Tôhoku Math. Journ.
38 (1986), 599–608.

ON THE GROWTH OF MEROMORPHIC SOLUTIONS OF SOME ALGEBRAIC DIFFERENTIAL EQUATIONS

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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(Received December 23, 1985)

1. Introduction. In this paper we shall study the growth of meromorphic solutions of some algebraic differential equations with the aid of the Nevanlinna theory of meromorphic functions (see [4], [6]). We denote by M the set of meromorphic functions in the complex plane, by E some subset of $[0, \infty)$ with $\operatorname{meas} E < \infty$ and by K some constant which is not always the same. The term "meromorphic" will mean meromorphic in the complex plane.

Let H be a differential polynomial of $w, w', \cdots, w^{(\mu)}$ $(\mu \geqq 1)$ with coefficients in M:

$$H = H(w, w', \cdots, w^{(\mu)}) = \sum_{\lambda \in I} c_\lambda(z) w^{q_0}(w')^{q_1} \cdots (w^{(\mu)})^{q_\mu} ,$$

where $c_\lambda \in M$ with $c_\lambda \neq 0$ and where I is a finite set of multi-indices $\lambda = (q_0, q_1, \cdots, q_\mu)$ of nonnegative integers $q_0, q_1, \cdots, q_\mu$. Let $Q_i(w)$ be a polynomial in w with coefficients in M:

$$Q_i = Q_i(w) = \sum_{j=0}^{m_i} a_{ij} w^j \quad (a_{ij} \in M,\ i = 0, 1, \cdots, n) .$$

Consider the differential equation (D.E., for short):

$$F(w, H) = Q_n(w)H^n + \cdots + Q_1(w)H + Q_0(w) = 0 , \tag{1}$$

where $Q_n(w) \neq 0$ and $F(w, H)$ is irreducible over M as a polynomial in w and H. A meromorphic solution $w = w(z)$ is said to be admissible if

$$T(r, f) = o(T(r, w)) \quad (r \to \infty,\ r \notin E)$$

for all coefficients $f = a_{ij}$, c_λ in (1).

Eremenko [1] gave the following:

"Suppose that the D.E. (1) has an admissible solution. Then,

(i) $m_n = 0$;

(ii) When $H = w^{(\mu)}$,

$$m_j \leqq (\mu + 1)(n - j) \quad (j = 0, 1, \cdots, n) ." \tag{2}$$

As a special case, Gackstatter and Laine [2] and Steinmetz [8] proved the following:

"When $n = 1$, if the D.E. (1) has an admissible solution, then

$$m_1 = 0 \quad \text{and} \quad m_0 \leqq \Delta ,$$

where

$$\Delta = \max_{\lambda \in I}(q_0 + 2q_1 + \cdots + (\mu + 1)q_\mu) ."$$

(See also [10]).

Further, Gackstatter and Laine [2] studied the D.E.

$$(3) \qquad (w')^n = \sum_{j=0}^{m} a_j w^j \quad (a_j \in M \text{ and } m \leqq 2n)$$

and conjectured that the D.E. (3) has no admissible solutions when $1 \leqq m \leqq n-1$. To this conjecture, partial answers were given in [7] and [9]:

"When $1 \leqq m \leqq n-1$, the D.E. (3) except

$$(w')^n = a(w + \alpha)^m \quad (\alpha;\ \text{constant},\ (n-m)\,|\,n)$$

has no admissible solutions."

Here, we shall consider the D.E. (1) when $H = w^{(\mu)}$ under the condition (2) and prove that some of them have no admissible solutions.

2. Lemmas. We shall give some lemmas for later use in this section. For nonconstant $f \in M$, we denote by $S(r, f)$ any quantity satisfying

$$S(r, f) = o(T(r, f)) \quad (r \to \infty,\ r \notin E)$$

as usual (see [4, p. 55]). It is well-known that

$$(4) \qquad m(r, f^{(\mu)}/f) = S(r, f)$$

(see [4], [6]).

LEMMA 1. *Let f, g be nonzero meromorphic functions linearly independent over $\boldsymbol{C}$. Put*

$$(5) \qquad f + g = h .$$

Then we have

$$T(r, f) \leqq T(r, h) + \bar{N}(r, h) + \bar{N}'(r, g) + N(r, D) + S(r, f) + S(r, g) ,$$

where $\bar{N}'(r, g)$ is the $\bar{N}$-function of the poles of g other than the poles of h and $D = g'/g - f'/f$.

