Tôhoku Math. Journ.
39 (1987), 589-596.

ESTIMATES OF THE AREA INTEGRALS BY THE NON-TANGENTIAL MAXIMAL FUNCTIONS

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(Received October 1, 1986)

Introduction. In this paper, we give a method to estimate the area integrals by the non-tangential maximal functions.

For a harmonic function u on the upper half-space $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n \times (0, \infty)$, the area integrals $A_\alpha u$ and the non-tangential maximal functions $N_\alpha u$ are defined by

$$(A_\alpha u)(x) = \left\{\iint_{\Gamma_\alpha(x)} |(\nabla u)(y, s)|^2 s^{1-n} dyds\right\}^{1/2} \quad \text{and}$$
$$(N_\alpha u)(x) = \sup\{|u(y, s)|; (y, s) \in \Gamma_\alpha(x)\} ,$$

respectively, for $\alpha > 0$ and $x \in \boldsymbol{R}^n$. In the above formulas, $|\nabla u|^2 = \sum |\partial u/\partial x_j|^2 + |\partial u/\partial t|^2$ with the summation taken over $j = 1, \cdots, n$, and $\Gamma_\alpha(x) = \{(y, s) \in \boldsymbol{R}_+^{n+1}; |x - y| < \alpha s\}$, where $|x|$ is the Euclidean norm of $x \in \boldsymbol{R}^n$. We denote the points of $\boldsymbol{R}_+^{n+1}$ by the ordered pairs (x, t), where $x \in \boldsymbol{R}^n$ and t is a positive real.

The norm equivalence between $A_\alpha u$ and $N_\alpha u$ was obtained by Burkholder-Gundy [1] and Fefferman-Stein [4].

If u is harmonic in $\boldsymbol{R}_+^{n+1}$, then $|\nabla u|^2 = \Delta |u|^2/2$, where Δ denotes the Laplacian on $\boldsymbol{R}_+^{n+1}$, $\sum \partial^2/\partial x_j^2 + \partial^2/\partial t^2$ with the summation taken over $j = 1, \cdots, n$. Furthermore, $|u|^2$ is subharmonic in $\boldsymbol{R}_+^{n+1}$.

McConnell [7] introduced certain functions $S_\alpha v$ on $\boldsymbol{R}^n$ for a subharmonic function v on $\boldsymbol{R}_+^{n+1}$. It can be written as

$$(S_\alpha v)(x) = \int_{\Gamma_\alpha(x)} (\Delta v)(y, s) s^{1-n} dyds ,$$

if $v \in C^2(\boldsymbol{R}_+^{n+1})$. He proved $\|S_\alpha v\|_p \leqq C\|N_\alpha v\|_p$ for a limited range of p, where $\|\cdot\|_p$ denotes the $L^p(\boldsymbol{R}^n)$-norm and C is a constant depending only on n, α, p. Recently, Uchiyama [9] obtained the same result for all $0 < p < \infty$.

We shall give another proof for this latter result and show that our method is applicable to more general area integrals including the ones

This work was partly supported by the Grant-in-Aid for Scientific Research, the Ministry of Education, Science and Culture, Japan.

induced from temperatures, which were studied in [2].

The author thanks Professor A. Uchiyama and the referee for their useful advice.

1. Definitions and the main theorem. Let $A=(a_{jk})$ be an $(n+1)$-dimensional complex square matrix and $b_1, \cdots, b_{n+1}$ be complex numbers. We define the differential operator L by

$$L=\sum_{j,k=1}^{n+1} a_{jk}\frac{\partial^2}{\partial x_j \partial x_k}+\sum_{j=1}^{n+1} b_j t^{-1}\frac{\partial}{\partial x_j}, \tag{1.1}$$

where, and hereafter, x_{n+1} denotes the $(n+1)$-st variable t.

