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DUALITY OF PROJECTIVE LIMIT SPACES AND INDUCTIVE LIMIT SPACES OVER A NONSPHERICALLY COMPLETE NONARCHIMEDEAN FIELD

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Abstract. A duality theorem of projective and inductive limit spaces over a nonspherically complete valued field is obtained under a certain condition, and topologies of spaces of locally analytic functions are studied.

Introduction. Morita obtained in [5] a duality theorem of projective limit spaces and inductive limit spaces over a spherically complete nonarchimedean valued field, and Schikhof studied in [8] locally convex spaces over a nonspherically complete nonarchimedean valued field. In this paper, we use the results of [8] and study the duality of such spaces over a nonspherically complete nonarchimedean valued field.

The duality theorem of [5] was obtained as a generalization of the results of Komatsu [3] by Morita using the theory of van Tiel [10] about locally convex spaces over a spherically complete nonarchimedean valued field. There the following two facts are used essentially: (i) The Mackey topology is the topology of uniform convergence on weakly c-compact sets; (ii) Any absolutely convex weakly c-compact set is strongly closed. Though we can generalize the notion of c-compactness to our case, it is difficult to obtain good analogues of these two facts over a nonspherically complete valued field. Hence we restrict our attention to a more restricted class than in [5], and prove a duality theorem by making use of van der Put's duality theorem of sequence spaces $c_0 = \{(a_1, a_2, a_3, \cdots) \in K^N;\ |a_m| \to 0\ (m \to \infty)\}$ and $l^\infty = \{(b_1, b_2, b_3, \cdots) \in K^N;\ \sup |b_m| < \infty\}$ over a nonspherically complete valued field K.

We prove a general duality theorem over such a field in Section 1, and apply the theorem to some examples in Section 2. In particular, we generalize the results of Morita [6] to any complete nonarchimedean valued field, and give a positive answer to a question of P. Robba.

We use the notation and terminology of Schikhof [8] throughout this paper.

1. Duality theorem.

1.1. Let K be a complete field with a nontrivial nonarchimedean valuation $|\ |$. We assume in Section 1 that K is *not* spherically complete. For each positive integer m, let $(X_m, |\ |_m)$ and $(Y_m, |\ |_m)$ be Banach spaces over K. We assume that X_m is of countable type. Hence X_m is reflexive (cf. e.g. van Rooij [7, Corollary 4.18]). Let

$$(\ ,\ )_m: X_m \times Y_m \to K$$

be a nondegenerate bicontinuous K-bilinear form such that X_m and Y_m become mutually dual locally convex spaces with respect to $(\ ,\ )_m$. Let $\{u_{m,n}: X_n \to X_m\ (m < n)\}$ be a projective system, and $\{v_{n,m}: Y_m \to Y_n\ (m < n)\}$ be an inductive system such that (i) the $u_{m,n}$'s are K-linear continuous maps, (ii) the $v_{n,m}$'s are K-linear continuous injective maps, and (iii) the equality $(u_{m,n}(x_n), y_m)_m = (x_n, v_{n,m}(y_m))_n$ holds for any $x_n \in X_n$ and $y_m \in Y_m$. Let (X, u_m) be the locally convex projective limit of $\{X_m, u_{m,n}\}$ and let (Y, v_m) be the locally convex inductive limit of $\{Y_m, v_{n,m}\}$. We assume further that (iv) the projection map $u_m: X \to X_m$ has a dense image for each m.