PROOF. The relations (5) and $f' + g' = h'$ give

$$f = (hg'/g - h')/(g'/g - f'/f)\,,$$

from which we obtain

$$(6)\qquad \begin{aligned} m(r, f) &\leqq m(r, hg'/g - h') + m(r, 1/D) + O(1) \\ &\leqq m(r, h) + m(r, g'/g) + m(r, h'/h) + m(r, D) \\ &\quad + N(r, D) - N(r, 1/D) + O(1) \end{aligned}$$

and

$$(7)\qquad N(r, f) \leqq N(r, h) + \bar{N}(r, h) + \bar{N}'(r, g) + N(r, 1/D)\,.$$

Using (4), (6), (7) and the inequality

$$m(r, D) \leqq m(r, f'/f) + m(r, g'/g) + O(1)\,,$$

we have the desired inequality immediately.

LEMMA 2. *Let a_j, b_j be in M $(j = 0, \cdots, m)$ with $b_m \neq 0$ and put*

$$R(u) = (a_m u^m + \cdots + a_0)/(b_m u^m + \cdots + b_0)\,.$$

If

$$|u(z)| \geqq 2\left\{\sum_{j=0}^{m-1} (|b_j(z)| + |a_j(z)|)\right\}\Big/|b_m(z)| + 1 \quad \text{for} \quad u \in M\,,$$

then

$$|R(u(z))| \leqq 2|a_m(z)|/|b_m(z)| + 1$$

([1, Lemma 1]).

LEMMA 3. *Let a_j $(j = 0, \cdots, t)$ and f be in M such that $a_t \neq 0$. Then*

$$\begin{aligned} t\{T(r, f) - \sum_{j=0}^{t} T(r, a_j)\} + O(1) &\leqq T\Big(r, \sum_{j=0}^{t} a_j f^j\Big) \\ &\leqq tT(r, f) + \sum_{j=0}^{t} T(r, a_j) + O(1) \end{aligned}$$

(see [5]).

LEMMA 4. *Let $X = u$ and $Y = v$ be a nonzero meromorphic solution of the functional equation*

$$(8)\qquad X^n = aY^m + \sum_{\nu=0}^{k} \sum_{i+j=\nu} a_{ij} X^i Y^j \quad (n,\ m,\ k;\ \text{integers};\ a, a_{ij} \in M,\ a \neq 0)$$

such that $u^n \neq av^m$. If $n \geqq m > k + 2 + m/n$, then there exists a constant K such that

$$T(r, u) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + S(r, u) + S(r, a) + \sum S(r, a_{ij})\,;$$

$$T(r, v) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + S(r, v) + S(r, a) + \sum S(r, a_{ij}) .$$

PROOF. As (u, v) is a solution of (8), we have

$$u^n = av^m + \sum_{\nu=0}^{k} \sum_{i+j=\nu} a_{ij} u^i v^j . \tag{9}$$

We rewrite (9) as

$$u^n = au^m(v/u)^m + \sum_{\nu=0}^{k} \{\sum_{i+j=\nu} a_{ij}(v/u)^j\} u^\nu .$$

Dividing this by $(v/u)^m$, we have

$$u^n(u/v)^m - au^m - \sum_{\nu=0}^{k} \left\{\sum_{j=0}^{\nu} a_{\nu-j\,j}(u/v)^{m-j}\right\} u^\nu = 0 ,$$

which reduces to

$$\begin{aligned} &\left(u^n - \sum_{\nu=0}^{k} a_{\nu 0} u^\nu\right)(u/v)^m \\ &\qquad - \left(\sum_{\nu=1}^{k} a_{\nu-1\,1} u^\nu\right)(u/v)^{m-1} - \cdots - a_{0k} u^k (u/v)^{m-k} - au^m = 0 . \end{aligned} \tag{10}$$

Case 1. $u^n - \sum_{\nu=0}^{k} a_{\nu 0} u^\nu = 0$. In this case, from the relation

$$u^n = \sum_{\nu=0}^{k} a_{\nu 0} u^\nu , \tag{11}$$

by Lemma 3 we have

$$nT(r, u) \leqq kT(r, u) + \sum_{\nu=0}^{k} T(r, a_{\nu 0}) + O(1) ,$$

which reduces to

$$T(r, u) \leqq \frac{1}{n-k} \sum_{\nu=0}^{k} T(r, a_{\nu 0}) + O(1) . \tag{12}$$

