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QUASI-PRODUCT ACTIONS OF A COMPACT ABELIAN GROUP ON A C^*-ALGEBRA

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Let α be an action of a compact abelian group on a separable prime C^*-algebra A, such that also the fixed point subalgebra, A^α, is prime. Several conditions on α are shown to be equivalent, among which are the following:

for each $g \in G$, either $\alpha_g = 1$ or α_g is properly outer;

there exists a faithful irreducible representation of A which is also irreducible on A^α;

there exists a faithful irreducible representation of A which is covariant.

An example of a nontrivial action satisfying these conditions is the infinite tensor product action on $M_{2^\infty} = \bigotimes_{n=1}^{\infty} M_2$ obtained from a sequence of nontrivial inner actions on M_2, each one appearing infinitely often. In earlier work, this example was shown to be, in a certain sense, typical of nontrivial actions satisfying the third condition. This fact is the key to deducing the first two conditions from the third.

The second condition is noteworthy in two respects. First, it involves only the fixed point subalgebra $A^\alpha \subseteq A$, not the action α itself. (This is not evident in the case of the other two conditions.) Second, while a representation verifying the third condition is required to be covariant, a representation verifying the second condition must in fact be as far as possible from being covariant.

1. In [26], Olesen, Pedersen, and Størmer obtained results concerning the system consisting of a prime C^*-algebra and a compact abelian group of automorphisms such that the fixed point subalgebra is prime. They showed that if the group is either the circle group or is finite of prime order, then

(i) the only multipliers commuting with the fixed point subalgebra are the scalars, and

(ii) the only automorphism in the group that is determined by a multiplier is the identity.

In addition, assuming that the group is finite but not necessarily of prime order, and that the C^*-algebra is simple, they showed that the properties (i) and (ii), which need no longer hold, are equivalent. (The nontrivial implication is (ii) $\Rightarrow$ (i).)

In the present paper, we shall improve these two results substantially. Our methods require that the C^*-algebra be separable. We shall formulate properties (i)′ and (ii)′ (14 and 15 below) which are stronger than (i) and (ii), but reduce to these in the case that the

C^*-algebra is simple. We shall show that the stronger properties still hold if the group is the circle group or is finite of prime order, and that, in any case, they are equivalent. The latter result is new even in the case that the C^*-algebra is simple, there being no restriction on the compact abelian group. Furthermore, and in fact as part of the proof, we shall show that the properties (i)′ and (ii)′ are equivalent to a number of other properties (1 to 13 below).

Properties (i)′ and (ii)′ are stated in terms of the limit multiplier C^*-algebra, which was used in [13], after a suggestion by G. K. Pedersen, and was considered further by Pedersen in [29]. (See also [26, Appendix].) Recall that the limit multiplier C^*-algebra, $M^\infty(A)$, of a C^*-algebra A is defined as the inductive limit of the net of multiplier C^*-algebras of essential closed two-sided ideals of A. In this connection, note that if $I \supseteq J$ are two such ideals, then $M(I) \subseteq M(J)$, and that if I and J are any two such ideals, then also $I \cap J$ is such.

Two of the properties (4 and 15) involve the proper outerness of certain automorphisms (either $\hat{\alpha}_\gamma$, $\gamma \neq 0$, or α_g, $g \neq 0$). Proper outerness of an automorphism of a C^*-algebra A was defined in [14] to mean that the restriction to any nonzero invariant closed two-sided ideal is at distance two from any automorphism of that ideal determined by a multiplier. It was shown in [13] and [14] that, at least in the case that A is an AF algebra (i.e., a separable approximately finite-dimensional C^*-algebra), the condition for an automorphism of A to be properly outer fails—and, moreover, with respect to an essential ideal—if, and only if, the canonical extension of the automorphism to $M^\infty(A)$ is inner. Various other reformulations of proper outerness in the case of AF algebras were also given in [13] and [14], and most of these are now known to be valid for any separable C^*-algebra, as a result of work of Kishimoto in [17] and Brown in [6]—see also [18] and [25] (a complete summary is given in Theorem 6.6 of [25]). The reformulation in terms of $M^\infty(A)$ follows from a result of Pedersen in [29]—see Proposition 3.2 below. In Propositions 3.1, 3.2, 3.3, 3.4, and 3.5 we also establish other facts concerning $M^\infty(A)$ that we shall need.

Two other properties (1 and 12) refer to the action on the algebra of the unitary group of the fixed point subalgebra; the first is topological transitivity of this action, in the sense of [21], and the second is strong topological transitivity, in the sense of [3]. (It is open in general whether these two properties are equivalent.)