As the following examples show, it may be natural to consider the following area integrals. To a locally integrable function v on $\boldsymbol{R}_+^{n+1}$, we apply the operator L in the sense of distributions. We consider the case where Lv is a positive Borel measure μ_{Lv} on $\boldsymbol{R}_+^{n+1}$. For such v, we define the area integrals $S_\alpha v$ by

$$(S_\alpha v)(x)=\int_{\Gamma_\alpha(x)} s^{1-n} d\mu_{Lv}(y, s).$$

If A is non-negative and $b_1, \cdots, b_{n+1}$ are real, and if $u\in C^2(\boldsymbol{R}_+^{n+1})$ is a solution of $Lu=0$, then

$$L|u|^2=([A+\bar{A}]\partial u/\partial x, \partial u/\partial x)\geqq 0,$$

where $\bar{A}=(\bar{a}_{jk})$ and $\bar{a}_{jk}$ is the complex conjugate of a_{jk}, $\partial u/\partial x$ denotes the column vector ${}^t(\partial u/\partial x_1, \cdots, \partial u/\partial x_{n+1})$ and $(\cdot, \cdot)$ is the usual inner product in $\boldsymbol{C}^{n+1}$.

EXAMPLE 1. If u is harmonic in $\boldsymbol{R}_+^{n+1}$ and $L=\Delta$, then $Lu=0$, $L|u|^2=2|\nabla u|^2$ and $S_\alpha|u|^2=2(A_\alpha u)^2$.

EXAMPLE 2. When u_0 is a temperature on $\boldsymbol{R}_+^{n+1}$, that is to say, u_0 is infinitely differentiable and satisfies $\partial u_0(x, t)/\partial t=\Delta_x u_0(x, t)=\sum_{j=1}^n \partial^2 u_0(x, t)/\partial x_j^2$ in $\boldsymbol{R}_+^{n+1}$, by setting $u(x, t)=u_0(x, t^2/4\pi)$ an dtaking $L=\Delta_x-2\pi t^{-1}\partial/\partial t$, we have $Lu=0$ and $L|u|^2=2|\nabla_x u|^2=2\sum_{j=1}^n|\partial u/\partial x_j|^2$. Making use of notation in [2], we have $S_\alpha|u|^2=2\omega_n\alpha^n\sum_{j=1}^n S_\alpha^2(x, K_j)$, where $K_j(x, t)=t\partial u(x, t)/\partial x_j$ $(j=1, \cdots, n)$ and ω_n is the measure of the unit sphere in $\boldsymbol{R}^n$.

EXAMPLE 3. If v is subharmonic and not identically $-\infty$ in $\boldsymbol{R}_+^{n+1}$, and if $L=\Delta$, then $S_\alpha v$ is identical with that in McConnell [7].

We define the non-tangential maximal functions $N_\alpha v$ by

$$(N_\alpha v)(x)=\operatorname{ess\,sup}|v\chi_{\Gamma_\alpha(x)}|=\|v\chi_{\Gamma_\alpha(x)}\|_\infty,$$

where $\chi_{\Gamma_\alpha(x)}$ is the characteristic function of $\Gamma_\alpha(x)$ and ess sup is the essential supremum with respect to the Lebesgue measure in $\boldsymbol{R}_+^{n+1}$.

For technical reasons, we introduce

$$[S_\alpha(W)v](x) = \int_{W\cap\Gamma_\alpha(x)} s^{1-n}d\mu_{Lv}(y, s)$$

for Borel sets $W\subset\boldsymbol{R}_+^{n+1}$. For $E\subset\boldsymbol{R}^n$, $|E|$ denotes the Lebesgue measure of E. We denote by $B_n(x, R)$ the open ball in $\boldsymbol{R}^n$ of radius R centered at x. For $R>0$, we set $T_R = B_n(o, R)\times(1/R, R)\subset\boldsymbol{R}_+^{n+1}$, where o denotes the origin in $\boldsymbol{R}^n$.

Our main purpose is to obtain the following good λ inequalities, which lead to the norm inequalities stated in Introduction.

THEOREM. *Let L be the operator defined by* (1.1) *with an* $(n+1)$-*dimensional complex square matrix A and complex numbers $b_1, \cdots, b_{n+1}$, and let v be a locally integrable function on $\boldsymbol{R}_+^{n+1}$ such that Lv is a positive Borel measure on $\boldsymbol{R}_+^{n+1}$. If $0<\alpha<\beta<\infty$, then*

$$|\{x\in\boldsymbol{R}^n;\,[S_\alpha(T_R)v](x)>\gamma,\ (N_\beta v)(x)\leqq 1\}|\leqq c_1\exp(-c_2\gamma)|\{x\in\boldsymbol{R}^n;\,[S_{2\alpha}(T_R)v](x)>1\}|$$

for all $\gamma>1$, where c_1 and c_2 are positive constants depending only on n, α, β and L.