Next, we estimate $T(r, v)$ in this case. From (9) and (11)

$$av^m = -a_{0k} v^k - \cdots - \left(\sum_{i=0}^{k-1} a_{i1} u^i\right) v$$

and by Lemma 3

$$\begin{aligned} (m-k)T(r, v) &\leqq T(r, a_{0k}) + \cdots + T\left(r, \sum_{i=0}^{k-1} a_{i1} u^i\right) + T(r, a) + O(1) \\ &\leqq \frac{k(k-1)}{2} T(r, u) + \sum_{\nu=0}^{k} \sum_{\substack{i+j=\nu \\ j \geqq 1}} T(r, a_{ij}) + T(r, a) + O(1) . \end{aligned}$$

Further, by (12) we have for a constant K

$$T(r, v) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + O(1) .$$

Case 2. $u^n - \sum_{\nu=0}^{k} a_{\nu 0} u^\nu \neq 0$. In this case, (10) reduces to

$$(u/v)^m - R_1(u)(u/v)^{m-1} - \cdots - R_k(u)(u/v)^{m-k} - R_{k+1}(u) = 0 , \tag{13}$$

where each $R_j(u)$ satisfies the condition of Lemma 2 as $n \geqq m > k$. Applying Lemma 2 to the estimate of the roots of (13):

$$|u/v| \leqq 1 + \max_{1 \leqq j \leqq k+1} |R_j(u)|$$

as in [1], we obtain

$$|u| \leqq 2\left\{(2|a| + 2)|v| + \sum_{\nu=0}^{k} \sum_{i+j=\nu} |a_{ij}| + |a| + 1\right\} . \tag{14}$$

Now, in (9) put

$$f = -av^m , \quad g = u^n \quad \text{and} \quad h = \sum_{\nu=0}^{k} \sum_{i+j=\nu} a_{ij} u^i v^j ,$$

then

$$h \neq 0 \quad \text{and} \quad f + g = h .$$

Here, we apply Lemma 1.

(I) When f and g are linearly dependent, from (9) we obtain

$$\alpha a v^m = \sum_{\nu=0}^{k} \sum_{i+j=\nu} a_{ij} u^i v^j \quad (\alpha \neq 0, \text{ constant}) ,$$

so that by (14) and Lemma 3

$$T(r, v) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + O(1) .$$

Further, as $u^n = (\alpha + 1) a v^m$,

$$nT(r, u) \leqq mT(r, v) + T(r, a) + O(1) .$$

Therefore, we obtain

$$T(r, u) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + O(1) .$$

(II) When f and g are linearly independent, by Lemma 1 we obtain

$$T(r, f) \leqq T(r, h) + \bar{N}(r, h) + \bar{N}'(r, g) + N(r, D) + S(r, f) + S(r, g) . \tag{15}$$

Here, we estimate each term of (15).

$$mT(r, v) - T(r, a) + O(1) \leqq T(r, f) , \tag{16}$$

$$T(r, h) \leqq kT(r, v) + K\{\sum T(r, a_{ij}) + T(r, a)\} + O(1) \tag{17}$$

(by (14) and Lemma 3),

$$\bar{N}(r, h) + \bar{N}'(r, g) \leqq \bar{N}(r, v) + \sum \bar{N}(r, a_{ij}) + \bar{N}(r, a) \quad \text{(by (14))} , \tag{18}$$

$$(19) \qquad N(r, D) \leqq \bar{N}(r, 1/u) + \bar{N}(r, 1/v) + \sum \{\bar{N}(r, 1/a_{ij}) + \bar{N}(r, a_{ij})\} + \bar{N}(r, 1/a) + \bar{N}(r, a) .$$

This is because, if u (resp. v) has a pole at $z = c$ which is neither a pole nor a zero of a_{ij} and a, then v (resp. u) has a pole at $z = c$, and f and g have a pole of the same order at $z = c$, which shows that D has no pole at $z = c$.