Another property (13) is an analogue of Tannaka duality. It is stated for automorphisms of $M^\infty(A)$, instead of just for automorphisms of A, in order to deduce the other properties from it. (Stated just for A, it is already known to follow from the property 1—see [21].)

In Section 4, a more general form of this analogue of Tannaka duality is given, in which some of the automorphisms are allowed to be inner. (In 13, none of the automorphisms can be inner, as follows from 13 $\rightarrow$ 15.)

In Section 5, we prove that strong topological transitivity and ergodicity are equivalent notions for an action on a von Neumann algebra. This is used for proving the implication $10 \Rightarrow 12$ in Theorem 1; this result also yields a new proof of the Tannaka duality theorem for von Neumann algebras given in [2].

In the following theorem, $\hat{G}$ denotes the dual group of G, $\hat{G}(\alpha)$ denotes the Connes spectrum of the action α ([30]), π_ω denotes the cyclic representation defined by the state ω, and $\int_G^\oplus \pi\alpha_g dg$ is viewed in the canonical way as a representation on $H_\pi \otimes L^2(G)$, where H_π is the Hilbert space of the representation π.

THEOREM. *Let A be a separable C^*-algebra, and let α be a faithful action of a compact abelian group G on A. Suppose that $G \neq 0$.*

The following fifteen conditions are equivalent.

1. *If $x, y \in A \setminus \{0\}$, then $xA^\alpha y \neq 0$.*
2. *Any sub-C^*-algebra of A containing A^α is prime.*
3. *For any closed subgroup H of G such that $G/H \cong \boldsymbol{T}$ or $G/H \cong \boldsymbol{Z}/n\boldsymbol{Z}$ for some $n = 1, 2, \cdots$, the fixed point algebra A^H is prime.*
4. *A^α is prime and the dual automorphisms $\hat{\alpha}_\gamma$, $\gamma \in \hat{G} \setminus \{0\}$, of the crossed product C^*-algebra $A \rtimes_\alpha G$ are properly outer.*
5. *A^α is prime and there exists an α-invariant pure state ω of A such that π_ω is faithful.*
6. *$\hat{G}(\alpha) = \hat{G}$ and there exists an α-invariant pure state ω of A such that π_ω is faithful.*
7. *For any sequence (ξ_n) of finite-dimensional unitary representations of G there exists an α-invariant sub-C^*-algebra B of A and a closed α^{**}-invariant projection q in the bidual A^{**} of A such that*
   (i) *$q \in B'$,*
   (ii) *$qAq = Bq$,*
   (iii) *$q \in J^{**} \subseteq A^{**}$ for any nonzero closed two-sided ideal J of A,*
   (iv) *the C^*-dynamical system $(Bq, G, \alpha^{**} \mid Bq)$ is isomorphic to the product system $(\bigotimes_{n=1}^\infty M_{\dim \xi_n}, G, \bigotimes_{n=1}^\infty \mathrm{Ad}\, \xi_n)$.*
8. *B and q exist as in 7 in the case that $\dim \xi_n = 2$ and $\xi_n = 1 \oplus \chi_n$, where (χ_n) is a sequence in $\hat{G}$ in which each element of $\hat{G}$ appears infinitely many times.*
9. *There exists an α-invariant state ω of A such that π_ω is faithful and*
$$\pi_\omega(A^\alpha)' \cap \pi_\omega(A)'' = \boldsymbol{C}.$$
10. *There exists a faithful representation π of A such that*
$$\pi(A^\alpha)' \cap \pi(A)'' = \boldsymbol{C}.$$
11. *There exists a faithful irreducible representation π of A such that*
$$\pi(A^\alpha)'' = \pi(A)''.$$

11′. *There exists a faithful irreducible representation π of A such that*

$$\left(\left(\int_G^{\oplus} \pi\alpha_g\,dg\right)(A)\right)'' = \pi(A)''\otimes L^\infty(G)\,.$$

12. *For each pair* $(x_1, \cdots, x_n)$ *and* $(y_1, \cdots, y_n)$ *of finite sequences in* $M^\infty(A)$ *such that* $\sum_{i=1}^n x_i\otimes y_i \neq 0$, *there exists* $a\in A^\alpha$ *such that* $\sum_{i=1}^n x_i a y_i \neq 0$.

13. A *and* A^α *are prime, and if* β *is an automorphism of* $M^\infty(A)$ *such that* $\beta\,|\,A^\alpha = 1$ *then* $\beta=\alpha_g$ *for some* $g\in G$.