COROLLARY 1. *Let v be the same as in Theorem. For any $0<\alpha$, $\beta<\infty$ and $0<p<\infty$, there exists a constant C depending only on n, α, β, L and p such that $\|S_\alpha v\|_p \leqq C\|N_\beta v\|_p$.*

COROLLARY 2. Let *v be the same as in Theorem and suppose $0<\alpha<\beta<\infty$. Then, for any $0<p<\infty$, there exist positive constants c and C depending only on n, α, β, L and p such that*

$$\int_{\boldsymbol{R}^n}\exp\{c(S_\alpha v)(x)/(N_\beta v)(x)\}\{(S_\alpha v)(x)\}^p dx \leqq C\|S_\alpha v\|_p^p\ .$$

When $L=\Delta$, $v=|u|^2$ and u is harmonic, the above Corollary 2 was obtained by Murai-Uchiyama [8], as an improvement of Fefferman-Gundy-Silverstein-Stein [5].

We shall prove Theorem following Uchiyama [9]. In the process, the following lemma plays an important role. It is an extension of Lemma 1 in [9] but is simpler: We do not have to control ∇v by $N_\beta v$ as in [9].

To state the lemma, we let $W_\alpha(E) = \cup\{\Gamma_\alpha(x); x\in E\}$ for $E\subset\boldsymbol{R}^n$.

LEMMA. *Let v, α and β be the same as in Theorem, and let $E=\{x\in\boldsymbol{R}^n; (N_\beta v)(x)\leqq 1\}$. If ν is defined by*

$$\nu(W) = \int_{W_\alpha(E)\cap W} t\,d\mu_{Lv}(x, t)$$

for Borel sets $W \subset \boldsymbol{R}_+^{n+1}$, then ν is a Carleson measure and the Carleson norm $\|\nu\|_C = \sup\{\nu[I\times(0, |I|^{1/n})]/|I|$; I is a cube in $\boldsymbol{R}^n\}$ is bounded by a constant depending only on n, α, β, and L.

2. Proof of Lemma. Let I be a cube in $\boldsymbol{R}^n$ and let $l(I)$ be its side length. It is clear that

$$\nu[I\times(0, l(I))] \leqq \lim_{h\to+0}\int_{W_\alpha(E)\cap W_\alpha(I)\cap\{\boldsymbol{R}^n\times(h,l(I))\}} td\mu_{Lv}(x, t) . \tag{2.1}$$

For small $h > 0$, we define W by

$$W = W_{(\alpha+\beta)/2}(E)\cap W_{(\alpha+\beta)/2}(I)\cap\{\boldsymbol{R}^n\times([1-\tau]h, [1+\tau]l(I))\} ,$$

where we take sufficiently small $\tau > 0$ so that the following relations hold:

$$B_{n+1}((x, t), \tau t)\subset\Gamma_{(\alpha+\beta)/2}(o) \quad \text{for} \quad (x, t)\in\Gamma_\alpha(o) , \quad \text{and} \tag{2.2}$$

$$B_{n+1}((x, t), \tau t)\cap\Gamma_{(\alpha+\beta)/2}(o) = \varnothing \quad \text{for} \quad (x, t)\notin\Gamma_\beta(o) . \tag{2.3}$$

Let ρ be a non-negative infinitely differentiable function on $\boldsymbol{R}^{n+1}$ such that $\operatorname{supp}\rho\subset B_{n+1}(o, 1)\subset\boldsymbol{R}^{n+1}$ and its integral over $\boldsymbol{R}^{n+1}$ is equal to 1. Let $\rho_\varepsilon(x, r) = \varepsilon^{-n-1}\rho(\varepsilon^{-1}x, \varepsilon^{-1}r)$ for $(x, r)\in\boldsymbol{R}^{n+1}$ and $\varepsilon > 0$, and let χ be

$$\chi(x, t) = (\chi_W * \rho_{\tau t})(x, t) = \int_W (\tau t)^{-n-1}\rho((x-y)/\tau t, (t-s)/\tau t)dyds$$

for $(x, t)\in\boldsymbol{R}_+^{n+1}$, where χ_W is the characteristic function of W. By the property (2.3) and the definition of W, we have

$$\operatorname{supp}\chi\subset[I^*\times(\{1-\tau\}h/\{1+\tau\}, \{1+\tau\}l(I)/\{1-\tau\})]\cap[W_\beta(E)\cap W_\beta(I)] , \tag{2.4}$$

where I^* is the cube which we obtain by expanding I $\{1+2\beta(1+\tau)/(1-\tau)\}$-times and has the same center as I. We also see that χ is a non-negative infinitely differentiable function. Furthermore, by (2.2), we have

$$\chi(x, t) = 1 \quad \text{for} \quad (x, t)\in W_\alpha(E)\cap W_\alpha(I)\cap\{\boldsymbol{R}^n\times(h, l(I))\} . \tag{2.5}$$