By definition, any element x of the projective limit X can be written as $x = (x_m)$ with $x_m \in X_m$ satisfying $u_{m,n}(x_n) = x_m$ for any m and n with $m < n$, and any element y of the inductive limit space Y can be written as $y = v_m(y_m)$ with some $y_m \in Y_m$. By our assumption (iii), the equality $(u_{m,n}(x_n), y_m)_m = (x_n, v_{n,m}(y_m))_n$ holds for any $m < n$. Hence $(u_m(x), y_m)_m$ does not depend on a special choice of m with $y = v_m(y_m) \in v_m(Y_m)$, and we can define a pairing

$$(\ ,\ ): X \times Y \to K$$

by $(x, y) = (u_m(x), y_m)_m$ with such a $y_m \in Y_m$. It is easy to see that this pairing $(\ ,\ )$ is K-bilinear. Since the projection map $u_m: X \to X_m$ is continuous, our pairing $(\ ,\ )$ is bicontinuous on $X \times Y_m$ for each m. Hence, by the universal mapping property of the inductive limit topology, $(\ ,\ )$ is bicontinuous on $X \times Y$.

Let $x = (x_m)$ be a nonzero element of X. Then $x_m \neq 0$ for some m. Since $(\ ,\ )_m$ is nondegenerate, $(x_m, y_m)_m \neq 0$ for some $y_m \in Y_m$. Hence $(x, y) = (x_m, y_m)_m \neq 0$ for some $y = v_m(y_m) \in v_m(Y_m) \subset Y$. Let $y = v_m(y_m)$ $(y_m \in Y_m)$ be a nonzero element of Y. Then $\{x_m \in X_m; (x_m, y_m)_m \neq 0\}$ is a non-empty open subset of X_m. Since the image of the projection map $u_m: X \to X_m$ is dense, there is an element $x = (x_m) \in X$ such that $(x, y) = (x_m, y_m)_m \neq 0$. Therefore our pairing $(\ ,\ )$ is nondegenerate. Hence we have proved the following:

PROPOSITION 1. *Let* $X = \operatorname{proj\,lim} X_m$ *and* $Y = \operatorname{ind\,lim} Y_m$ *be as before. Then we have a nondegenerate bicontinuous K-bilinear form*

$$(\, , \,) \colon X \times Y \to K \, .$$

1.2.

LEMMA 1. *Let* E *and* F *be locally convex K-vector spaces, and let* $(\, , \,) \colon E \times F \to K$ *be a nondegenerate bicontinuous K-bilinear form. Let* $\sigma(E, F)$ *be the weakest locally convex topology on* E *such that* $E \ni e \mapsto (e, f) \in K$ *is continuous for each* $f \in F$. *Then for any continuous K-linear form* $L \colon E \to K$ *with respect to* $\sigma(E, F)$, *there exists an element* $f \in F$ *such that*

$$L(e) = (e, f)$$

holds for any $e \in E$. *In particular,* $(E, \sigma(E, F))' = F$.

PROOF. Let $L \colon E \to K$ be as in the lemma. Since L is continuous, there are a finite number of elements $f_1, \cdots, f_n$ in F such that for all $e \in E$

$$|L(e)| \leqq \sup_{1 \leqq i \leqq n} |(e, f_i)| \, .$$

Let $E^* = \{e \in E; (e, f_i) = 0 \text{ for any } i = 1, \cdots, n\}$. Then E^* is contained in the kernel of L, and L factors through E/E^*. Since $(\, , \,)$ induces a nondegenerate K-bilinear form on $(E/E^*) \times (Kf_1 + \cdots + Kf_n)$, the algebraic dual of E/E^* can be identified with $Kf_1 + \cdots + Kf_n$. Hence there are $a_1, \cdots, a_n \in K$ such that

$$L(e) = \left(e, \sum_{i=1}^{n} a_i f_i\right)$$

holds for any $e \in E$. Then $f = \sum a_i f_i$ satisfies the condition of the lemma.

We apply this lemma to our case. Let $E = X$ and $F = Y$. Then our bilinear form $(\, , \,)$ satisfies the condition of the lemma. Let $\sigma(X, Y)$ (resp. $\sigma(Y, X)$) be the weakest locally convex topology on X (resp. on Y) such that $X \ni x \mapsto (x, y) \in K$ is continuous for each $y \in Y$ (resp. $Y \ni y \mapsto (x, y) \in K$ is continuous for each $x \in X$). Then it follows from Lemma 1 that $(X, \sigma(X, Y))' = Y$ and $(Y, \sigma(Y, X))' = X$.