14. A^α *is prime, and* $(A^\alpha)'\cap M^\infty(A)=\boldsymbol{C}$.

15. A *and* A^α *are prime, and* α_g *is properly outer for each* $g\in G\setminus\{0\}$.

Furthermore, if G is the circle group or a finite cyclic group of prime order, then these conditions are equivalent to the following one.

16. A *and* A^α *are prime.*

(If $G=0$, the equivalence of all the conditions, with the exception of 7 and 8, remains valid, but the theorem then reduces to the well-known fact that a separable C^*-algebra is primitive if and only if it is prime.)

2. Proof of Theorem 1. We shall prove the following implications:

$$1\Rightarrow 2\Rightarrow\cdots\Rightarrow 10\Rightarrow 12\Rightarrow\cdots\Rightarrow 15\Rightarrow 4\Rightarrow 1\,;\quad 8\Rightarrow 11\Rightarrow 10\,;\quad 11\Leftrightarrow 11'\,.$$

The implications $1\Rightarrow 2$ and $2\Rightarrow 3$ are obvious.

Ad $3\Rightarrow 4$. Assume 3. Putting $H=G$ we see that A^α is prime. Now fix $0\neq\gamma\in\hat{G}$, and let us show that $\hat{\alpha}_\gamma$ is properly outer.

With $H=\operatorname{Ker}\gamma$, we have that G/H is isomorphic to $\gamma(G)$, which by compactness of G is a closed subgroup of $\boldsymbol{T}$, and therefore either $\boldsymbol{T}$ itself, or a cyclic group of finite order $n=2, 3, \cdots$. In particular, A^H is prime by 3.

It follows, as we shall now show, that the double crossed product $(A\rtimes_\alpha G)\rtimes_{\hat{\alpha}|H^\perp} H^\perp$ is prime.

LEMMA 2.1. *Let G be a compact abelian group, let* α *be a faithful action of G on a* C^**-algebra A, and let H be a closed subgroup of G. The following two conditions are equivalent*:

1. A^H *is prime.*
2. $(A\rtimes_\alpha G)\rtimes_{\hat{\alpha}} H^\perp$ *is prime.*

PROOF. Note that $(A\rtimes_\alpha G)\rtimes_{\hat{\alpha}} H^\perp$ is the fixed point subalgebra of $(A\rtimes_\alpha G)\rtimes_{\hat{\alpha}}\hat{G}$ under $\hat{\hat{\alpha}}_H$. By [32], the system $((A\rtimes_\alpha G)\rtimes_{\hat{\alpha}}\hat{G}, \hat{\hat{\alpha}})$ is isomorphic to the system $(A\otimes K(L^2(G)), \alpha\otimes\lambda)$, where $K(L^2(G))$ denotes the algebra of compact operators on $L^2(G)$, and λ is the representation of G on $K(L^2(G))$ determined by the left regular representation. Since G is compact, we have a canonical system of matrix units $(e_{\chi\chi'})_{\chi,\chi'\in\hat{G}}$ for $K(L^2(G))$, with $e_{\chi\chi'}\in K^\lambda(\chi-\chi')$, i.e. $\alpha_g(e_{\chi\chi'})=\langle\chi-\chi', g\rangle e_{\chi\chi'}$. Clearly

$$(1\otimes e_{00})(A\otimes K)^H(1\otimes e_{00})=A^H\otimes e_{00}\,.$$

Denote by I the closed two-sided ideal of $(A\otimes K)^H$ generated by $A^H\otimes e_{00}$. By Theorem 1.6 of [28], I and the hereditary subalgebra $A^H\otimes e_{00}$ have the same spectrum, and, in particular, one is prime if and only if the other is.

The implication $2 \rightarrow 1$ is now immediate: if $(A\otimes K)^H$ is prime, then I is prime and hence A^H is prime.

To prove the implication $1 \rightarrow 2$, moreover, it is now enough to show that if A^H is prime, then the ideal I of $(A\otimes K)^H$ is essential. Suppose that A^H is prime, and denote the largest closed two-sided ideal of $(A\otimes K)^H$ orthogonal to $A^H\otimes e_{00}$ by J. We must show that $J=0$.

