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5-DIMENSIONAL CONTACT MANIFOLDS WITH SECOND BETTI NUMBER $b_2=0$

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Abstract. In this paper we improve some results of S. I. Goldberg ([3], [4]) in the 5-dimensional case. As consequences we obtain:

(a) the sphere S^5 is the only compact simply connected normal homogeneous contact manifold of dimension 5 with $b_2=0$;

(b) if a 5-dimensional compact simply connected regular Sasakian manifold is μ-holomorphically pinched with $\mu>1/2$, then it is homeomorphic with a sphere.

1. Introduction. Recently Goldberg [4] has proved the following: Let M be a compact simply-connected regular Sasakian ($=$normal contact Riemannian) manifold of dimension $2n+1$, $n\geqslant 2$, with positive sectional curvature. Then M is homeomorphic with a sphere.

Moreover the same author proved in [3] that, if in addition the scalar curvature r of M is constant, then M is isometric to a sphere. These results of Goldberg are relative to the problem posed in [2]. In this paper we improve these results in the 5-dimensional case. Precisely we obtain:

THEOREM 1. *Let M be a 5-dimensional simply-connected compact regular Sasakian manifold with $b_2=0$ and with scalar curvature $r>-4$. Then M is homeomorphic with a sphere.*

THEOREM 2. *Let M be a 5-dimensional simply-connected compact regular Sasakian manifold with $b_2=0$ and with scalar curvature $r>-4$. Then M is isometric to a sphere S^5; M is isometric to $S^5(1)$ of constant sectional curvature* 1 *when* $r=20$.

Note that all compact Sasakian manifolds with positive sectional curvature have the second Betti number equal to zero (cf. [8], p. 41–5).

The same result as in Theorem 2 is obtained by replacing "Sasakian manifold" with "η-Einstein contact manifold" (see Theorem 3 in Section 4). Theorems 1 and 2 give the following interesting consequences.

COROLLARY 1. *A 5-dimensional compact simply-connected regular Sasakian man-*

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ifold either of positive curvature or μ-holomorphically pinched with $\mu > 1/2$, is homemorphic with a sphere.

COROLLARY 2. *A 5-dimensional compact simply-connected normal homogeneous contact manifold with $b_2 = 0$ is, with respect to the invariant metric, isometric with a sphere.*

COROLLARY 3. *A 5-dimensional compact simply-connected isotropy irreducible homogeneous contact manifold with $b_2 = 0$ is, with respect to the invariant metric, isometric to the sphere $S^5(1)$ of constant sectional curvature 1.*

COROLLARY 4. *A 5-dimensional compact simply-connected homogeneous contact manifold either of positive curvature or μ-holomorphically pinched with $\mu > 1/2$ in the invariant metric is isometric with a sphere.*

REMARK. If the torsion part of the integral second homology group vanishes, the condition on the regularity is not necessary in Corollary 1 (see Theorem 11.6 and Corollary 11.9 of [13]). Corollary 4 extends Corollary 2 of [3] in the 5-dimensional case.

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2. Contact manifolds. A $(2n+1)$-dimensional C^∞ manifold is said to have a *contact structure*, and is called a contact manifold, if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere; η is defined up to a non-zero factor.

An *almost contact structure* $(\varphi, X_\circ, \eta)$ on a $(2n+1)$-dimensional C^∞ manifold M is given by a tensor field φ of type $(1, 1)$, a vector field $X_\circ$ called characteristic field, and a 1-form η on M such that

$$\eta(X_\circ) = 1\,, \qquad \varphi(X_\circ) = 0 \quad \text{and} \quad \varphi^2 = -I + \eta(\cdot)X_\circ$$

where I is the identity transformation. If M has such a structure, it is called an almost contact manifold. In this case, a Riemannian metric g can be found so that

$$\eta = g(X_\circ, \cdot)\,, \qquad g(\varphi X, Y) = -g(X, \varphi Y) \quad \text{and} \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields X and Y on M. Such a metric is called an associated metric and is however clearly not unique. The resulting structure $(\varphi, X_\circ, \eta, g)$ is then called an *almost contact metric structure*. An almost contact metric structure $(\varphi, X_\circ, \eta, g)$ is called a *contact metric structure* (or contact Riemannian structure) if $g(X, \varphi Y) = d\eta(X, Y)$ for any vector fields X, Y on M.

It has been shown by Sasaki that a contact manifold M with contact form η has an underlying contact Riemannian structure $(\varphi, X_\circ, \eta, g)$. A contact metric structure $(\varphi, X_\circ, \eta, g)$ is said to be *K-contact* if $X_\circ$ is a Killing field with respect to g, while it is said to be *normal* if the almost complex structure J on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X,\ -fX_\circ, \eta(X)\frac{d}{dt}\right)$$

is integrable, where f is a real-valued function.