Let $\tau(X)$ be the projective limit topology on X, and let $\tau(Y)$ be the inductive limit topology of Y. Since our pairing is bicontinuous, $\sigma(X, Y) \leqq \tau(X)$ and $\sigma(Y, X) \leqq \tau(Y)$. Hence we have

$$Y = (X, \sigma(X, Y))' \subset (X, \tau(X))' \quad \text{and} \quad X = (Y, \sigma(Y, X))' \subset (Y, \tau(Y))' \, .$$

Let $f: X \to K$ be a K-linear continuous map with respect to $\tau(X)$. Then $f^{-1}(\{x \in X; |f(x)| < 1\})$ is open in X. It follows from the definition of the projective limit topology that there exist a positive integer m and a positive number ε such that $f^{-1}(\{x \in X; |f(x)| < 1\}) \supset \{x \in X; |u_m(x)|_m < \varepsilon\}$. This shows that $u_m(X) \ni u_m(x) \mapsto f(x) \in K$ is continuous. Since $u_m(X)$ is dense in X_m, this map can be extended to a continuous K-linear map $f_m: X_m \to K$. Since X_m and Y_m are mutually dual with respect to $(\ ,\ )_m$, there is an element $y_m \in Y_m$ such that $f_m(x_m) = (x_m, y_m)_m$ holds for any $x_m \in X_m$. Then $f(x) = f_m(x_m) = (x_m, y_m)_m = (x, y)$ holds for any $x = (x_m) \in X$ with $y = v_m(y_m) \in v_m(Y_m) \subset Y$. Therefore $(X, \tau(X))' = Y$.

Let $g: Y \to K$ be a continuous K-linear map with respect to $\tau(Y)$. Since the natural injection $v_m: Y_m \to Y$ is continuous, g induces a continuous map $g_m: Y_m \to K$ for each m. Since X_m is the dual of Y_m, there is a unique element $x_m \in X_m$ such that $g_m(y_m) = (x_m, y_m)$ holds for any $y_m \in Y_m$. If $n > m$, then $(x_m, y_m)_m = g_m(y_m) = g(v_m(y_m)) = g_n(v_{n,m}(y_m)) = (x_n, v_{n,m}(y_m))_n = (u_{m,n}(x_n), y_m)_m$ holds for any $y_m \in Y_m$. Since the pairing $(\ ,\ )_m$ is nondegenerate, $u_{m,n}(x_n) = x_m$ holds for $n > m$. Hence $x = (x_m)$ is an element of $\operatorname{proj\,lim} X_m = X$ such that $g(y) = (x, y)$ holds for any $y \in \operatorname{ind\,lim} Y_m = Y$. Hence $(Y, \tau(Y))' = X$. Therefore we have proved the following:

PROPOSITION 2. *We have $(X, \tau(X))' = Y$ and $(Y, \tau(Y))' = X$ as sets.*

1.3. Since each X_m is a Banach space of countable type, it follows from Schikhof [8, 4.12] that $X = \operatorname{proj\,lim} X_m$ is a Fréchet space of countable type. Since K is not spherically complete, it follows from [8, Corollary 9.8] that X is reflexive. Hence we have the following:

PROPOSITION 3. *X is a Fréchet space of countable type. In particular, X is reflexive.*

Let y be a nonzero element of Y. Since the pairing $(\ ,\ ): X \times Y \to K$ is nondegenerate, there is an element x of X such that $(x, y) \neq 0$. Then $|(x, y)| \neq 0$. Since $p_x(y) = |(x, y)|$ is a continuous seminorm for $\sigma(Y, X)$, it follows that $(Y, \sigma(Y, X))$ is Hausdorff. Since $\tau(Y)$ is stronger than $\sigma(Y, X)$, $(Y, \tau(Y))$ is also Hausdorff. Hence we have proved the following:

PROPOSITION 4. *$(Y, \tau(Y))$ is a Hausdorff space.*

REMARK. If the maps $u_{m,n}$'s are compact maps, then we can show that $X = \operatorname{proj\,lim} X_m$ is a nuclear Montel space. In general, since each Y_m is barreled, $Y = \operatorname{ind\,lim} Y_m$ is also barreled.