Since G is abelian, both $(A\otimes K)^H$ and $A^H\otimes e_{00}$ are invariant under $\alpha\otimes\lambda$, and therefore J is invariant. Hence (as G is compact), if there exists a nonzero element in J then there exists one of the form $a\otimes e_{\chi\chi'}$ with $a\in A^\alpha(\chi'-\chi)$. (By the proof of Proposition 3.3, below, A^α contains an approximate unit for A. The tensor product of this with finite sums $\sum e_{\chi\chi}$ is an approximate unit for $A\otimes K$, invariant under $\alpha\otimes\lambda$. Hence, if $0\neq x\in J^{\alpha\otimes\lambda}$, then, after multiplication on the left by an element of $A^\alpha\otimes e_{\chi\chi}$ for some χ, and similarly on the right, x has the desired form.) Fix χ, $\chi'\in\hat{G}$ and $a\in A^\alpha(\chi'-\chi)$ with $a\otimes e_{\chi\chi'}\in J$, and let us show that $a=0$. Since $A^\alpha(\chi+H^\perp)\otimes e_{0\chi}\subseteq(A\otimes K)^H$, we have

$$0=(A^H\otimes e_{00})(A^\alpha(\chi+H^\perp)\otimes e_{0\chi})(a\otimes e_{\chi\chi'})=A^H A^\alpha(\chi+H^\perp)a\otimes e_{0\chi'}\,.$$

Since H is compact, A^H contains an approximate unit for A. (See proof of Proposition 3.3.) Hence,

$$A^\alpha(\chi+H^\perp)a=0\,,\quad \text{i.e.,}$$

$$A^\alpha(\chi+H^\perp)^*A^\alpha(\chi+H^\perp)aa^*=0\,.$$

But $A^\alpha(\chi+H^\perp)^*A^\alpha(\chi+H^\perp)$ is a two-sided ideal of A^H, and aa^* belongs to A^H (in fact to A^α). Hence, since A^H is prime, either $A^\alpha(\chi+H^\perp)=0$, or $a=0$. But by Lemma 2.2, below, with $(A, H, \alpha\,|\,H)$ and $\chi+H^\perp\in\hat{H}$ in place of (A, G, α) and χ, since α is faithful and A^H is prime, $A^\alpha(\chi+H^\perp)\neq 0$. Therefore $a=0$, as desired.

LEMMA 2.2. *Let G be a compact abelian group, let α be a faithful action of G on a C^*-algebra A, and suppose that A^α is prime. It follows that* $\operatorname{Sp}\alpha=\hat{G}$, *i.e., for every $\chi\in\hat{G}$, $A^\alpha(\chi)\neq 0$.*

PROOF. First, let us show that $\operatorname{Sp}\alpha$ is a subgroup of $\hat{G}$. If $A^\alpha(\chi)\neq 0$ and $A^\alpha(\chi')\neq 0$, then

$$A^\alpha(\chi-\chi')\supseteq A^\alpha(\chi)A^\alpha(\chi')^*\,,$$

and $A^\alpha(\chi)^*A^\alpha(\chi)$, $A^\alpha(\chi')^*A^\alpha(\chi')$ are nonzero two-sided ideals of the prime algebra A^α, so have nonzero product. This shows that $A^\alpha(\chi-\chi')\neq 0$.

Second, as $\operatorname{Sp}\alpha$ is a subgroup of $\hat{G}$, we have $\operatorname{Sp}\alpha=H^\perp$ where $H=(\operatorname{Sp}\alpha)^\perp\subseteq G$. Hence, in $\hat{H}$,

$$\mathrm{Sp}(\alpha\,|\,H) = (\mathrm{Sp}\,\alpha)/H^\perp = H^\perp/H^\perp = 0\,.$$

In other words, $\alpha\,|\,H$ is trivial. Since α is faithful, $H=0$, i.e. $\mathrm{Sp}\,\alpha = \hat{G}$.

Returning to the proof of the implication $3 \Rightarrow 4$, we now have that $(A \rtimes_\alpha G) \rtimes_{\hat\alpha} H^\perp$ is prime. Hence by Theorem 5.8 of [24] (with $A \rtimes_\alpha G$ in place of A, $H^\perp$ in place of G, and $\hat\alpha\,|\,H^\perp$ in place of α),

