0',
\end{aligned}$$

where

$$N = N_1 \oplus N_2 \oplus N_3 \oplus N_4, \quad N_0 = N_1 \oplus N_3, \quad N' = N_1 \oplus N_2, \quad N_0' = N_1.$$

By putting

$$\begin{aligned}
&S = (N + E) \cap (\mathrm{Ad}(GL(V))D)^-, \quad S_0 = (N_0 + E) \cap (\mathrm{Ad}(G(V))D)^-, \\
&S' = (N' + E) \cap (\mathrm{Ad}(GL(W) \times GL(V'))D)^-, \\
&S_0' = (N_0' + E) \cap (\mathrm{Ad}(G(W) \times G(V'))D)^-,
\end{aligned}$$

we get cross sections of the closures of the orbits containing D at E under the actions of $GL(V)$, $G(V)$, $GL(V')$ and $G(V')$, respectively. Moreover we have

$$\dim_E S = \operatorname{codim}((\operatorname{Ad}(GL(V))D)^-, \operatorname{Ad}(GL(V))E) ,$$
$$\dim_E S' = \operatorname{codim}(\operatorname{Ad}(GL(W)\times GL(V'))D)^-, \operatorname{Ad}(GL(W)\times GL(V'))E)$$

by Remark 7. By [KP2, Proposition 3.1], we have $\dim_E S' = \dim_E S$. By the normality of a closure of a conjugacy class in $\mathfrak{gl}(V)$, $(\operatorname{Ad}(GL(V))D)^-$ is normal at E ([KP1]). But since $\operatorname{Sing}(S, E) = \operatorname{Sing}((\operatorname{Ad}(GL(V))D)^-, E)$, S is normal at E (cf., Remark 4). Since S' is a closed subset of S, S' and S coincide in a suitable neighbourhood of E.

On the other hand, we have

$$\begin{aligned} S'\cap \mathfrak{p}(V) &= \{\mathfrak{p}(V)\cap(N'+E)\}\cap\{\mathfrak{p}(V)\cap(\operatorname{Ad}(GL(W)\times GL(V'))D)^-\} \\ &= \{\mathfrak{p}(V)\cap N'+E\}\cap\mathfrak{p}(V)\cap\{\mathfrak{gl}(W)\oplus\mathfrak{gl}(V')\}\cap\{\operatorname{Ad}(GL(W)\times GL(V'))D\}^- \\ &= \{\mathfrak{p}(V)\cap N'+E\}\cap\{\mathfrak{p}(W)\oplus\mathfrak{p}(V')\cap((\operatorname{Ad}(GL(W)\times GL(V'))D)^-\} \\ &= (N'_0+E)\cap(\operatorname{Ad}(G(W)\times G(V'))D)^- = S'_0 . \end{aligned}$$

Hence $S\cap\mathfrak{p}(V)\supset S_0\supset S'_0 = S'\cap\mathfrak{p}(V)$. Therefore, S_0 and S'_0 also coincide in a suitable neighbourhood of E. Thus we get

$$\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, E) = \operatorname{Sing}(\bar{C}_{\varepsilon,\nu}\times\bar{C}_{\varepsilon,\eta'}, (F, E')) .$$

But since F is a smooth point of $\bar{C}_{\varepsilon,\nu}$, we have

$$\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, E) = \operatorname{Sing}(\bar{C}_{\varepsilon,\eta'}, E') . \qquad \text{q.e.d.}$$

(2.4) Singularities of minimal degenerations.

DEFINITION. Let $\sigma < \eta$ be an ε-degeneration.

(i) We say that $\sigma < \eta$ is minimal if there is no ε-diagram ν such that $\sigma < \nu < \eta$.

(ii) We say that $\sigma < \eta$ is irreducible if it cannot be obtained by addition of rows and columns in a nontrivial way.

Here we shall give a description of smooth equivalence classes of minimal ε-degenerations.

REMARK 8. (i) Let X be an element of $\mathfrak{p}(V)$. Then we have

$$\dim \mathfrak{z}_{\mathfrak{g}(V)}(X) - \dim \mathfrak{z}_{\mathfrak{p}(V)}(X) = \dim \mathfrak{g}(V) - \dim \mathfrak{p}(V)$$

by [KR, Proposition 5], where $\mathfrak{z}_{\mathfrak{g}(V)}(X)$ and $\mathfrak{z}_{\mathfrak{p}(V)}(X)$ are the centralizers of X in $\mathfrak{g}(V)$ and $\mathfrak{p}(V)$, respectively. It follows from this that

$$\dim GL(V)X = 2\dim G(V)X .$$

(ii) In the setting of Theorem 2, we have

$$\operatorname{codim}(\bar{C}_{\varepsilon,\eta'}, C_{\varepsilon,\sigma'}) = \operatorname{codim}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma})$$

by (i) above and [KP2, Proposition 3.1]. Moreover $\sigma' \leqq \eta'$ is minimal if and only if $\sigma \leqq \eta$ is minimal.

(iii) Any ε-degeneration is obtained in a unique way from an irreducible ε-degeneration by addition of rows and columns.

From the view-point of Remark 8, for the classification of the minimal ε-degerations, one should first describe the minimal irreducible ε-degenerations. They are given in Table 1.

TABLE 1

ε	1	1	-1	-1
η	(n)	$(2, 1^{n-2})$	(m, m)	$(2^2, 1^{2m-4})$
σ	$(n-1, 1)$	1^n	$(m-1, m-1, 1, 1)$	1^{2m}
$\operatorname{codim}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma})$	1	$n-1$	4	$4(m-1)$
$\operatorname{Sing}(\bar{C}_{\varepsilon,\eta}, C_{\varepsilon,\sigma})$	x_n	$x_n{}^*$	y_m	$y_m{}^*$

The notations x_n, x_n^*, y_m and y_m^* in Table 1 are defined as follow.