Now we can prove the following key lemma:

PROPOSITION 5. *The strong topology $b(Y, X)$ on $(X, \tau(X))' = Y$ and the inductive limit topology $\tau(Y)$ of Y coincide.*

PROOF. Since any bounded set of $(X, \tau(X))$ is contained in a bounded set of the form $B = \{x = (x_m) \in X; |x_m| \leqq M_n\}$ with a sequence (M_n) of positive numbers, the subsets of Y of the form $U_B = \{y \in Y; |(x, y)| \leqq 1$ for all $x \in B\}$ make a fundamental system of neighbourhoods of $0 \in Y$ with respect to $b(Y, X)$. Since the pairing $(\ ,\ )_m: X_m \times Y_m \to K$ makes X_m and Y_m into mutually dual Banach spaces, for any positive number M_m, there is a positive number N_m such that, if $y_m \in Y_m$ satisfies $|y_m| \leqq N_m$, then the condition $|(x_m, y_m)_m| \leqq 1$ is satisfied for any $x_m \in X_m$ with $|x_m| \leqq M_m$. Then $y = v_m(y_m) \in v_m(Y_m)$ is contained in U_B if $|y_m| \leqq N_m$. Hence U_B contains

$$\bigcup_{m \geqq 1} v_m(\{y_m \in Y_m; |y_m| \leqq N_m\}) .$$

Since the subsets of Y of this form make a fundamental system of neighbourhoods of $0 \in Y$ with respect to $\tau(Y)$, we have $b(Y, X) \leqq \tau(Y)$. Since we can prove the opposite inequality $\tau(Y) \leqq b(Y, X)$ in the same way, the strong topology $b(Y, X)$ and the inductive limit topology $\tau(Y)$ of Y coincide.

Since $(X, \tau(X))$ is reflexive, the following corollary follows from Proposition 5.

COROLLARY. *$(Y, \tau(Y))$ is reflexive, and the strong dual space of $(Y, \tau(Y))$ is isomorphic to $(X, \tau(X))$.*

Since X is a Fréchet space, X is bornologic (cf. proofs of van Tiel [10, Théorèmes 3.17 and 4.30]). It follows from Schikhof [8, Proposition 6.8] that $(Y, \tau(Y)) \simeq ((X, \tau(X))', b(Y, X))$ is complete. Therefore we have proved the following theorem:

THEOREM 1. *Let $X = \operatorname{proj\,lim} X_m$, $Y = \operatorname{ind\,lim} Y_m$ and $(\ ,\ ): X \times Y \to K$ be as in* 1.1. *Then X is a Fréchet space of countable type, Y is Hausdorff and complete and the pairing $(\ ,\ )$ makes X and Y into mutually dual locally convex spaces over K.*

2. Examples.

2.1. Let k be an algebraically closed field with a nontrivial non-archimedean complete valuation $|\ \ |$. Let $\boldsymbol{P}^1(k) = k \cup \{\infty\}$ be the one-dimensional projective space over k, let K be a complete subfield of k, and let C be a compact subset of K. Put $V = \boldsymbol{P}^1(k)$. Let $\{r_n\}_{n=1}^{\infty}$ be a strictly decreasing sequence in $|K^*|$ such that $r_n \to 0$ ($n \to \infty$). Then, for each n, C is covered by a finite number of open balls of the form

$$C_{n,i} = \{z \in k; |z - c_{n,i}| < r_n\} \quad (c_{n,i} \in C) .$$

We assume that (i) C is covered by $C_{n,i}$ ($i = 1, \cdots, l_n$) and (ii) the $C_{n,i}$'s are mutually disjoint. Put

$$C_n = \coprod_{i=1}^{l_n} C_{n,i} .$$

Then $C = \cap C_n$.