Therefore, the integral in (2.1) is not greater than

$$J = \int_{\boldsymbol{R}_+^{n+1}} t\chi(x, t)d\mu_{Lv}(x, t) .$$

Since $t\chi(x, t)$ is an infinitely differentiable function with compact support on $\boldsymbol{R}_+^{n+1}$, in other words, a test function, we get

$$J = \int_{W_0} L^*[t\chi(x, t)]v(x, t)dxdt ,$$

where $W_0 = \{(x, t); L^*[t\chi(x, t)] \neq 0\}$ and

$$L^* = \sum_{j,k=1}^{n+1} a_{jk}\frac{\partial^2}{\partial x_j\partial x_k} - \sum_{j=1}^{n+1} b_j\frac{\partial}{\partial x_j}\frac{1}{t} .$$

Elementary calculus shows that

$$L^*[t\chi(x, t)] = t\sum_{j,k=1}^{n+1} a_{jk}\frac{\partial^2\chi}{\partial x_j\partial x_k}(x, t) + \sum_{j=1}^{n+1} c_j\frac{\partial\chi}{\partial x_j}(x, t) ,$$

where $c_j = a_{j,n+1} + a_{n+1,j} - b_j$ $(j = 1, \cdots, n + 1)$. Hence, by (2.4) and (2.5), the domain W_0 of integration is narrow. We can easily check that $|\partial^2\chi(x, t)/\partial x_j\partial x_k| \leqq C/t^2$ and $|\partial\chi(x, t)/\partial x_j| \leqq C/t$ for every $(x, t) \in \boldsymbol{R}_+^{n+1}$ and all $j, k = 1, \cdots, n + 1$ with some constant C depending only on n, τ and ρ. For $(x, t) \in W_0 \subset \operatorname{supp}\chi$, there exists a point $z \in E$ such that $(x, t) \in \Gamma_\beta(z)$ by (2.4). This implies $\operatorname{ess\,sup}|v\chi_{W_0}| = \|v\chi_{W_0}\|_\infty \leqq 1$, where χ_{W_0} is the characteristic function of W_0. Therefore,

$$J \leqq C\int_{W_0} t^{-1}dxdt ,$$

with C depending only on n, τ, ρ and L, but independent of h. If we write $W_0(x) = \{t; (x, t) \in W_0\}$ for $x \in I^*$, the last integral is equal to

$$\int_{I^*} dx \int_{W_0(x)} t^{-1}dt$$

by Fubini's theorem. We divide I^* into three parts I_1^*, I_2^* and I_3^*;

$$\begin{aligned} I_1^* &= \{x; (x, h) \in W_\alpha(E) \cap W_\alpha(I)\} , \\ I_2^* &= \{x; (x, l(I)) \in W_\alpha(E) \cap W_\alpha(I)\} - I_1^* \quad \text{and} \\ I_3^* &= I^* - (I_1^* \cup I_2^*) . \end{aligned}$$

The following inclusion relations are easily checked:

$$\begin{aligned} &W_0(x) \subset [\{(1 - \tau)/(1 + \tau)\}h, h] \cup [l(I), \{(1 + \tau)/(1 - \tau)\}l(I)] \quad \text{for} \quad x \in I_1^* , \\ &W_0(x) \subset [d(x)/\beta, d(x)/\alpha] \cup [l(I), \{(1 + \tau)/(1 - \tau)\}l(I)] \quad \text{for} \quad x \in I_2^* , \quad \text{and} \\ &W_0(x) \subset [\{\alpha/\beta\}l(I), \{(1 + \tau)/(1 - \tau)\}l(I)] \quad \text{for} \quad x \in I_3^* . \end{aligned}$$

The function d appearing in the second relation is defined by $d(x) = \max\{$the distance of x from E, the distance of x from $I\}$. By these relations, we have $\int_{W_0(x)} t^{-1}dt \leqq C'$ for a constant C'. Therefore,

$$J \leqq CC'|I^*| \leqq C''|I|$$

with a constant C'' depending only on n, α, β, τ, ρ and L, but independent of h. This and (2.1) imply $\nu[I\times(0, l(I))] \leqq C''|I|$.