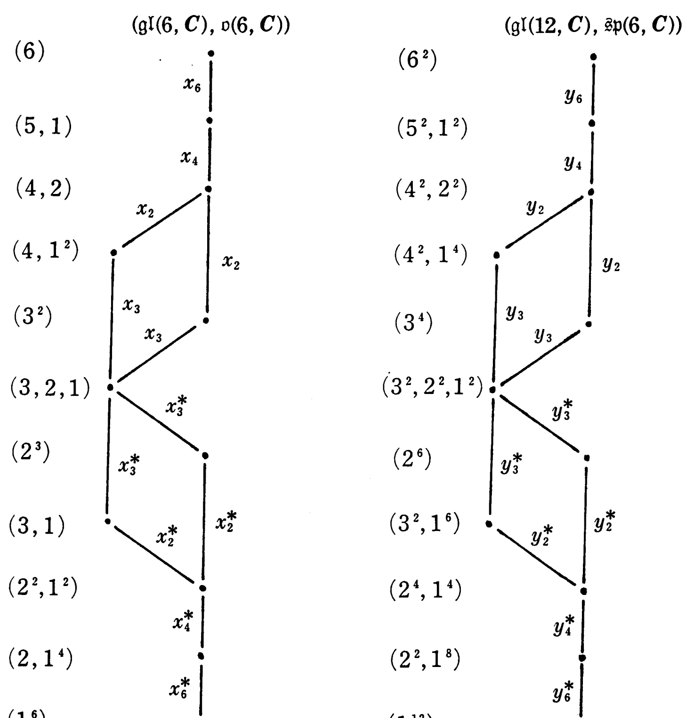

FIGURE 3

As in Sekiguchi [S], x_n (resp. y_m) is the smooth equivalence class of the variety defined by $x^n + y^2 = 0$ in C^2 (resp. $x^m + y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0$ in C^5) at the origin. On the other hand, x_n^* (resp. y_m^*) is the smooth equivalence class of the closure of the nonzero minimal nilpotent orbit in $\mathfrak{p}(V)$ at 0, where $\dim V = n$ and $\varepsilon = 1$ (resp. $\dim V = 2m$ and $\varepsilon = -1$). Since the origin of the variety defined by $x^n + y^2 = 0$ is not a normal point, the closure of a nilpotent orbit in $\mathfrak{p}(V)$ is not normal in general when $\varepsilon = 1$.

EXAMPLE. The closure relation and the minimal singularities of the closures of nilpotent orbits in $(\mathfrak{gl}(6, C), \mathfrak{o}(6, C))$ and $(\mathfrak{gl}(12, C), \mathfrak{sp}(6, C))$ are given as in Figure 3. (Note that $x_2^* = x_2$ and $y_2^* = y_2$.)

3. Normality of the closures of nilpotent orbits in $(\mathfrak{gl}(2m, C), \mathfrak{sp}(m, C))$.

(3.1) Dimension formula. In this section, we prove that the closures of nilpotent orbits in $\mathfrak{p}(V)$ are normal in case $\varepsilon = -1$. For this, we need a certain dimension formula. The normality is not true in case $\varepsilon = 1$ as in (2.4). But we will also give this formula in case $\varepsilon = 1$, since the formula suggests the difficulty in giving a sufficient condition for the closure of a nilpotent orbit to be normal.

Let U and V be two quadratic spaces of type ε. By putting $(\ ,\ )_{U\oplus V} = (\ ,\ )_U + (\ ,\ )_V$, $U \oplus V$ is a quadratic space of type ε. Put $\tilde{\mathfrak{g}} = \mathfrak{gl}(U \oplus V)$ and define two involutions σ and θ of $\tilde{\mathfrak{g}}$ as a Lie algebra by $\sigma(X) = -X^*$ and $\theta(X) = JXJ^{-1}$ for $X \in \tilde{\mathfrak{g}}$, where

$$J = \begin{bmatrix} 1_U & 0 \\ 0 & -1_V \end{bmatrix}.$$

Note that $X \mapsto X^*$ gives a linear anti-involution (i.e., $(XY)^* = Y^*X^*$) of $\tilde{\mathfrak{g}}$ as an associative algebra and θ is a linear involution of $\tilde{\mathfrak{g}}$ as an associative algebra. Since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}$$

for $A \in \mathfrak{gl}(U)$, $B \in L(V, U)$, $C \in L(U, V)$, $D \in \mathfrak{gl}(V)$, we have $\sigma \cdot \theta = \theta \cdot \sigma$. Hence we have a direct sum decomposition

$$\tilde{\mathfrak{g}} = (\tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^\theta) \oplus (\tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^{-\theta}) \oplus (\tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^\theta) \oplus (\tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^{-\theta}),$$

where $\tilde{\mathfrak{g}}^\tau = \{X \in \tilde{\mathfrak{g}}; \tau(X) = X\}$ for a linear map $\tau: \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$. Here $\tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^{-\theta}$ and $\tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^{-\theta}$ are given by

$$\tilde{\mathfrak{g}}^{\sigma}\cap\tilde{\mathfrak{g}}^{-\theta}=\left\{\begin{bmatrix}0 & B\\ -B^* & 0\end{bmatrix};\ B\in L(V,U)\right\},$$
$$\tilde{\mathfrak{g}}^{-\sigma}\cap\tilde{\mathfrak{g}}^{-\theta}=\left\{\begin{bmatrix}0 & B\\ B^* & 0\end{bmatrix};\ B\in L(V,U)\right\}.$$

Define $\mathfrak{g}'$, $\mathfrak{g}$, $\tilde{G}$, G' and G by

$$\mathfrak{g}':=\tilde{\mathfrak{g}}^{\theta}=\mathfrak{gl}(U)\oplus\mathfrak{gl}(V)\,,\quad \mathfrak{g}:=\tilde{\mathfrak{g}}^{\sigma}\cap\tilde{\mathfrak{g}}^{\theta}=\mathfrak{g}(U)\oplus\mathfrak{g}(V)\,,$$
$$\tilde{G}:=GL(U\oplus V)\,,\quad G':=\tilde{G}^{\theta}=GL(U)\times GL(V)\,,$$
$$G:=\{g\in G';\, g^*=g^{-1}\}=G(U)\times G(V)\,.$$

Then the group G' acts on $\tilde{\mathfrak{g}}^{-\theta}$ and the group G acts on $\tilde{\mathfrak{g}}^{\sigma}\cap\tilde{\mathfrak{g}}^{-\theta}$ by the adjoint action. Since the map

$$L(V,U)\to\tilde{\mathfrak{g}}^{\sigma}\cap\tilde{\mathfrak{g}}^{-\theta},\ B\mapsto\begin{bmatrix}0 & B\\ -B^* & 0\end{bmatrix}$$

is a G-equivariant isomorphism, we can identify $\tilde{\mathfrak{g}}^{\sigma}\cap\tilde{\mathfrak{g}}^{-\theta}$ with $L(V,U)$ as G-modules.