A normal contact Riemannian structure is called *Sasakian structure*. Naturally a Sasakian structure is also of K-contact. A contact structure η on a compact manifold M is called *regular* if $X_\circ$ is a regular vector field on M, that is, every point $y \in M$ has a cubical coordinate neighborhood U such that the integral curves of $X_\circ$ passing through U pass through the neighborhood only once. In this case M is a principal circle bundle over a sympletic manifold B whose fundamental 2-form Ω has integral periods and the corresponding fibration $p: M \to B$ is known as the *Boothby-Wang fibration* of M (cf. [2]).

3. Proof of Theorem 1. Let $(\varphi, X_\circ, \eta, g)$ be the regular normal contact metric structure on the 5-dimensional compact simply-connected manifold M. Then the base manifold B of the Boothby-Wang fibration $p: M \to B$ is a compact Kaehlerian manifold of complex dimension 2 and with complex structure J and Kaehler metric h defined by

$$g = p^*h + \eta \otimes \eta\,, \qquad d\eta = p^*\Omega$$

$$\Omega(X, Y) = h(X, JY) \qquad \text{for any vector fields} \quad X,\ Y \text{ on } B\,.$$

Functions on B can be considered as functions on M in a natural way, i.e., if f is a function on B we shall write f rather than p^*f when it is lifted to M. Locally, the Kaehler metric h on B and the Riemann metric g on M are given by

$$h = \sum (w^i)^2 \quad \text{and} \quad g = \sum (\theta^\alpha)^2$$

where w^i are 1-forms defined on a small open set U of B, $\theta^0 = \eta$ and $\theta^i = p^*w^i$. If we put $\Omega = \sum J_{ij} w^i \wedge w^j$, the components K_{ijkh} of the curvature tensor of B with respect to the basis (w^i) and the components $R_{\alpha\beta\gamma\delta}$ of the curvature tensor of M with respect to the basis (θ^α) are related by (cf. Lemma 4 of [3]):

$$(3.1) \qquad \begin{cases} R_{ijkh} = K_{ijkh} - (2J_{ij}J_{kh} + J_{ik}J_{jh} - J_{ih}J_{jk})\,, \\ R_{i0k0} = -R_{i00k} = -R_{0ik0} = R_{0i0k} = \delta_{ik} \\ R_{\alpha\beta\gamma\delta} = 0 \qquad \text{otherwise} \\ \forall\, \alpha, \beta, \gamma, \delta = 0, 1, \cdots, 4 \quad \text{and} \quad \forall\, i, j, k, h = 1, \cdots, 4\,. \end{cases}$$

Then, the components $R_{\alpha\beta}$ and K_{ij} of the Ricci tensors S of M and S' of B, are related by

$$(3.2) \qquad R_{ij} = K_{ij} - 2\delta_{ij}\,, \qquad R_{i0} = 0\,, \qquad R_{00} = 4\,.$$

Consequently the scalar curvatures r and r' of M and B respectively, are related by

$$(3.3) \qquad r = r' - 4\,.$$

Moreover the second Betti numbers of M and B are related by (cf. [2], p. 733)

$$b_2(B)=b_2(M)+1 .$$

Therefore, the assumption $b_2(M)=0$ implies $b_2(B)=1$.

Now if (β^1, β^2) is a local field of unitary coframes on B, then the Ricci form γ and the scalar curvature r' are given respectively by

$$\gamma=\frac{\sqrt{-1}}{4\pi}\sum K_{i\bar{j}}\beta^i \wedge \bar{\beta}^j , \qquad r'=2\sum K_{i\bar{i}} .$$

It is well-known that the first Chern class $c_1(B)$ of B is represented, in the de Rham cohomology, by the Ricci form γ. Moreover $b_2(B)=1$ implies that

$$c_1(B)=a[\phi] \qquad \text{for some real number} \quad a , \tag{3.4}$$

where ϕ is the 2-form given by $\phi=(\sqrt{-1}/8\pi)\sum \beta^i \wedge \bar{\beta}^i$.

On the other hand, by a direct computation we find

$$\phi \wedge \gamma=\frac{r'}{2}\phi^2 .$$

Thus by integrating both sides of this equation, we obtain

$$a=\frac{1}{2\,\mathrm{vol}(B)}\int_B r' * 1 \tag{3.5}$$

where $*$ is the Hodge star operator.