$$(H^\perp)^\wedge(\hat\alpha\,|\,H^\perp) = (H^\perp)^\wedge\,.$$

Since $H^\perp$ is the cyclic subgroup of $\hat{G}$ generated by γ, it follows, either by Remark 2.5 of [18] or by Theorem 6.6 of [25], that $\hat\alpha_\gamma$ is properly outer, as desired. (Let us expand on Remark 2.5 of [18]: If β is an automorphism of a C^*-algebra which is not properly outer, then to show that the Connes spectrum $\boldsymbol{T}(\beta)$ (or the Borchers spectrum $\boldsymbol{T}_B(\beta)$) is equal to $\{1\}$, it is enough by 1.3 (or 2.1) of [18] to consider the case $\beta = \exp\delta$ where δ is a derivation. Since $\mathrm{Sp}\,\beta^n = (\mathrm{Sp}\,\beta)^n$, we have $\boldsymbol{T}(\beta^n) \subseteq \boldsymbol{T}(\beta)^n$ (and $\boldsymbol{T}_B(\beta^n) \subseteq \boldsymbol{T}_B(\beta)^n$), and so to prove that $\boldsymbol{T}(\beta) = \{1\}$ we may replace δ by $n^{-1}\delta$ and suppose that $\|\beta - 1\| < |e^{2\pi i/3} - 1|$, so that $\mathrm{Sp}\,\beta$ does not contain any nontrivial subgroup of $\boldsymbol{T}$. But then $\boldsymbol{T}(\beta)$ equals $\{1\}$ because it is a subgroup of $\boldsymbol{T}$ ([30], 8.8.4; to get $\boldsymbol{T}_B(\beta) = \{1\}$ use [30], 8.8.5). Incidentally, combining this argument with Lemma 3.6 of [25] and using compactness of $\boldsymbol{T}$, we have a different proof of Lemma 4.1 of [25], that every derivation is close to zero on some invariant hereditary sub-C^*-algebra. However, this proof does not seem to give a subalgebra which is invariant under all automorphisms commuting with the derivation, as does that in [25].)

Ad $4 \Rightarrow 5$. (We prove this implication by combining ideas from the proof of Theorem 2.1 in [17] and the argument on page 161 in [19].) Assume 4. In particular, A^α is prime, and it follows by Lemma 2.1, with G in place of H, that $A \rtimes_\alpha G$ is prime. (This does not use the hypothesis of proper outerness.)

Since $A \rtimes_\alpha G$ is separable, there exists a sequence (J_n) of nonzero closed two-sided ideals of $A \rtimes_\alpha G$ such that every nonzero closed two-sided ideal contains some J_n ([9], 3.3.4). (If (x_n) is a dense sequence in the unit sphere of $A \rtimes_\alpha G$, we may take J_n to be the smallest closed two-sided ideal of $A \rtimes_\alpha G$ such that $\|x_n + J_n\| \leqslant 1/2$, for if J is any nonzero closed two-sided ideal there is some n such that $\|x_n + J\| \leqslant \|x_n\|/2 = 1/2$.) Since $A \rtimes_\alpha G$ is prime, we may replace J_n by $J_1 \cap \cdots \cap J_n$ and suppose that the sequence (J_n) is decreasing.

Denote by T the set of $a \in A \rtimes_\alpha G$ such that $a \geqslant 0$, $\|a\| = 1$, and there exists $0 \neq b \in A \rtimes_\alpha G$ with $ab = b$. By spectral theory, T is not empty, and if $a \in T$ then there exists $b \in T$ such that $ab = b$.

Choose a dense sequence of unitaries (u_m) in $(A \rtimes_\alpha G)^\sim$, the C^*-algebra $A \rtimes_\alpha G$ with unit adjoined, and let (σ_n) be an enumeration of the automorphisms $(\mathrm{Ad}\,u_m)\hat\alpha_\gamma$, $m = 1, 2, \cdots$, $\gamma \in \hat{G} \setminus \{0\}$.

Construct as follows a sequence (e_n) in T such that

$$e_n e_{n+1} = e_{n+1}, \qquad e_n \in J_n, \quad \text{and} \quad \|e_n \sigma_n(e_n)\| \leqslant n^{-1}.$$

Suppose that we have constructed $e_k \in T$ for $1 \leqslant k < n$ such that $e_{k-1}e_k = e_k$, $e_k \in J_k$, and $\|e_k \sigma_k(e_k)\| \leqslant k^{-1}$. Choose $x \in T$ such that $e_{n-1}x = x$. (If $n=1$, just choose $x \in T$.) By Proposition 6.4 of [25], applied to the properly outer automorphism σ_n and the hereditary sub-C^*-algebra $J_n \cap (x(A \rtimes_\alpha G)x)^-$, there exists e_n in this subalgebra such that $0 \leqslant e_n$, $\|e_n\| = 1$, and $\|e_n \sigma_n(e_n)\| < n^{-1}$. Necessarily, $e_{n-1}e_n = e_n$, and modifying e_n slightly using spectral theory ensures that, in addition, $e_n \in T$, as desired.