PROPOSITION 4. *Let k be an algebraically closed field with* $\mathrm{char}(k)\neq 2$ *and $\tilde{\mathfrak{g}}=\mathfrak{gl}(n,k)$. Let θ be a linear involution of the associative algebra $\tilde{\mathfrak{g}}$ and $X\mapsto X^*$ a linear anti-involution of the associative algebra $\tilde{\mathfrak{g}}$ commuting with θ. Put*

$$\mathfrak{g}'=\tilde{\mathfrak{g}}^{\theta}\,,\quad G'=\mathfrak{g}'\cap GL(n,k)\quad\text{and}\quad G=\{g\in G';\, g^*=g^{-1}\}\,.$$

Then G' acts on $\tilde{\mathfrak{g}}^{-\theta}$ and G acts on $\tilde{\mathfrak{g}}^{\sigma}\cap\tilde{\mathfrak{g}}^{-\theta}$ by the adjoint action, where $\sigma(X)=-X^*$. For $X, Y\in\tilde{\mathfrak{g}}^{\sigma}\cap\tilde{\mathfrak{g}}^{-\theta}$, X and Y are conjugate under G if and only if they are conjugate under G'

PROOF. Suppose that $Y=gXg^{-1}$ for some $g\in G'$. Then we have $gXg^{-1}=(g^*)^{-1}Xg^*$ and hence

$$gg^*\in Z_{G'}(X):=\{h\in G';\, hX=Xh\}\,.$$

Put $v=g^{-1}(g^*)^{-1}\in Z_{G'}(X)$. We note the following fact which is easily checked by the Chinese remainder theorem; for a non-singular matrix $A\in GL(n,k)$, there exists a polynomial $f(T)\in k[T]$ such that $A=f(A)^2$.

Take a polynomial $f(T)\in k[T]$ so that $v=f(v)^2$. It is easy to see that $f(v)\in Z_{G'}(X)$ and $f(v)^*=f(v)$. Hence $g^{-1}(g^*)^{-1}=v=f(v)^2=f(v)f(v)^*$ and hence $gf(v)\in G$. Thus $Y=gXg^{-1}=(gf(v))X(gf(v))^{-1}$ with $gf(v)\in G$.

q.e.d.

In order to classify G-orbits in $\{A\in L(V,U);\ A^*A$ is nilpotent$\}$, we first describe the classification of nilpotent $G'=GL(U)\times GL(V)$-orbits in

$$\tilde{\mathfrak{g}}^{-\theta} = \left\{ \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix};\ A \in L(V, U),\ B \in L(U, V) \right\}$$

due to [KP1]. For any nilpotent element X of $\tilde{\mathfrak{g}}^{-\theta}$, we can take a Jordan basis

$$\{X^a u_i;\ 1 \leqq i \leqq r_1,\ 0 \leqq a \leqq \lambda_i - 1\} \cup \{X^b v_j;\ 1 \leqq j \leqq r_2,\ 0 \leqq b \leqq \mu_j - 1\}$$

of X such that $u_i \in U$ and $v_j \in V$. By letting a string

$$\overbrace{abab\cdots}^{\lambda_i}. \qquad (\text{resp. } \overbrace{baba\cdots}^{\mu_j}.)$$

correspond to $\{X^a u_i;\ 0 \leqq a \leqq \lambda_i - 1\}$ (resp. $\{X^b v_j;\ 0 \leqq b \leqq \mu_j - 1\}$), we get a diagram τ_X which is the sum of such strings. For example, if $\lambda_1 = 3$ $(r_1 = 1)$ and $\mu_1 = 5$, $\mu_2 = 2$ $(r_2 = 2)$, then

$$\begin{aligned} \tau_X = {} & babab \\ & aba \\ & ba\ . \end{aligned}$$

Such a diagram is called an *ab*-diagram. It is easy to see that the *ab*-diagram τ_X is independent of the choice of a Jordan basis. Therefore, we call τ_X the *ab*-diagram of X. If X and Y are nilpotent elements of $\tilde{\mathfrak{g}}^{-\theta}$, we see that $\tau_X = \tau_Y$ if and only if X and Y are conjugate under G'. Thus we have a one-to-one correspondence between the set of nilpotent G'-orbits in $\tilde{\mathfrak{g}}^{-\theta}$ and the set of *ab*-diagrams τ such that $n_a(\tau) = \dim U$ and $n_b(\tau) = \dim V$, where $n_a(\tau)$ (resp. $n_b(\tau)$) is the number of a's (resp. b's) in τ.

By Proposition 4 and the above classification, G-orbits in $\{A \in L(V, U);\ A^*A \text{ is nilpotent}\} \simeq \{X \in \tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta};\ X \text{ is nilpotent}\}$ are classified by the *ab*-diagrams τ such that $n_a(\tau) = \dim U$ and $n_b(\tau) = \dim V$. The following dimension formula plays an important role in proving the normality of the closures of nilpotent orbits.

PROPOSITION 5. *Let X be an element of $L(V, U)$ such that X^*X is nilpotent. Let O_X be the $G = G(U) \times G(V)$-orbit of X and τ the ab-diagram of X. Also let*

$$\begin{array}{ccc} L(V, U) & \xrightarrow{\ \pi\ } & \mathfrak{p}(U) \\ \rho \big\downarrow & & \\ \mathfrak{p}(V) & & \end{array}$$

be the maps introduced in (2.2). *Denote by a_i (resp. b_i) the number of the rows of τ of length i starting with a (resp. b) and put*

$$\Delta_\tau = \sum_{i;\,\mathrm{odd}} a_i b_i \,.$$

Then we have

$$\dim O_X = \frac{1}{2}(\dim \pi(O_X) + \dim \rho(O_X) + nm - \Delta_\tau) - \frac{\varepsilon}{4}(n + m - o(\tau)) \,,$$

where $o(\tau)$ is the number of the rows of τ having odd length, $m = \dim U$ and $n = \dim V$.

The proof of this proposition is given in (3.3).