If we assume $r>-4$, from (3.3)–(3.5), we deduce that B has positive first Chern class. Then, using a classification theorem of Yau [14] for complex surfaces with positive first Chern class, B is biholomorphic to $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$ or to a surface obtained from $\boldsymbol{CP}^2$ by blowing up k points, $0 \leqslant k \leqslant 8$, in general position. Since $b_2(B)=1$, B cannot be biholomorphic to $\boldsymbol{CP}^1 \times \boldsymbol{CP}^1$. On the other hand, by blowing up a point of $\boldsymbol{CP}^2$ the second Betti number increases by one. Therefore $b_2(B)=1$ implies B is biholomorphic to the complex projective space $\boldsymbol{CP}^2$. Consequently, from the homotopy sequences of the fiberings

$$S^1 \to S^5 \to \boldsymbol{CP}^2 \quad \text{and} \quad S^1 \to M \to B ,$$

$$\pi_i(M)=\pi_i(S^5) \qquad \text{for all} \quad i>1 .$$

Since M is simply connected we see that it is of the same homotopy type as the 5-sphere. Applying a result due to Smale [10] we conclude that M is homeomorphic with S^5.

4. Proof of Theorem 2. Since $b_2(M)=0$, from Lemma 6 of [5] we have

(4.1) $$\int_M \left(|S|^2-\frac{r^2}{2}+2r\right)dv+\frac{1}{4\,\mathrm{vol}(M)}\left(\int_M r\,dv\right)^2-20\,\mathrm{vol}(M)=0$$

where $|S|$ denotes the length of the Ricci tensor and dv the volume element. Because $r=$ const., (4.1) becomes

(4.2) $$\int_M \left(|S|^2-\frac{r^2}{4}+2r-20\right)dv=0\,.$$

Moreover, if we put $E=S-(r/4-1)g-(5-r/4)\eta\otimes\eta$, then

(4.3) $$|S|^2-\frac{r^2}{4}+2r-20=|E|^2\geqslant 0$$

where the equality holds if and only if M is η-Einstein. From (4.1)–(4.3), we deduce that M is η-Einstein (see also Corollary 5.7 of [12]).

If the scalar curvature $r=20$, from (4.2) it follows that M is an Einstein manifold. Then, from (3.2) and (3.3), (B, h) is a Kaehler-Einstein manifold with scalar curvature $r'=24$. Now, from the proof of Theorem 1, the base manifold B of the Boothby-Wang fibration is biholomorphic with CP^2. Thus we may consider the Fubini-Study metric $h_\circ$ on B with constant holomorphic sectional curvature $c=4$. Because h is a Kaehler-Einstein metric and B is biholomorphic with CP^2, a result of Berger (cf. for example [6], p. 74) implies

(4.4) $$h=ah_\circ \qquad \text{for some constant} \quad a>0\,.$$

Since (B, h) has scalar curvature $r'=24$, from (4.4) we obtain $a=1$, i.e., (B, h) has constant holomorphic sectional curvature $c=4$. Thus formulas (3.1) imply that (M, g) has constant sectional curvature 1. Finally the simple connectedness of M implies that it is isometric with $S^5(1)$.

If $r=\text{const.}>-4$, we put

$$g'=ag+(a^2-a)\eta\otimes\eta\,, \qquad \varphi'=\varphi\,, \qquad X'_\circ=\frac{1}{a}X_\circ\,, \qquad \eta'=a\eta$$

with $a=(r+4/24)=\text{const.}>0$. Then, by Lemma 2.1 of [13], $(\varphi', X'_\circ, \eta', g')$ is a regular Sasakian structure on M; moreover by direct computation we see that the new metric g' is an Einstein metric with scalar curvature equal to 20. As before (M, g') has constant sectional curvature 1 and if $(\varphi_\circ, X_\circ, \eta_\circ, g_\circ)$ denotes the standard Sasakian structure on $S^5(1)$, by Theorem 2 of [11] there is a diffeomorphism $f\colon M\to S^5$ such that

$$f^*g_\circ=g'\,, \qquad f_*X'_\circ=X_\circ\,, \qquad f^*\eta_\circ=\eta'\,, \qquad f_*\circ\varphi'=\varphi_\circ\circ f_*\,.$$

Thus if we consider on S^5 the following metric

$$g'_\circ = \frac{1}{a} g_\circ + \frac{1-a}{a^2} \eta_\circ \otimes \eta_\circ ,$$

then

$$f^* g'_\circ = \frac{1}{a} f^* g_\circ + \frac{1-a}{a^2} f^* \eta_\circ \otimes f^* \eta_\circ = g .$$

Therefore (M, g) is isometric to $(S^5, g'_\circ)$. This concludes the proof of Theorem 2.