As (e_n) is a decreasing sequence of positive elements of $A \rtimes_\alpha G$ of norm one, the set of states of $A \rtimes_\alpha G$ with value 1 on e_n for all n is a nonempty compact face in the state space of $A \rtimes_\alpha G$. Therefore, there exists a pure state ϕ_0 of $A \rtimes_\alpha G$ such that $\phi_0(e_n) = 1$ for all n. Denote by ϕ the unique $\hat{\hat{\alpha}}$-invariant extension of ϕ_0 to a state of $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$. We shall show that π_ϕ is faithful and that ϕ is pure.

Since $\phi_0(e_n) = 1$ and $e_n \in J_n$, J_n is not contained in $\operatorname{Ker} \pi_{\phi_0}$ for any n. Hence $\operatorname{Ker} \pi_{\phi_0} = 0$. Since $\operatorname{Ker} \pi_\phi$ is $\hat{\hat{\alpha}}$-invariant, if it were nonzero its intersection with the fixed point subalgebra $A \rtimes_\alpha G$ would be nonzero, but this intersection is clearly contained in $\operatorname{Ker} \pi_{\phi_0}$, which is zero. Therefore $\operatorname{Ker} \pi_\phi = 0$.

Since ϕ_0 is pure, to show that ϕ is pure, it suffices to show that ϕ is the unique extension of ϕ_0 to a state of $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$. Let ψ be a state of $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ such that $\psi | A \rtimes_\alpha G = \phi_0$. To show that $\psi = \phi$, we must show that ψ is $\hat{\hat{\alpha}}$-invariant, i.e. that $\psi(bu(\gamma)) = 0$ for any $b \in A \rtimes_\alpha G$ and any $0 \neq \gamma \in \hat{G}$. Here $u(\gamma)$ denotes the unitary multiplier of the crossed product by $\hat{G}$ canonically associated with $\gamma \in \hat{G}$. Since any C^*-algebra is spanned linearly by its unitary elements (Proposition 3, page 4 of [8]), it is enough to suppose that b is unitary, and then of course b can be approximated by a subsequence of (u_m). In the enumeration, above, of $(\operatorname{Ad} u_m)\hat{\alpha}_\gamma$ as σ_n, let us denote by $\sigma_{m,\gamma}$ and $e_{m,\gamma}$ the σ_n and e_n corresponding to $(\operatorname{Ad} u_m)\hat{\alpha}_\gamma$. Thus, for fixed $0 \neq \gamma \in \hat{G}$,

$$e_{m,\gamma}\sigma_{m,\gamma}(e_{m,\gamma}) \to 0 \qquad (m \to \infty).$$

In other words, with $\gamma \in \hat{G} \setminus \{0\}$ fixed, we have

$$e_{m,\gamma} u_m u(\gamma) e_{m,\gamma} = e_{m,\gamma} \sigma_{m,\gamma}(e_{m,\gamma}) u_m u(\gamma) \to 0.$$

Since $\psi(e_{m,\gamma}) = \phi_0(e_{m,\gamma}) = 1$ for all m, we have

$$\psi(u_m u(\gamma)) = \psi(e_{m,\gamma} u_m u(\gamma) e_{m,\gamma}) \to 0,$$

whence $\psi(bu(\gamma)) = 0$. Therefore, $\psi = \phi$. This shows that ϕ is pure.

Let us again identify $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ with $A \otimes K(L^2(G))$, and $\hat{\hat{\alpha}}$ with $\alpha \otimes \lambda$. Then, with $e_{\gamma\gamma}$ as above, in the proof of Lemma 2.1, for any $\gamma \in \hat{G}$ the positive functional $\phi_\gamma = (1 \otimes e_{\gamma\gamma})\phi(1 \otimes e_{\gamma\gamma})$, if nonzero, is a scalar multiple of an $\hat{\hat{\alpha}}$-invariant pure state of $A \otimes e_{\gamma\gamma} \subseteq A \otimes K$, i.e. an α-invariant pure state, say ω_γ, of A. Furthermore, for some $\gamma \in \hat{G}$, ϕ_γ is nonzero, and then π_{ϕ_γ} is faithful, since π_ϕ is. Since $\pi_{\phi_\gamma}(a \otimes e_{\gamma\gamma}) = \pi_{\omega_\gamma}(a) \otimes e_{\gamma\gamma}$, it follows that, for such γ, π_{ω_γ} is a faithful representation of A. This shows that, with $\omega=\omega_\gamma$ for such a γ, ω is an α-invariant pure state of A, and π_ω is faithful, as desired.