(3.2) Normality of closures of nilpotent orbits. Let V be a quadratic space of type ε and $D \in \mathfrak{p}(V)$ be a nilpotent element with $G(V)$-orbit $C_D = C_{\varepsilon,\eta}$. In (2.2) we have canonically defined a non-degenerate ε-form (i.e., $(u, v) = \varepsilon(v, u)$) on $D(V)$ such that two maps

$$V \underset{I}{\overset{X^{\cdot}}{\rightleftarrows}} D(V) \quad (D = IX^{\cdot}\text{: the canonical decomposition})$$

are adjoint (i.e., $(X^{\cdot})^* = I$) and that $D|D(V) = X^{\cdot}I \in C_{\varepsilon,\eta'}$, where η' is the ε-diagram we obtain from η by erasing the first column. Repeating this we get a sequence of quadratic spaces

$$V_0 := V\,,\quad V_1 := D(V),\ \cdots,\quad V_i := D^i(V),\ \cdots,$$
$$V_t := D^t(V) \neq 0\,,\quad V_{t+1} := D^{t+1}(V) = 0\,,$$

of type ε and we have $D|V_i \in C_{\varepsilon,\eta^i} \subset \mathfrak{p}(V_i)$, where η^i is the ε-diagram we obtain from η by erasing the first i columns.

Now we consider the variety

$$Z \subset M := L(V_0, V_1) \times L(V_1, V_2) \times \cdots \times L(V_{t-1}, V_t)$$

defined by the following equations;

$$(*)\qquad X_1X_1^* = X_2^*X_2 \quad X_2X_2^* = X_3^*X_3,\ \cdots,\ X_{t-1}X_{t-1}^* = X_t^*X_t\,,\quad X_tX_t^* = 0\,.$$

The group $G(V_0) \times G(V_1) \times \cdots \times G(V_t)$ acts on M by the action

$$(g_0, g_1, \cdots, g_t)(X_1, X_2, \cdots, X_t) = (g_1X_1g_0^{-1}, g_2X_2g_1^{-1}, \cdots, g_tX_tg_{t-1}^{-1})\,.$$

Clearly Z is stable under $G(V_0) \times G(V_1) \times \cdots \times G(V_t)$. As in [KP3, 5.2], we have the following:

REMARK 8. (i) For any $(X_1, X_2, \cdots, X_t) \in Z$, we have

$$X_i^*X_i \in \bar{C}_{\varepsilon,\eta^i} \qquad (1 \leqq i)\,.$$

(ii) Put $X_i^{\cdot} := D|V_{i-1}\colon V_{i-1} \to V_i \in L(V_{i-1}, V_i)$. Then $(X_1^{\cdot}, X_2^{\cdot}, \cdots, X_t^{\cdot}) \in Z$.

(iii) By (i) we can define a map $\phi: Z \to \bar{C}_{\varepsilon,\eta}$ by $\phi(X_1, \cdots, X_t) = X_1^* X_1$. The map ϕ is clearly $G(V_0)$-equivariant and hence $\phi(Z) \supset C_{\varepsilon,\eta}$.

Let

$$\begin{array}{ccc} L(V_{i-1}, V_i) & \xrightarrow{\ \pi\ } & \mathfrak{p}(V_i) \supset \bar{C}_{\varepsilon,\eta^i} \\ \rho \big\downarrow & & \\ \mathfrak{p}(V_{i-1}) \supset \bar{C}_{\varepsilon,\eta^{i-1}} & & \end{array}$$

be the map introduced in (2.2) and put

$$N_{\varepsilon,\eta^{i-1}} := \pi^{-1}(\bar{C}_{\varepsilon,\eta^i}) .$$

Then $\rho(N_{\varepsilon,\eta^i}) = \bar{C}_{\varepsilon,\eta^i}$ by Lemma 8. We see that Z is the iterated fibre product as in Figure 4:

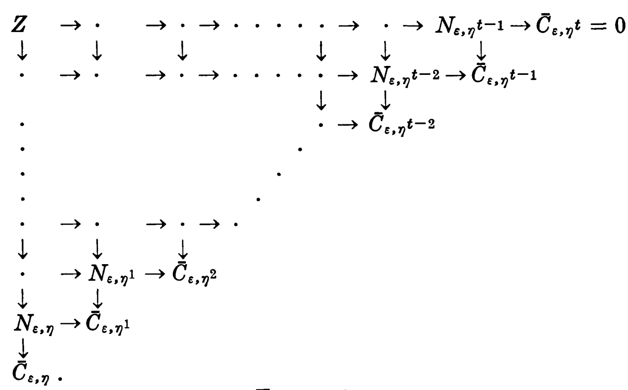

FIGURE 4

Note that $G(V_i) \times G(V_{i-1})$-orbits in $\{X \in L(V_{i-1}, V_i);\ X^*X \text{ is nilpotent}\}$ are classified by the ab-diagrams τ such that $n_a(\tau) = \dim V_i$ and $n_b(\tau) = \dim V_{i-1}$ as in (3.1). Hence for such an ab-diagram τ, we denote by O_τ the corresponding orbit. We also denote by $\pi(\tau)$ (resp. $\rho(\tau)$) the Young diagram we obtain from τ by erasing b's (resp. a's). Then we have $\pi(O_\tau) = C_{\varepsilon,\pi(\tau)}$ and $\rho(O_\tau) = C_{\varepsilon,\rho(\tau)}$ as in [KP3, 6.4]. Consider the finite set Λ of strings $\lambda = (\tau_1, \tau_2, \cdots, \tau_t)$ of ab-diagrams τ_i corresponding to a nonempty orbit $O_{\tau_i} \subset L(V_{i-1}, V_i)$ satisfying the following:

(a) $\pi(\tau_i) = \rho(\tau_{i+1})$ (i.e., $\pi(O_{\tau_i}) = \rho(O_{\tau_{i+1}})$)

for $i = 1, \cdots, t$.

(b) $\pi(\tau_t) = 0$ (i.e., $\pi(O_{\tau_t}) = 0$).