$$(20) \qquad S(r, f) + S(r, g) \leqq S(r, v) + S(r, a) + \sum S(r, a_{ij}) \quad \text{(by (14))} ,$$

$$(21) \qquad \bar{N}(r, v) \leqq T(r, v) + O(1) ,$$

$$(22) \qquad \bar{N}(r, 1/u) \leqq T(r, u) + O(1) ,$$

$$(23) \qquad T(r, u) \leqq \frac{m}{n} T(r, v) + K\{\sum T(r, a_{ij}) + T(r, a)\} + O(1) \quad \text{(by (9) and (14))} ,$$

$$(24) \qquad \bar{N}(r, 1/v) \leqq T(r, v) + O(1) .$$

From (15)–(24), we obtain the inequality

$$(25) \qquad (m - k - 2 - m/n)T(r, v) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + S(r, v) + S(r, a) + \sum S(r, a_{ij}) ,$$

and, as $m - k - 2 - m/n > 0$ by assumption,

$$(26) \qquad T(r, v) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + S(r, v) + S(r, a) + \sum S(r, a_{ij}) .$$

Next we estimate $T(r, u)$. From (9) and (14), we have

$$mT(r, v) \leqq nT(r, u) + kT(r, v) + K\{\sum T(r, a_{ij}) + T(r, a)\} + O(1) ,$$

that is,

$$(27) \qquad T(r, v) \leqq \frac{n}{m - k} T(r, u) + K\{\sum T(r, a_{ij}) + T(r, a)\} + O(1) .$$

From (23), (26) and (27), we obtain

$$T(r, u) \leqq K\{\sum T(r, a_{ij}) + T(r, a)\} + S(r, u) + S(r, a) + \sum S(r, a_{ij}) .$$

Combining Case 1 and Case 2, we complete the proof.

COROLLARY. *If*

$$T(r, a_{ij}) = S(r, u) \quad \textit{and} \quad T(r, a) = S(r, u)$$

or

$$T(r, a_{ij}) = S(r, v) \quad \textit{and} \quad T(r, a) = S(r, v) ,$$

then

$$m \leqq k + 2 + m/n .$$

REMARK. Especially when $n = m$,

$$m - 3 \leqq k .$$

This is an improvement of Theorem II in [3].

3. Theorem. As an application of Lemma 4, we consider the growth of meromorphic solutions of the differential equation

$$(w^{(\mu)})^n + Q_{n-1}(w)(w^{(\mu)})^{n-1} + \cdots + Q_1(w)w^{(\mu)} + Q_0(w) = 0 , \tag{28}$$

where $n \geqq 1$, $\mu \geqq 1$ and

$$Q_i = Q_i(w) = \sum_{j=0}^{m_i} a_{ij}w^j \quad (i = 0, 1, \cdots, n; a_{ij} \in M; m_i = \deg Q_i) .$$

We put

$$k = \begin{cases} \max\{(i + m_i); 1 \leqq i \leqq n - 1 \text{ and } Q_i \neq 0\} \\ 0 , \quad \text{when all } Q_i = 0 \quad (1 \leqq i \leqq n - 1) \end{cases}$$

and $m_0 = m$.

THEOREM. *Let $w = w(z)$ be a meromorphic solution of the* D.E. (28) *for which $w^{(\mu)} \neq 0$.*

(I) *When $k + 2 \leqq m \leqq n - 1$, w satisfies either*

$$(w^{(\mu)})^n + a_{0m}(w + a_{0m-1}/ma_{0m})^m = 0 \tag{29}$$

or

$$T(r, w) \leqq K \sum T(r, a_{ij}) + S(r, w) + \sum S(r, a_{ij}) .$$

(II) *When $k + 3 \leqq m = n$ and $a_{0m-1} = a_{0m-2} = 0$, w satisfies either*

$$(w^{(\mu)})^n + a_{0n}w^n = 0$$

or

$$T(r, w) \leqq K \sum T(r, a_{ij}) + S(r, w) + \sum S(r, a_{ij}) .$$

(III) *When $k + 3 \leqq n \leqq m - 1$ and $a_{0m-1} = \cdots = a_{0k+1} = 0$, w satisfies either*

$$(w^{(\mu)})^n + a_{0m}w^m = 0$$

or

$$T(r, w) \leqq K \sum T(r, a_{ij}) + S(r, w) + \sum S(r, a_{ij}) .$$

PROOF. (I) We rewrite Q_0 as follows:

$$Q_0(w) = a_{0m}(w + a_{0m-1}/ma_{0m})^m + \sum_{j=0}^{m-2} b_{0j}w^j ,$$

where b_{0j} is rational in a_{0m}, a_{0m-1} and a_{0j}. Put

$$u = w^{(\mu)} \quad \text{and} \quad v = w + a_{0m-1}/ma_{0m} .$$

Then, as $k + 2 \leqq m$, (28) becomes

$$u^n = -a_{0m}v^m + \sum_{\nu=0}^{m-2} \sum_{i+j=\nu} c_{ij}u^i v^j , \tag{30}$$

where c_{ij} is rational in a_{pq}. Suppose that $u^n \neq -a_{0m}v^m$. We may suppose $v \neq 0$. Then, we may apply the method of the proof of Lemma 4 to (30). We change only (21) of Case 2, (II) in the proof of Lemma 4 as follows: Instead of (21), we use the inequality

$$\begin{aligned} \bar{N}(r, v) &\leqq \bar{N}(r, w) + \bar{N}(r, 1/a_{0m}) + \bar{N}(r, a_{0m-1}) \\ &\leqq 2 \sum \bar{N}(r, a_{ij}) + \bar{N}(r, 1/a_{0m}) . \end{aligned} \tag{21'}$$

To obtain the last inequality, we apply the method used for (14) to

$$(w^{(\mu)})^n = -a_{0m}w^m - \sum_{\nu=0}^{m-1} \sum_{i+j=\nu} a_{ij}(w^{(\mu)})^i w^j .$$

Then, we have the inequality

$$|w^{(\mu)}| \leqq 2\left\{(2|a_{0m}| + 2)|w| + \sum_{\nu=0}^{m-1} \sum_{i+j=\nu} |a_{ij}| + |a_{0m}| + 1\right\} ,$$

which shows that

$$\bar{N}(r, w) \leqq \sum \bar{N}(r, a_{ij}) .$$

In this case, instead of (25), we obtain

$$\begin{aligned} (m - (m - 2) - 1 - m/n)T(r, v) &= (1 - m/n)T(r, v) \\ &\leqq K \sum T(r, a_{ij}) + S(r, v) + \sum S(r, a_{ij}) , \end{aligned}$$

which reduces to

$$T(r, w) \leqq K \sum T(r, a_{ij}) + S(r, w) + \sum S(r, a_{ij}) ,$$

as c_{ij} is rational in a_{pq}.

(II) Put $w^{(\mu)} = u$ and $w = v$. Then (28) becomes

$$u^n = -a_{0n}v^n - \sum_{\nu=0}^{k} \sum_{i+j=\nu} a_{ij}u^i v^j . \tag{31}$$

Suppose that $u^n \neq -a_{0n}v^n$. Then as in the case (I) of this proof, we obtain

$$(m - k - 2)T(r, v) \leqq K \sum T(r, a_{ij}) + S(r, v) + \sum S(r, a_{ij}) ,$$

which reduces to

$$T(r, w) \leqq K \sum T(r, a_{ij}) + S(r, w) + \sum S(r, a_{ij})$$

as $k+3 \leqq m$.

(III) Put $w=u$ and $w^{(\mu)}=v$. Then (28) becomes

$$u^m = bv^n + \sum_{\nu=0}^{k} \sum_{i+j=\nu} b_{ij} v^i u^j ,$$

where $b=-1/a_{0m}$ and $b_{ij}=-a_{ij}/a_{0m}$.

Suppose that $u^m \neq bv^n$. Then, as $k+3 \leqq n \leqq m-1$, we may apply Lemma 4 to this case and obtain

$$T(r, w) \leqq K \sum T(r, a_{ij}) + S(r, w) + \sum S(r, a_{ij})$$

immediately.

COROLLARY. *When $\mu=1$ and $k+2 \leqq m \leqq n-1$, if all coefficients a_{ij} are rational, any meromorphic solution $w=w(z)$ of the* D.E. (28) *is rational.*

PROOF. Suppose $w' \neq 0$. When w does not satisfy (29) for $\mu=1$, we have

$$T(r, w) \leqq K \sum T(r, a_{ij}) + S(r, w) + \sum S(r, a_{ij}) ,$$

from which we obtain

$$\liminf_{r\to\infty} \frac{T(r, w)}{\log r} < \infty .$$

This shows that w is rational ([6, p. 40]).

When w satisfies (29) for $\mu=1$, then it is well-known that w is rational ([9, Corollary to Theorem 1] or [11, Theorem 3]).

If $w'=0$, then w is a polynomial.

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