Let f be a k-valued function on $V - C = \{z \in V; z \notin C\}$. Then f is called a K-analytic function on $V - C$ if and only if the restriction of f to each $V - C_n$ is given by a convergent series of the form

$$f_n(z) = a_\infty + \sum_{i=1}^{l_n} \sum_{m=-1}^{-\infty} a_m^{(i)}(z - c_{n,i})^m$$

with a_∞, $a_m^{(i)} \in K$ and $|a_m^{(i)}| r_n^m \to 0$ ($m \to -\infty$) (cf. Morita [4], Gerritzen-van der Put [1], etc.). Let $\mathscr{O}_K(V - C_n)$ be the space of all functions f_n: $V - C_n \to k$ of this form. Then the equality

$$\max(|a_\infty|, \max_{i,m} |a_m^{(i)}| r_n^m) = \max_{z \in V - C_n} |f_n(z)|$$

holds. If we define a norm $|f_n|_n$ by this formula, then the K-vector space $\mathscr{O}_K(V - C_n)$ becomes a complete Banach space with this norm. Further, we can identify the quotient space $\mathscr{O}_K(V - C_n)/K$ ($K = \{f_n(z) = a_\infty; a_\infty \in K\}$) with the subspace $\{\sum_{i=1}^{l_n} \sum_{m=-1}^{-\infty} a_m^{(i)}(z - c_{n,i})^m; a_m^{(i)} \in K, |a_m^{(i)}| r_n^m \to 0 \ (m \to -\infty)\}$ of $\mathscr{O}_K(V - C_n)$. Since the set of all finite sums of this form is dense in $\mathscr{O}_K(V - C_n)/K$, $\mathscr{O}_K(V - C_n)/K$ is a Banach space of countable type.

Let $\mathscr{O}_K(V - C)$ be the set of all K-analytic functions on $V - C$, and put $\mathscr{B}_K(C) = \mathscr{O}_K(V - C)/K$. Then $\mathscr{B}_K(C)$ can be identified with the locally convex projective limit of the $\mathscr{O}_K(V - C_n)/K$ with respect to the restriction maps. Obviously the restriction maps $u_{n,l}$: $\mathscr{O}_K(V - C_l)/K \to \mathscr{O}_K(V - C_n)/K$ ($n < l$) are K-linear and continuous. Since any finite sum of the form $\sum_i \sum_m a_m^{(i)}(z - c_{n,i})^m$ ($a_m^{(i)} \in K$) is in $\mathscr{O}_K(V - C)/K = \mathscr{B}_K(C)$, the image of the projection map $\mathscr{B}_K(C) \to \mathscr{O}_K(V - C_n)/K$ is dense for each n.

Put

$$\mathscr{O}_{b,K}(C_n) = \left\{g(z) = \sum_{i=1}^{l_n} \sum_{m=0}^{+\infty} b_m^{(i)}(z - c_{n,i})^m; b_m^{(i)} \in K, \sup_{0 \leq m < \infty} |b_m^{(i)}| r_n^m < +\infty\right\} .$$

Then $\mathscr{O}_{b,K}(C_n)$ becomes a Banach space with

$$|g(z)|_n = \sup_{1 \leq i \leq l_n} \sup_{0 \leq m < +\infty} |b_m^{(i)}| r_n^m .$$

If $n < l$ and $|c_{n,i} - c_{l,j}| < r_n$, then $\sum_{0\leq m<+\infty} b_m^{(i)}(z - c_{n,i})^m$ can be written in the form $\sum_{0\leq m<+\infty} b_m'^{(j)}(z-c_{l,j})^m$ with $b_m'^{(j)} \in K$, and we have $\sup_{0\leq m<+\infty} |b_m^{(i)}| r_n^m = \sup_{0\leq m<+\infty} |b_m'^{(j)}| r_n^m$. Hence we have an injective K-linear continuous map $u_{l,n}$: $\mathcal{O}_{b,K}(C_n) \to \mathcal{O}_{b,K}(C_l)$ $(n < l)$. Let $\mathscr{A}_K(C)$ be the locally convex inductive limit space of the Banach spaces $\mathcal{O}_K(C_n)$.