Let $\lambda = (\tau_1, \cdots, \tau_t) \in \Lambda$ and put $\sigma_i = \pi(\tau_i) = \rho(\tau_{i+1})$ $(i = 1, \cdots, t)$, $\sigma = \sigma_0 = \rho(\tau_1)$ and $\sigma_t = \pi(\tau_t)$. Then we have $C_{\varepsilon,\sigma_i} \subset \bar{C}_{\varepsilon,\eta^i} = \bar{C}_{D|V_i}$ as in [KP3, 8.1]. Put

$$Z_\lambda = \{(X_1, \cdots, X_t) \in Z;\, X_i \in O_{\tau_i}\} = Z \cap (O_{\tau_1} \times \cdots \times O_{\tau_t}) \,.$$

Then we can see that Z_λ is the iterated fibre product as in Figure 5, where $\lambda_i := (\tau_{i+1}, \cdots, \tau_t)$.

$$\begin{array}{llllllll}
Z_\lambda \to & Z_{\lambda_1} \to & Z_{\lambda_2} \to & \cdots\cdots \to & Z_{\lambda_{t-2}} \to & O_{\tau_t} \to & C_{\varepsilon,\sigma_t} = 0 \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \\
\cdot \to & \cdot \to & \cdot \to & \cdots\cdots \to & O_{\tau_{t-1}} \to & C_{\varepsilon,\sigma_{t-1}} & \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \\
\cdot \to & \cdot \to & \cdot \to & \cdots\cdots \to & C_{\varepsilon,\sigma_{t-2}} & & \\
\downarrow & \downarrow & \downarrow & & & & \\
\vdots & \vdots & \vdots & \iddots & & & \\
\downarrow & \downarrow & \downarrow & & & & \\
\cdot \to & O_{\tau_2} \to & C_{\varepsilon,\sigma_2} & & & & \\
\downarrow & \downarrow & & & & & \\
O_{\tau_1} \to & C_{\varepsilon,\sigma_1} & & & & & \\
\downarrow & & & & & & \\
C_{\varepsilon,\sigma_0} & & & & & &
\end{array}$$

FIGURE 5

Since the maps

$$\begin{array}{ll}
O_{\tau_i} \to & C_{\varepsilon,\sigma_i} \\
\downarrow & \\
C_{\varepsilon,\sigma_{i-1}} &
\end{array}$$

are smooth, all the maps and the varieties in this diagram are smooth. If $\varepsilon = -1$, then $G(V_i)$ is isomorphic to $Sp(m_i, C)$ ($m_i = \dim V_i/2$) and hence Z_λ is irreducible. Now Z is a disjoint union

$$Z = \bigcup_{\lambda \in \Lambda} Z_\lambda \,.$$

As in [KP3, 8.1], the dimension of Z_λ is given as follows by Proposition 5.

PROPOSITION 6. *For any* $\lambda = (\tau_1, \cdots, \tau_t) \in \Lambda$, *we have*

$$\dim Z_\lambda = \frac{1}{2}\dim C_{\varepsilon,\sigma} + \sum_{i=0}^{t-1}\left\{\frac{1}{2}n_i n_{i+1} - \frac{\varepsilon}{4}(n_i + n_{i+1})\right\} - \frac{1}{2}\Delta_\lambda + \frac{\varepsilon}{4}o(\lambda) \,,$$

where

$$\Delta_\lambda = \sum_{i=1}^{t} \Delta_{\tau_i} \,, \quad o(\lambda) = \sum_{i=1}^{t} o(\tau_i) \quad \textit{and} \quad n_i = \dim V_i \,.$$

PROPOSITION 7. *Let σ be an ε-diagram such that $C_{\varepsilon,\sigma} \subset \bar{C}_D = \bar{C}_{\varepsilon,\eta}$ (i.e., $\sigma \leqq \eta$) and $\phi: Z \to \bar{C}_D$ the map in Remark* 8. *Then*

(i) $\dim \phi^{-1}(C_{\varepsilon,\sigma}) = (1/2)\dim C_{\varepsilon,\sigma} + \sum_{i=0}^{t-1}\{(1/2)n_i n_{i+1} - (\varepsilon/4)(n_i + n_{i+1})\} + \max\{(\varepsilon/4)o(\lambda) - (1/2)\Delta_\lambda;\, \lambda = (\tau_1, \cdots, \tau_t) \in \Lambda,\, \rho(\tau_1) = \sigma\}$.

(ii) *If* $\varepsilon = -1$, we have

$$\operatorname{codim}(Z, \phi^{-1}(C_{\varepsilon,\sigma})) \geqq \frac{1}{2}\operatorname{codim}(\bar{C}_D, C_{\varepsilon,\sigma}) .$$

PROOF. Since $\phi^{-1}(C_{\varepsilon,\sigma})$ is the union of Z_λ for $\lambda = (\tau_1, \cdots, \tau_t) \in \Lambda$ with $\rho(\tau_1) = \sigma$, (i) follows from Proposition 6.

(ii) Let $\tau_i^\cdot$ be the ab-diagram of $X_i^\cdot = D|V_{i-1} \in L(V_{i-1}, V_i)$ and $\lambda^\cdot = (\tau_1^\cdot, \cdots, \tau_t^\cdot) \in \Lambda$. For any $\lambda = (\tau_1, \cdots, \tau_t) \in \Lambda$ with $\rho(\tau_1) = \sigma$, we have

$$\dim Z_{\lambda^\cdot} - \dim Z_\lambda = \frac{1}{2}(\dim C_{\varepsilon,\eta} - \dim C_{\varepsilon,\sigma}) - \frac{1}{2}(\Delta_{\lambda^\cdot} - \Delta_\lambda) + \frac{1}{4}(o(\lambda) - o(\lambda^\cdot))$$

by Proposition 6. Since $X_i^\cdot: V_{i-1} \to V_i$ is surjective, each row of $\tau_i^\cdot$ starts with b (cf. [KP1, Remark 2]). Thus we have $a_i = 0$ and hence $\Delta_{\lambda^\cdot} = 0$.

Now we claim that $o(\lambda) \geqq o(\lambda^\cdot)$. Since $\rho(\tau_i^\cdot) = \eta^{i-1}$, $\pi(\tau_i^\cdot) = \eta^i$ and η^i is the ε-diagram we obtain from η^{i-1} by erasing the first column, in each row of $\tau_i^\cdot$ the number of the a's is one fewer than the number of b's. Therefore, the length of each row of $\tau_i^\cdot$ is odd and hence we have $o(\tau_i^\cdot) = |\eta^{i-1}| - |\eta^i| = n_b(\tau_i^\cdot) - n_a(\tau_i^\cdot)$ where $|\eta^i|$ is the size of the Young diagram η^i. Let A_j (resp. B_j) be the number of the a's (resp. b's) in the j-th row of τ_i. Then

$$o(\tau_i) = {}^\#\{j; B_j - A_j \neq 0\} = \sum_j |B_j - A_j| \geqq |\sum_j (B_j - A_j)| = n_b(\tau_i^\cdot) - n_a(\tau_i^\cdot) = o(\tau_i^\cdot)$$

and hence $o(\lambda) \geqq o(\lambda^\cdot)$.