Now, let $(\varphi, X_\circ, \eta, g)$ be a regular contact metric structure on a 5-dimensional simply-connected compact manifold M. By a result of Hatakeyama (cf. [1], p. 70) we may assume that $(\varphi, X_\circ, \eta, g)$ is a K-contact structure. If this contact metric structure is also η-Einstein with scalar curvature $r > -4$, then (cf. Prop. 5.4 of [12]) the base manifold B of the Boothby-Wang fibration is an Einstein-almost Kaehler manifold whose scalar curvature $r' = r + 4 > 0$. But, recently Sekigawa [9] proved that a 4-dimensional compact Einstein-almost Kaehler manifold whose scalar curvature is nonnegative is necessarily a Kaehler manifold. Then by a result of Hatakeyama (cf. [1], p. 87), $(\varphi, X_\circ, \eta, g)$ is an η-Einstein Sasakian structure on M. So from Theorem 2 we obtain:

THEOREM 3. *A 5-dimensional compact simply connected regular K-contact η-Einstein manifold with $b_2 = 0$ and scalar curvature $r > -4$, is isometric with a sphere; M is isometric to $S^5(1)$ when $r = 20$.*

5. Proof of the corollaries. Let M be a 5-dimensional compact Sasakian manifold.

If M has positive curvature, then $r > -4$ and from Theorem 41.2 of [8] we have $b_2 = 0$. Now we assume that M is μ-holomorphically pinched with $\mu > 1/2$. Then by Theorem 8.3 of [13] we have $b_2 = 0$. Moreover $r > -4$, in fact by (38.12) and (38.7) of [8] we have

$$r = \sum_i (R_{ii} + R_{i^*i^*}) + 4 = 2 \sum_i \left\{ 1 + \sum_j (K_{ij} + K_{ij^*}) \right\} + 4$$

where the sectional curvatures $K_{\alpha\beta}$ relative to a φ-basis $(e_i, e_{i^*} = \varphi e_i, X_\circ)$ satisfy (cf. (7.2) and Prop. 6.2 of [13])

$$1 + \sum_j (K_{ij} + K_{ij^*}) \geq 2(4\mu - 2) > 0 .$$

So Corollary 1 follows from Theorem 1.

A contact manifold M is called homogeneous if there is a connected Lie group G which acts transitively and effectively on M as a group of diffeomorphisms and leaves the contact form η invariant. If M is also compact and simply-connected, then the contact form is regular and M is a principal circle bundle over a homogeneous Kaehler manifold B (cf. [2]). Moreover the curvature form of the connection is $p^*\Omega = d\eta$ where $p: M \to B$ is the bundle projection map and Ω the fundamental 2-form of B. Hence the contact metric structure is normal (cf. for example [1], pp. 86–87). Since M is compact and simply-connected, then according to a theorem of Montgomery (cf. [2]) we may suppose G to be compact. If K is the isotropy group of a point $p_\circ$ in M, then $M = G/K$.

Let $(X_\circ, \eta, g)$ be a regular Sasakian structure of the homogeneous contact manifold $M = G/K$, such that η and g are G-invariant on M.

PROOF OF COROLLARY 2. The metric g on M generally comes from a left-invariant Riemannian metric $\tilde{g}$ on G. Now we assume that the metric g is a normal homogeneous metric, i.e., it is induced by a bi-invariant metric $\tilde{g}$ on G (cf. [15]). So from Samelson's theorem [7], we obtain that the sectional curvatures are non-negative. Moreover, since the metric g is invariant, its scalar curvature is a constant (non-negative). Therefore Corollary 2 follows from Theorem 2.

PROOF OF COROLLARY 3. Assume that $M = G/K$ is an isotropy irreducible homogeneous space, i.e., the isotropy linear group K^* acts irreducibly on $T_{p_\circ}(M)$. Then M is an Einstein space in the invariant metric g. In fact if K^* acts irreducibly on $T_{p_\circ}(M)$, the Ricci tensor $S_{p_\circ}$ and the metric $g_{p_\circ}$ are proportional and hence $S = \lambda g$. Since (M, g) is a Sasakian-Einstein space with $b_2(M) = 0$, the formula (4.2) becomes

$$\int_M (r-20)^2 dv = 0 .$$

Therefore $r = 20$ and consequently Corollary 3 follows from Theorem 2.

PROOF OF COROLLARY 4. By the proof of Corollary 1, we get $b_2(G/K) = 0$ and $r > -4$. On the other hand r is constant because the metric is invariant. So Corollary 4 follows from Theorem 2.

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Added in proof. Combining Theorems 2, 3 of the paper [I. Hasegawa and M. Seino, J. Hokkaido Univ. Ed. Sect. II A 32 (1981/82), No. 1, 1–7; MR 84j: 53055] with the Theorem 1 of our paper, we obtain the following new Theorem: *If a 5-dimensional complete simply-connected regular Sasakian manifold with vanishing contact Bochner tensor is μ-holomorphically pinched with $\mu>0$, then it is homeomorphic with a sphere.*