For any

$$f(z) = \sum_{i=1}^{l_n} \sum_{m=-1}^{-\infty} a_m^{(i)}(z - c_{n,i})^m \in \mathcal{O}_K(V - C_n)/K \quad \text{and}$$

$$g(z) = \sum_{i=1}^{l_n} \sum_{m=0}^{+\infty} b_m^{(i)}(z - c_{n,i})^m \in \mathcal{O}_{b,K}(C_n) ,$$

we define

$$(f(z),\, g(z))_n = \sum_{i=1}^{l_n} \sum_{s+t=-1} a_s^{(i)} b_t^{(i)} .$$

Since $|a_m^{(i)}| r_n^m \to 0$ $(m \to \infty)$, and since the $|b_m^{(i)}| r_n^m$'s are bounded, this pairing $(\ ,\ )_n$ is a well defined bicontinuous K-bilinear nondegenerate pairing. If $n < l$ and $f(z) \in \mathcal{O}_K(V - C_l)/K$, then $u_{n,l}f(z) \in \mathcal{O}_K(V - C_n)/K$ and $f(z)g(z)$ is a K-analytic function on $C_n - C_l$. Since $(u_{n,l}f(z),\, g(z))_n$ can be regarded as the sum of residues of $f(z)g(z)$ in C_l, it is equal to $(f(z),\, v_{l,n}g(z))_l$.

For any $f(z) \in \mathscr{B}(C)$ and $g(z) \in \mathscr{A}_K(C)$, we choose a positive integer n and a unique element $g_n(z)$ of $\mathcal{O}_{b,K}(C_n)$ such that $g(z) = v_n(g_n(z))$, and we define

$$(f(z),\, g(z)) = (u_n(f(z)),\, g_n(z))_n .$$

Then it follows from the arguments in 1.1 that this pairing $(\ ,\ )$ is well defined, bicontinuous, K-bilinear and nondegenerate.

Now we have the following theorem:

THEOREM 2. *Let C, $\mathscr{B}_K(C)$, $\mathscr{A}_K(C)$ and*

$$(\ ,\ ) \colon \mathscr{B}_K(C) \times \mathscr{A}_K(C) \to K$$

be as before. Then $\mathscr{B}_K(C)$ is a Fréchet space of countable type, $\mathscr{A}_K(C)$ is a complete Hausdorff space, $(\ ,\ )$ is a bicontinuous K-bilinear nondegenerate pairing, and $\mathscr{B}_K(C)$ and $\mathscr{A}_K(C)$ become mutually dual locally convex spaces with respect to $(\ ,\ )$.

PROOF. Let $r_n = |d|$ $(d \in K)$. Then the mapping

$$\mathcal{O}_K(V - C_n)/K \ni \sum_{i=1}^{l_n} \sum_{m=-1}^{-\infty} a_m^{(i)}(z - c_{n,i})^m$$
$$\mapsto (a_{-1}^{(1)}d^{-1},\, a_{-1}^{(2)}d^{-1},\, \cdots,\, a_{-2}^{(1)}d^{-2},\, a_{-2}^{(2)}d^{-2},\, \cdots,\, a_{-m}^{(1)}d^{-m},\, a_{-m}^{(2)}d^{-m},\, \cdots) \in c_0$$

and

$$\mathcal{O}_{b,K}(C_n) \ni \sum_{i=1}^{l_n} \sum_{m=0}^{+\infty} b_m^{(i)}(z - c_{n,i})^m$$
$$\mapsto (b_0^{(1)}, b_0^{(2)}, \cdots, b_1^{(1)}d, b_1^{(2)}d, \cdots, b_m^{(1)}d^m, b_m^{(2)}d^m, \cdots) \in l^\infty$$