Thus $\dim Z_{\lambda^\cdot} - \dim Z_\lambda \geqq (1/2)\operatorname{codim}(\bar{C}_D, C_{\varepsilon,\sigma})$ and hence $Z_{\lambda^\cdot}$ has the maximal dimension among all Z_λ with $\lambda \in \Lambda$. Since $Z = \cup_{\lambda \in \Lambda} Z_\lambda$ is a finite union, we have $\dim Z_{\lambda^\cdot} = \dim Z$. Then (ii) easily follows from this. q.e.d.

PROPOSITION 8. *Suppose that* $\varepsilon = -1$. *Then*:

(i) *The scheme* Z *defined by the equations* $(*)$ *is irreducible, reduced (hence* Z *is a variety) and a complete intersection in* M.

(ii) *The map* $\phi: Z \to \bar{C}_D$ *is the quotient map under* $G(V_1) \times \cdots \times G(V_t)$.

PROOF. Consider the map

$$\zeta: M = \prod_{i=1}^{t} L(V_{i-1}, V_i) \to \prod_{i=1}^{t} \mathfrak{p}(V_i) =: N$$

defined by $\zeta(X_1, \cdots, X_t) := (X_1X_1^* - X_2^*X_2, X_2X_2^* - X_3^*X_3, \cdots, X_tX_t^*)$. Then Z, as a scheme, is the scheme-theoretic fibre $\zeta^{-1}(0)$. As in [KP3, 5.5], ζ is smooth in $M' := \{(X_1, \cdots, X_t);$ all X_i are surjective$\} = \prod_{i=1}^t L'(V_{i-1}, V_i)$. In particular Z is smooth in $Z' := Z \cap M'$. Since $X_i^\cdot = D|V_{i-1} \in L(V_{i-1}, V_i)$ is surjective, $(X_1^\cdot, \cdots, X_t^\cdot)$ is contained in Z' and hence $Z' \neq \varnothing$. Thus $\operatorname{codim}(M, Z') = \dim N$. By the property of the ab-diagram $\tau_i^\cdot$ of $X_i^\cdot$ stated in the proof of Proposition 7, $\lambda^\cdot = (\tau_1^\cdot, \cdots, \tau_t^\cdot)$ is the only element of Λ

such that $\rho(\tau_1) = \eta$. Since

$$\dim Z_{\lambda^\cdot} - \dim Z_\lambda \geqq \frac{1}{2}\operatorname{codim}(\bar{C}_D, C_{\varepsilon,\sigma})$$

for $\lambda = (\tau_1, \cdots, \tau_r) \in \Lambda$ with $\sigma = \rho(\tau_1^\cdot)$, only $Z_{\lambda^\cdot}$ has the maximal dimension among all Z_λ. Since $Z_{\lambda^\cdot} \subset Z'$, we have $\operatorname{codim}(M, Z) = \dim N$ and hence Z is a complete intersection in M.

Since each Z_λ ($\lambda \in \Lambda$) is irreducible, the irreducible components of Z are of the form $\bar{Z}_\lambda$. But since $\dim \bar{Z}_\lambda < \dim Z = \dim M - \dim N$ for $\lambda \in \Lambda$ with $\lambda \neq \lambda^\cdot$ and since each irreducible component of the fibre $\zeta^{-1}(0) = Z$ must have dimension $\geqq \dim M - \dim N$, $\bar{Z}_{\lambda^\cdot}$ is the only irreducible component of Z and hence Z is irreducible.

Since Z is irreducible, smooth in Z' and a complete intersection in M, Z is reduced.

(ii) is proved by Theorem 3 as in the proof of [KP3, Theorem 5.3 (i)]. q.e.d.

Now we give the main result of this section.

THEOREM 4. *Suppose that* $\varepsilon = -1$. *Then the closure* $\bar{C}_D$ *of the nilpotent* $G(V)$*-orbit* C_D *in* $\mathfrak{p}(V)$ *is a normal variety.*

PROOF. Let $S(Z)$ be the singular locus of Z. Since $Z_{\lambda^\cdot}$ is smooth and Z is a disjoint union

$$Z = Z_{\lambda^\cdot} \cup \left(\bigcup_{\sigma<\eta} \phi^{-1}(C_{\varepsilon,\sigma})\right),$$

we have

$$S(Z) \subset \bigcup_{\sigma<\eta} \phi^{-1}(C_{\varepsilon,\sigma}).$$

Let σ_0 be an ε-diagram such that $\sigma_0 < \eta$ and $\dim \phi^{-1}(C_{\varepsilon,\sigma_0})$ is maximal. Then we have

$$\operatorname{codim}(Z, S(Z)) \geqq \operatorname{codim}(Z, \phi^{-1}(C_{\varepsilon,\sigma_0})) \geqq \frac{1}{2}\operatorname{codim}(\bar{C}_D, C_{\varepsilon,\sigma_0})$$

by Proposition 7 (ii). By Remark 8 (ii) and Table 1 in (2.4), it is easy to see that $\operatorname{codim}(\bar{C}_D, C_{\varepsilon,\sigma_0}) \geqq 4$. Thus Z is non-sigular in codimension 1 and a complete intersection in M. Hence Z is normal. Since $\phi: Z \to \bar{C}_D$ is a quotient map, $\bar{C}_D$ is also normal. q.e.d.

(3.3) Proof of the dimension formula. We now prove Proposition 5. We use the notations introduced in (3.1).