3. Proof of Theorem. We use the following known theorems.

THEOREM A (Murai-Uchiyama [8]). *If $f \in BMO(\boldsymbol{R}^n)$ and if $\|f\|_{BMO} \leqq 1$, then there exist positive constants c_1' and c_2' depending only on n such that*

$$|\{x;\,|f(x)|>\gamma\}|\leqq c_1'\exp(-c_2'\gamma)|\{x;\,|f(x)|>1\}|\quad(\gamma>1)\,.$$

THEOREM B. *Let ν be a finite Carleson measure on $\boldsymbol{R}_+^{n+1}$ with the Carleson norm $\|\nu\|_C$ and K be an integrable continuous function on $\boldsymbol{R}^n$ satisfying*

$$|K(x+y)-K(x)|\leqq B|y|(1+|x|)^{-n-1}\quad(|x|\geqq 2|y|)$$

for a constant B. Then

$$(S_K\nu)(x)=\int_{\boldsymbol{R}_+^{n+1}}K_t(x-y)d\nu(y,t)$$

exists for almost all $x\in\boldsymbol{R}^n$ and $\|S_K\nu\|_{BMO}\leqq C_n(\|K\|_1+B)\|\nu\|_C$, where C_n *is a constant depending only on n, and $K_t(x)=t^{-n}K(t^{-1}x)$ for $(x,t)\in\boldsymbol{R}_+^{n+1}$.*

When $K_t(x)$ is the Poisson kernel on $\boldsymbol{R}_+^{n+1}$, $S_K\nu$ is the balayage of ν. In this case, Theorem B is proved in [6, pp. 229–230]. The proof can be applied to our case.

Now, we begin the proof of our Theorem. Let ψ be an infinitely differentiable function on $\boldsymbol{R}^n$ satisfying $0\leqq\psi\leqq 1$, $\psi(x)=1$ $(|x|\leqq 1)$ and $\psi(x)=0$ $(|x|\geqq 2)$. Using this ψ, we set $K(x)=\psi(x/\alpha)$ and $K_t(x)=t^{-n}K(t^{-1}x)$ for $x\in\boldsymbol{R}^n$ and $t>0$. Then

$$t^nK_t(x)=1\quad\text{for}\quad(x,t)\in\Gamma_\alpha(o)\,,\quad\text{and}\quad t^nK_t(x)=0\quad\text{for}\quad(x,t)\notin\Gamma_{2\alpha}(o)\,.$$

We set $E=\{x;\,(N_\beta v)(x)\leqq 1\}$ and define the measure ν exactly as in Lemma. If we write $\nu_R(W)=\nu(W\cap T_R)$ for $W\subset\boldsymbol{R}_+^{n+1}$, then, by the above relations, we get

$$[S_\alpha(W_\alpha(E)\cap T_R)v](x)\leqq(S_K\nu_R)(x)\leqq[S_{2\alpha}(T_R)v](x)\tag{3.1}$$

for all $x\in\boldsymbol{R}^n$. Lemma shows that ν_R is a finite Carleson measure and there exists a constant C_1 depending only on n, α, β, and L such that $\|\nu_R\|_C\leqq\|\nu\|_C\leqq C_1$. Applying Theorem B to $S_K\nu_R$, we have $\|S_K\nu_R\|_{BMO}\leqq C_2\|\nu_R\|_C$ with some constant C_2 depending only on n and K. Therefore, Theorem A implies

$$|\{x;\,(S_K\nu_R)(x)>\gamma\}|\leqq c_1\exp(-c_2\gamma)|\{x;\,(S_K\nu_R)(x)>1\}|\tag{3.2}$$

for all $\gamma>1$, where c_1 and c_2 are positive constants depending only on n, C_1 and C_2. Since $[S_\alpha(T_R)v](x)=[S_\alpha(W_\alpha(E)\cap T_R)v](x)$ for $x\in E$, Theorem follows from (3.1) and (3.2).