are K-linear isometries of Banach spaces, and compatible with the pairing $(\ ,\ )_n$ and the standard pairing $\langle\ ,\ \rangle$ of c_0 and l^∞ up to a constant factor d^{-1}, where

$$c_0 = \{(a_1, a_2, a_3, \cdots); a_1, a_2, a_3, \cdots \in K, |a_m| \to 0\ (m \to \infty)\} ,$$
$$|(a_1, a_2, a_3, \cdots)| = \sup_{1 \leq m < \infty} |a_m| ,$$
$$l^\infty = \{(b_1, b_2, b_3, \cdots); b_1, b_2, b_3, \cdots \in K, \sup_{1 \leq m < \infty} |b_n| < \infty\} ,$$
$$|(b_1, b_2, b_3, \cdots)| = \sup_{1 \leq m < \infty} |b_m| ,$$
$$\langle (a_1, a_2, a_3, \cdots), (b_1, b_2, b_3, \cdots) \rangle = \sum_{m=1}^{\infty} a_m b_m .$$

If K is not spherically complete, then, by a theorem of van der Put (cf. e.g. van Rooij [7, p. 111]), c_0 and l^∞ are mutually dual Banach spaces with respect to $\langle\ ,\ \rangle$. Hence $\mathcal{O}_K(V - C_n)/K$ and $\mathcal{O}_{b,K}(C_n)$ are also mutualy dual Banach spaces with respect to $(\ ,\ )_n$. Therefore we apply Theorem 1 to our case and obtain Theorem 2 in this case.

If K is spherically complete, then this duality of c_0 and l^∞ does not hold. But we can use Lemma 3.5 and Theorems in Morita [5, 3-1] in this case. Since we can prove our theorem as in [5, 3-3~3-4], we omit the details.

REMARK. We can also show in the same way that the space $\mathscr{A}(U)$, Ind(P, G, χ) and D_χ of Morita [6, III] are complete Hausdorff spaces over any complete nonarchimedean field k. Further, we can construct the holomorphic discrete series π_s of Morita [6, I] and prove the duality of π_s and T_χ (cf. Morita [6, II, Theorem 3]) over any complete non-archimedean field.

2.2. Let $C = \{0\}$ and let d be an element of K^* whose absolute value is smaller than 1. Then

$$\mathscr{B}_K(C) \simeq \{(a_{-1}, a_{-2}, a_{-3}, \cdots) \in K^N;\ \text{for any positive integer } n, |a_m d^{mn}| \to 0\} ,$$
$$\mathscr{A}_K(C) \simeq \{(b_0, b_1, b_2, \cdots) \in K^N;\ \text{for some positive integer } n, \sup_m |b_m d^{mn}| < \infty\} ,$$
$$((a_{-1}, a_{-2}, a_{-3}, \cdots), (b_0, b_1, b_2, \cdots)) = a_{-1}b_0 + a_{-2}b_1 + a_{-3}b_2 + \cdots .$$

The duality of Theorem 2 in this case for a nonspherically complete field was first proved by Schikhof by means of the results of De Grande-De

Kimpe [2].

2.3. Let K be a field with a complete nontrivial nonarchimedean valuation $|\ \ |$. Let $\{r_n\}_{n=1}^{\infty}$ be a strictly increasing sequence in $|K^*|$ such that $r_n \to 1$ $(n \to \infty)$. Let W be the K-vector space consisting of all Laurent series $\sum_{m=-\infty}^{+\infty} a_m z^m$ $(a_m \in K)$ such that $|a_m| r^m \to 0$ $(m \to +\infty)$ for any r with $0 < r < 1$, and $|a_m| r^m \to 0$ $(m \to -\infty)$ for some r with $0 < r < 1$. Then W is the direct sum $W_1 \oplus W_2$ of two subspaces:

$$W_1 = \left\{\sum_{m=0}^{+\infty} a_m z^m;\ a_m \in K,\ |a_m| r^m \to 0\ (m \to +\infty) \text{ for any } r \text{ with } 0 < r < 1\right\}$$

$$W_2 = \left\{\sum_{m=-1}^{-\infty} b_m z^m;\ b_m \in K,\ |b_m| r^m \to 0\ (m \to -\infty) \text{ for some } r \text{ with } 0 < r < 1\right\}.$$