Let $X \in \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^{-\theta}$ be a non-zero nilpotent element. Since $\theta|\tilde{\mathfrak{g}}^\sigma: \tilde{\mathfrak{g}}^\sigma \to \tilde{\mathfrak{g}}^\sigma$ is an involution, $(\tilde{\mathfrak{g}}^\sigma, \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^\theta)$ is a symmetric pair. By Kostant and Rallis [KR, Proposition 4], we can take a normal S-triple (H, X, Y) which contains X as a nilpositive element (i.e., $H \in \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^\theta$, $Y \in \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^{-\theta}$). Then $\tilde{\mathfrak{g}}$ is decomposed as

$$\tilde{\mathfrak{g}} = \bigoplus_{i \in \boldsymbol{Z}} \tilde{\mathfrak{g}}_i , \quad \tilde{\mathfrak{g}}_i := \{A \in \tilde{\mathfrak{g}}; [H, A] = iA\} .$$

Put

$$\tilde{\mathfrak{p}} = \bigoplus_{i \geqq 0} \tilde{\mathfrak{g}}_i \quad \text{and} \quad \tilde{\mathfrak{n}} = \bigoplus_{i > 0} \tilde{\mathfrak{g}}_i .$$

The $\tilde{\mathfrak{p}}$ is a parabolic subalgebra of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{n}}$ is a Levi decomposition. By the representation theory of $\mathfrak{sl}_2$, we have the following lemma.

LEMMA 10. (a) $\mathfrak{z}_{\tilde{\mathfrak{g}}}(X) := \{A \in \tilde{\mathfrak{g}}; [A, X] = 0\} \subset \tilde{\mathfrak{p}}$
(b) $X \in \tilde{\mathfrak{n}}_2 := \bigoplus_{i \geqq 2} \tilde{\mathfrak{g}}_i$ *and* ad $X: \tilde{\mathfrak{p}} \to \tilde{\mathfrak{n}}_2$ *is surjective.*

Since $\mathfrak{g}'$ and $\mathfrak{g}$ are H-stable, H defines the $\boldsymbol{Z}$-graduations of $\mathfrak{g}'$ and $\mathfrak{g}$, both induced by the $\boldsymbol{Z}$-graduation of $\tilde{\mathfrak{g}}$. Hence $\mathfrak{p}' := \tilde{\mathfrak{p}} \cap \mathfrak{g}'$ (resp. $\mathfrak{p} := \tilde{\mathfrak{p}} \cap \mathfrak{g}$) is a parabolic subalgebra of $\mathfrak{g}'$ (resp. $\mathfrak{g}$) with a Levi decomposition

$$\mathfrak{p}' = \mathfrak{g}_0' \oplus \mathfrak{n}' , \quad \mathfrak{g}_0' := \tilde{\mathfrak{g}}_0 \cap \mathfrak{g}' , \quad \mathfrak{n}' := \tilde{\mathfrak{n}} \cap \mathfrak{g}'$$
$$(\text{resp. } \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{n} , \quad \mathfrak{g}_0 := \tilde{\mathfrak{g}}_0 \cap \mathfrak{g} , \quad \mathfrak{n} := \tilde{\mathfrak{n}} \cap \mathfrak{g}) .$$

LEMMA 11. *Let O_X' (resp. O_X) be the orbit of X under G' (resp. G). Then we have*
(a) $\dim O_X' = \dim \mathfrak{n}' + \dim \mathfrak{n}_2'$, $\dim O_X = \dim \mathfrak{n} + \dim \mathfrak{n}_2$.
(b) $\dim \mathfrak{n}_2' = 2 \dim \mathfrak{n}_2$.

PROOF. (a) Since $H \in \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^\theta$, we have a direct sum decomposition

$$\tilde{\mathfrak{p}} = (\tilde{\mathfrak{p}} \cap \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^\theta) \oplus (\tilde{\mathfrak{p}} \cap \tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^\theta) \oplus (\tilde{\mathfrak{p}} \cap \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^{-\theta}) \oplus (\tilde{\mathfrak{p}} \cap \tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^{-\theta}) .$$

On the other hand, since $[X, \tilde{\mathfrak{g}}^\theta] \subset \tilde{\mathfrak{g}}^{-\theta}$, $[X, \tilde{\mathfrak{g}}^{-\theta}] \subset \tilde{\mathfrak{g}}^\theta$ and $\tilde{\mathfrak{n}}_2 = [X, \tilde{\mathfrak{p}}]$ (Lemma 10), we have

$$[X, \mathfrak{p}'] = [X, \tilde{\mathfrak{p}} \cap \tilde{\mathfrak{g}}^\theta] = \tilde{\mathfrak{n}}_2 \cap \tilde{\mathfrak{g}}^{-\theta} = \mathfrak{n}_2' ,$$
$$[X, \mathfrak{p}] = [X, \tilde{\mathfrak{p}} \cap \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^\theta] = \tilde{\mathfrak{n}}_2 \cap \tilde{\mathfrak{g}}^\sigma \cap \tilde{\mathfrak{g}}^{-\theta} = \mathfrak{n}_2 .$$

By Lemma 10 (a), we have

$$\mathfrak{z}_{\mathfrak{g}'}(X) \subset \tilde{\mathfrak{p}} \cap \mathfrak{g}' = \mathfrak{p}' , \quad \mathfrak{z}_{\mathfrak{g}}(X) := \{A \in \mathfrak{g}; [A, X] = 0\} \subset \tilde{\mathfrak{p}} \cap \mathfrak{g} = \mathfrak{p} .$$

Thus

$$\begin{aligned} \dim O_X' &= \dim[\mathfrak{g}', X] = \dim \mathfrak{g}' - \dim \mathfrak{z}_{\mathfrak{g}'}(X) = \dim \mathfrak{g}' - \dim \mathfrak{z}_{\mathfrak{p}'}(X) \\ &= \dim \mathfrak{g}' - (\dim \mathfrak{p}' - \dim \mathfrak{n}_2') = (\dim \mathfrak{g}' - \dim \mathfrak{p}') + \dim \mathfrak{n}_2' \\ &= \dim \mathfrak{n}' + \dim \mathfrak{n}_2' . \end{aligned}$$

Similarly we have $\dim O_X = \dim \mathfrak{n} + \dim \mathfrak{n}_2$.

(b) Since $JH = HJ$ and $J(\tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta}) = \tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^{-\theta}$, we have

$$J(\tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}_i) = \tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{g}}_i$$

for all $i \in \boldsymbol{Z}$. In particular,

$$J\mathfrak{n}_2 = J(\tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{n}}_2) = \tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{n}}_2 .$$

But since

$$\mathfrak{n}_2' = (\tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{n}}_2) \oplus (\tilde{\mathfrak{g}}^{-\sigma} \cap \tilde{\mathfrak{g}}^{-\theta} \cap \tilde{\mathfrak{n}}_2) = \mathfrak{n}_2 \oplus J\mathfrak{n}_2 ,$$

we have $\dim \mathfrak{n}_2' = 2 \dim \mathfrak{n}_2$. q.e.d.