3. Proofs of Corollaries 1 and 2. To prove Corollary 1, we may assume $0<\alpha<\beta<\infty$ without loss of generality (cf. Fefferman-Stein [4, Lemma 1]). It is easy to derive

$$\|S_\alpha(T_R)v\|_p^p\leqq\gamma^p\{\|N_\beta v\|_p^p+c_1\exp(-c_2\gamma)\|S_{2\alpha}(T_R)v\|_p^p\}$$

for any $\gamma > 1$ from Theorem with the same constants c_1 and c_2 as those in it (cf. Uchiyama [9]). For the second term in the braces, we have $\|S_{2\alpha}(T_R)v\|_p^p \leqq C_0\|S_\alpha(T_R)v\|_p^p$ for a constant C_0 depending only on n, α and p as in the proof of Proposition 4 in [3]. Our assumption on v and the fact that the support of $[S_\alpha(T_R)v]$ is compact give $\|S_\alpha(T_R)v_p\| < \infty$. Therefore, we have

$$\{1 - C_0c_1\gamma^p \exp(-c_2\gamma)\}\|S_\alpha(T_R)v\|_p^p \leqq \gamma^p\|N_\beta v\|_p^p .$$

Taking a suitable γ and letting R tend to infinity, we have the conclusion of Corollary 1.

Corollary 2 is an analogue of a result of Murai-Uchiyama [8]. We give here a proof slightly different from theirs.

Let c_2 be the same constant as that in Theorem. As the constant c in Corollary 2, we may take any number c such that $0 < c < c_2$. Take an infinitely differentiable function χ on $(0, \infty)$ such that $\chi(t) = 0$ for $0 < t < 1$, and $\chi(t) = 1$ for $t > 2$, and set $\phi_0(t) = \chi(t)\exp(ct)$ and $\phi_1(t) = \{1 - \chi(t)\}\exp(ct)$ for $0 < t < \infty$. Then

$$\begin{aligned}&\int_{R^n}\exp\{c(S_\alpha v)(x)/(N_\beta v)(x)\}\{(S_\alpha v)(x)\}^p dx\\ &\quad\leqq \int_{R^n}\Phi[(S_\alpha v)(x), (N_\beta v)(x)]dx + \int_{R^n}\phi_1[(S_\alpha v)(x)/(N_\beta v)(x)]\{(S_\alpha v)(x)\}^p\, dx ,\end{aligned}$$

where $\Phi(t, u) = \phi_0(t/u)t^p$. The second term is bounded by $\exp(2c)\|S_\alpha v\|_p^p$. As for the first term, by Fatou's theorem, it is sufficient to estimate

$$I_R = \int\Phi([S_\alpha(T_R)v](x), (N_\beta v)(x))dx .$$

If $(N_\beta v)(x) = 0$, then $[S_\alpha(T_R)v](x) = 0$. Therefore, we may regard the integral of I_R to be taken over $\{x; (N_\beta v)(x) \neq 0\}$. By the definition of Φ, we have

$$I_R = \iint_{0<u<t} -\Phi_{tu}(t, u)|\{x; [S_\alpha(T_R)v](x) > t, (N_\beta v)(x) \leqq u\}|dtdu ,$$

where $\Phi_{tu} = \partial^2\Phi/\partial t\partial u$. By Theorem, the right hand side is bounded by

$$c_1\int_0^\infty|\{x; [S_{2\alpha}(T_R)v](x) > u\}|du\int_u^\infty|\Phi_{tu}(t, u)|\exp(-c_2t/u)dt .$$

Since $\Phi_{tu}(t, u) = -u^{p-1}\{(1 + p)(t/u)^p\phi_0'(t/u) + (t/u)^{p+1}\phi_0''(t/u)\}/u$, the inner integral is equal to

$$C_1 = u^{p-1}\int_1^\infty|(1 + p)t^p\phi_0'(t) + t^{p+1}\phi_0''(t)|\exp(-c_2t)dt .$$

Because of $0 < c < c_2$, the last integral is finite, and so

$$I_R \leqq C_1 c_1 p^{-1} \|S_{2\alpha}(T_R)v\|_p^p \leqq C_0 C_1 c_1 p^{-1} \|S_\alpha v\|_p^p .$$

This completes the proof of Corollary 2.

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