Put

$$W_{1,n} = \left\{\sum_{m=0}^{+\infty} a_m z^m;\ a_m \in K,\ |a_m| r_n^m \to 0\ \ (m \to +\infty)\right\}$$

and

$$W_{2,n} = \left\{\sum_{m=-1}^{-\infty} b_m z^m;\ b_m \in K,\ \sup_m |b_m| r_n^m < \infty\right\}.$$

Then they become Banach spaces with the following norms:

$$\left|\sum_{m=0}^{+\infty} a_m z_m\right|_{1,n} = \sup_m |a_m| r_n^m \quad \text{and}$$

$$\left|\sum_{m=-1}^{-\infty} b_m z^m\right|_{2,n} = \sup_m |b_m| r_n^m .$$

Put

$$\left(\sum_{m=0}^{+\infty} a_m z^m,\ \sum_{m=-1}^{-\infty} b_m z^m\right) = \sum_{m+n=-1} a_m b_n .$$

Let d be an element of K with $|d| = r_n$. Then

$$\sum_{m=0}^{+\infty} a_m z^m \mapsto (a_0,\ a_1 d^1,\ a_2 d^2,\ \cdots) \quad \text{and}$$

$$\sum_{m=-1}^{-\infty} b_m z^m \mapsto (b_{-1} d^{-1},\ b_{-2} d^{-2},\ b_{-3} d^{-3},\ \cdots)$$

induce isometries $W_{1,n} \xrightarrow{\sim} c_0$ and $W_{2,n} \xrightarrow{\sim} l^\infty$ preserving the pairings up to a constant factor. Hence $W_{1,n}$ is of countable type, and $W_{1,n}$ and $W_{2,n}$ become mutually dual Banach spaces with respect to $(\ ,\ )_n$. Further W_1 and W_2 can be identified with $\operatorname{proj\,lim} W_{1,n}$ and $\operatorname{ind\,lim} W_{2,n}$ with respect to the natural maps $u_{n,l}\colon \sum a_m z^m \mapsto \sum a_m z^m$ $(n < l)$ and $v_{l,n}\colon \sum b_m z^m \mapsto \sum b_m z^m$ $(n < l)$. Let τ_1 and τ_2 be the projective limit topology of W_1 and the inductive limit topology of W_2. By Morita [5, Lemma 3.5], the $v_{l,n}$'s are c-compact maps and the projective system $\{W_{1,n}, u_{n,l}\}$ can be replaced by a cofinal system $\{W'_{1,n}, u'_{n,l}\}$ so that the resulting maps $u'_{n,l}$ are also c-compact if K is spherically complete (cf. the arguments in [5, 3-3~3-4]). Since W_1 contains all finite sums of the form $\sum_{m=0}^{+\infty} a_m z^m$, the image of the projection map u_n: $W_1 \to W_{1,n}$ is dense for each n. Hence it follows from Theorem 1 of this paper and theorems in Morita [5, 3-1] that (i) W_1 is a Fréchet space of countable type, (ii) W_2 is a complete Hausdorff space, and (iii) the pairing

$$\left(\sum_{m=0}^{+\infty} a_m z^m, \sum_{m=-1}^{-\infty} b_m z^m\right) = \sum_{m+n=-1} a_m b_m$$

makes W_1 and W_2 into mutually dual spaces. Hence the direct sum $W = W_1 \oplus W_2$ is a complete Hausdorff space, and the inner product

$$\left(\sum_{m=-\infty}^{+\infty} a_m z^m, \sum_{m=-\infty}^{+\infty} b_m z^m\right) = \sum_{m+n=-1} a_m b_n$$

makes W into a self dual space. This selfduality of W was conjectured by P. Robba.

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