Ad 5 ➞ 6. Condition 5 implies that both A and A^α are prime, whence by 8.10.4 of [30], $\hat{G}(\alpha)=\mathrm{Sp}\,\alpha$. In particular, as $\hat{G}(\alpha)$ is group ([30], 8.8.4), so also is $\mathrm{Sp}\,\alpha$, and since α is faithful this implies $\mathrm{Sp}\,\alpha=\hat{G}$.

Ad 6 ➞ 7. Except for the property (iii), this is exactly Theorem 2.1 of [5]. Referring to the proof of that theorem, we ensure that B and q have the extra property as follows.

Since A is separable and prime there exists a decreasing sequence (J_n) of nonzero closed two-sided ideals of A such that any nonzero closed two-sided ideal of A contains some J_n (see proof of 4 ➞ 5 above). Also, since A is prime and G is compact we have

$$\bigcap_{g\in G} \alpha_g(J)\neq 0$$

for any nonzero closed two-sided ideal J of A. (First, by strong continuity of α, for any $h\in G$ there is a neighbourhood U_h of h in G such that $I_h=\bigcap_{g\in U_h}\alpha_g(J)\neq 0$; by compactness of G, there are $h_1,\cdots,h_k\in G$ such that $U_{h_1}\cup\cdots\cup U_{h_k}=G$; finally, by primeness of A, $I_{h_1}\cap\cdots\cap I_{h_1}\neq 0$, i.e. $\bigcap_{g\in G}\alpha_g(J)\neq 0$.) It follows that

$$J\cap A^\alpha\neq 0$$

for any nonzero closed two-sided ideal J of A. In particular, $(J_n\cap A^\alpha)$ is a decreasing sequence of nonzero closed two-sided ideals of A^α.

Now note that the quasimatrix system (e_n), $(v_{n,i})_{i=1}^{d_n}$ of Lemma 2.7 of [5] can be constructed so that

$$e_n\in J_n\cap A^\alpha\,,$$

$n=1,2,\cdots$. Then the projection $q_n\in A^{**}$ defined in the proof of Theorem 2.1 of [5] is contained in J_n^{**}, and hence the limit $q=\lim q_n$ is contained in $\bigcap J_n^{**}$. Since any nonzero closed two-sided ideal J contains some J_n, (iii) holds.

Ad 7 ➞ 8. 8 is a special case of 7.

Ad 8 ➞ 9. Assume 8. Denote by τ the unique tracial state of the C^*-algebra qAq, which is isomorphic to the Glimm algebra M_{2^∞}, and denote by ω the corresponding state of A,

$$A\ni a\longmapsto\tau(qaq)\,.$$

Since q is α^{**}-invariant, and τ is unique, ω is α-invariant.

Since $\omega(q)=1$, $q\notin\mathrm{Ker}(\pi_\omega^{**})\supseteq(\mathrm{Ker}\,\pi_\omega)^{**}$, and hence by 8(iii), $\mathrm{Ker}\,\pi_\omega=0$.

That $\pi_\omega(A^\alpha)'\cap\pi_\omega(A)''=\boldsymbol{C}$ was shown in the proof of Theorem 3.1 of [15]. (The case $q=1$ is Lemma 4.2 of [4].)

Ad 9 ➡ 10. This is evident.

Ad 8 ➡ 11. Assume 8. Let (ϕ_n) be any sequence of pure states of M_2 such that, for each $\chi \in \hat{G}$, the (infinite) subsequence $(\phi_n)_{\chi_n=\chi}$ contains a subsequence that converges to a nondiagonal pure state, i.e. to a pure state with density matrix not equal to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For instance, ϕ_n may be taken to be a fixed pure state with density matrix different from $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ (e.g. the pure state with density matrix $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$).

Denote by ϕ the pure state of qAq obtained from $\bigotimes_{n=1}^{\infty} \phi_n$ by identifying qAq with $\bigotimes_{n=1}^{\infty} M_2$ as in 8, and denote also by ϕ the corresponding pure state of A,

$$A \ni a \longmapsto \phi(qaq) = (\bigotimes_{n=1}^{\infty} \phi_n)(qaq)\,.$$

Since $\phi(q)=1$, $q \notin \operatorname{Ker}(\pi_\phi^{**}) \supseteq (\operatorname{Ker}\pi_\phi)^{**}$, and hence by 8(iii), $\operatorname{Ker}\pi_\phi = 0$.