As in (3.1), we identify $L(V, U)$ with $\tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta}$ via the $G(U) \times G(V)$-equivariant isomorphism

$$L(V, U) \simeq \tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta} , \quad B \mapsto \begin{bmatrix} O & B \\ -B^* & O \end{bmatrix} .$$

The following lemma easily follows from [KP1, Proposition 5.3] and Remark 8, (i).

LEMMA 12. *Let O_X' (resp. O_X) be the orbit of*

$$X = \begin{bmatrix} O & X \\ -X^* & O \end{bmatrix} \in \tilde{\mathfrak{g}}^{\sigma} \cap \tilde{\mathfrak{g}}^{-\theta}$$

under $G' = GL(U) \times GL(V)$ (resp. $G = G(U) \times G(V)$) and τ be the ab-diagram of X. Let $C'_{a(X)}$ (resp. $C'_{b(X)}$) be the orbit of $\pi(X) = X^*X \in \mathfrak{gl}(U)$ (resp. $\rho(X) = XX^* \in \mathfrak{gl}(V)$) under $GL(U)$ (resp. $GL(V)$). Then we have

(a) $\dim O_X' = (1/2)(\dim C'_{a(X)} + \dim C'_{b(X)}) + nm - \Delta_\tau$, *where* $n = \dim V$ *and* $m = \dim U$.

(b) $\dim C'_{a(X)} = 2 \dim \pi(O_X)$, $\dim C'_{b(X)} = 2 \dim \rho(O_X)$.

Now we prove Proposition 5.

PROOF OF PROPOSITION 5. If $U = \bigoplus_i U_i$ and $V = \bigoplus_j V_j$ are the weight space decompositions of U and V with respect to H (i.e., $U_i = \{u \in U; Hu = iu\}$ and $V_j = \{v \in V; Hv = jv\}$) we find

$$\mathfrak{g}_0' = (\bigoplus_i \mathfrak{gl}(U_i)) \oplus (\bigoplus_j \mathfrak{gl}(V_j)) ,$$

$$\mathfrak{g}_0 \simeq (\bigoplus_{i>0} \mathfrak{gl}(U_i)) \oplus \mathfrak{g}(U_0) \oplus (\bigoplus_{j>0} \mathfrak{gl}(V_j)) \oplus \mathfrak{g}(V_0) \quad \text{(as vector spaces)}$$

as in [KP3, 7.7]. Put $d_a = \dim U_0$ and $d_b = \dim V_0$. Then we have

$$\begin{aligned} 2 \dim \mathfrak{g}_0 - \dim \mathfrak{g}_0' &= 2 \dim \mathfrak{g}(U_0) + 2 \dim \mathfrak{g}(V_0) - \dim \mathfrak{gl}(U_0) - \dim \mathfrak{gl}(V_0) \\ &= d_a(d_a - \varepsilon) + d_b(d_b - \varepsilon) - d_a^2 - d_b^2 = -\varepsilon(d_a + d_b) . \end{aligned}$$

By Lemma 11, we have

$$4\dim O_X - 2\dim O'_X = 4\dim \mathfrak{n} - 2\dim \mathfrak{n}' .$$

Since $\mathfrak{g} \simeq \mathfrak{n} \oplus \mathfrak{g}_0 \oplus \mathfrak{n}$ and $\mathfrak{g}' \simeq \mathfrak{n}' \oplus \mathfrak{g}'_0 \oplus \mathfrak{n}'$ as vector spaces, we have

$$\begin{aligned} 4\dim O_X - 2\dim O'_X &= 2(\dim \mathfrak{g} - \dim \mathfrak{g}_0) - (\dim \mathfrak{g}' - \dim \mathfrak{g}'_0) \\ &= 2\dim \mathfrak{g} - \dim \mathfrak{g}' + \varepsilon(d_a + d_b) . \end{aligned}$$

Since $\mathfrak{g} = \mathfrak{g}(U) \oplus \mathfrak{g}(V)$ and $\mathfrak{g}' = \mathfrak{gl}(U) \oplus \mathfrak{gl}(V)$,

$$4\dim O_X - 2\dim O'_X = -\varepsilon(m + n - d_a - d_b) .$$

Hence we have

$$\dim O_X = \frac{1}{2}(\dim \pi(O_X) + \dim \rho(O_X) + mn - \Delta_\tau) - \frac{\varepsilon}{4}(m + n - d_a - d_b)$$

by using Lemma 12. But then $d_a + d_b = \dim U_0 + \dim V_0$ coincides with the number of the rows of odd length of the Young diagram of

$$X = \begin{bmatrix} O & X \\ -X^* & O \end{bmatrix}$$

and hence $d_a + d_b = o(\tau)$. q.e.d.

REFERENCES

[DP] C. DE CONCINI AND C. PROCESI, A characteristic free approach to invariant theory, Adv. Math. 21 (1976), 330–354.

[H] W. HESSLINK, Singularities in the nilpotent scheme of a classical group, Trans. Amer. Math. Soc. 222 (1976), 1–32.

[K] B. KOSTANT, Lie group representations on polynomial rings, Amer. J. Math. 85 (1963), 327–404.

[KR] B. KOSTANT AND S. RALLIS, Orbits and representations associated with symmetric spaces, Amer, J. Math. 93 (1971), 753–809.

[KP1] H. KRAFT AND C. PROCESI, Closures of conjugacy classes of matrices are normal, Invent. Math. 53 (1979), 227–247.

[KP2] H. KRAFT AND C. PROCESI, Minimal singularities in GL_n, Invent. Math. 62 (1981), 503–515.

[KP3] H. KRAFT AND C. PROCESI, On the geometry of conjugacy classes in classical groups, Comment. Math. Helv. 57 (1982), 539–602.

[MF] D. MUMFORD AND J. FOGARTY, Geometeic Invariant Theory, 2nd ed., Erg. der Math. 34, Berlin-Heidelberg-New York, Springer Verlag, 1982.

[S] J. SEKIGUCHI, The nilpotent subvariety of the vector space associated to a symmetric pair, Publ. RIMS. Kyoto Univ. 20 (1984), 155–212.

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