We shall show that $\pi_\phi(A^\alpha)'' = \pi_\phi(A)''$. We shall show this, or, equivalently, that $\pi_\phi(A^\alpha)' = \boldsymbol{C}$, in two steps: first, we shall show that Φ, the canonical cyclic vector for $\pi_\phi(A)$, is also cyclic for $\pi_\phi(A^\alpha)$, and thus separating for $\pi_\phi(A^\alpha)'$; and, second, we shall show that $\pi_\phi(A^\alpha)'\Phi = \boldsymbol{C}\Phi$.

Let us show that Φ is cyclic for $\pi_\phi(A^\alpha)$. To do this, we shall show that, for each $\chi \in \hat{G}$,

$$\Phi \in \pi_\phi(A^\alpha(\chi))^- \Phi\,,$$

where the bar denotes ultraweak closure. Then, for each $\chi \in \hat{G}$,

$$\begin{aligned}\pi_\phi(A^\alpha(-\chi))\Phi &\subseteq \pi_\phi(A^\alpha(-\chi))\pi_\phi(A^\alpha(\chi))^-\Phi \subseteq \pi_\phi(A^\alpha(-\chi)A^\alpha(\chi))^-\Phi \\ &\subseteq \pi_\phi(A^\alpha)^-\Phi \subseteq (\pi_\phi(A^\alpha)\Phi)^-\,,\end{aligned}$$

where the last bar denotes weak closure in H_ϕ, which on a linear subspace is the same as norm closure. Since the closed linear span of $\bigcup_{\chi\in\hat{G}} A^\alpha(-\chi)$ is equal to A, and Φ is cyclic for $\pi_\phi(A)$, this shows that Φ is cyclic for $\pi_\phi(A^\alpha)$.

Let, then, χ be an element of $\hat{G}$ and let us show that $\Phi \in \pi_\phi(A^\alpha(\chi))^-\Phi$. (If $\chi=0$ this follows from the fact that A^α contains an approximate unit for A,—see for example the proof of Proposition 3.3, below.) For each k such that $\chi_k=\chi$ denote by c_k the image of $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2$ under the k-th embedding of M_2 in $qAq = \bigotimes_{n=1}^{\infty} M_2$. Then, by the choice of (ϕ_n),

$$\phi(c_k) = \phi_k\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \not\to 0\,.$$

(Here k is such that $\chi_k=\chi$.) Passing to a subsequence of (c_k), we may suppose that

$$\phi(c_k) \to \lambda \neq 0 ,$$

and then, since (c_k) is a central sequence in qAq, and $\pi_\phi(qAq)$ is irreducible on $\pi_\phi(q)H_\phi$,

$$\pi_\phi(c_k) \to \lambda\pi_\phi(q) \quad \text{ultraweakly} .$$

Since $q \in B'$, the map

$$B \ni b \longmapsto bq \in Bq = qAq$$

is a morphism, and so we can choose $b_k \in B$ with $b_k q = c_k$, and $\|b_k\| \leqslant 2$ (even with $\|b_k\| = 1$). Replacing b_k by $\int_G \overline{\langle \chi, g \rangle} \alpha_g(b_k) dg$, we may suppose that $b_k \in B^\alpha(\chi)$. If now b is any ultraweak limit point of the sequence $\pi_\phi(b_k)$, we have

$$b\pi_\phi(q) = \pi_\phi(q)b = \lambda\pi_\phi(q) ,$$

and as $\pi_\phi(q)\Phi = \Phi$ we have

$$\Phi = \lambda^{-1} b\Phi \in \pi_\phi(A^\alpha(\chi))^- \Phi ,$$

as desired.

Now let us show that $\pi_\phi(A^\alpha)'\Phi = \mathbf{C}\Phi$. Since q is an α^{**}-invariant projection in A^{**} and also is closed, we have $q \in (A^\alpha)^{**} \subseteq A^{**}$. ($1-q$ is the unit of the ultraweak closure in A^{**} of an α-invariant hereditary sub-C^*-algebra of A, and so is the limit of an approximate unit of this subalgebra; this approximate unit may, as remarked above, be chosen to be α-invariant.) In particular, $\pi_\phi(q) \in \pi_\phi(A^\alpha)''$. Moreover, for each $\chi \in \hat{G}$, it was shown in the preceding paragraph that $\pi_\phi(q) \in \pi_\phi(A^\alpha(\chi))^-$ (where the bar denotes ultraweak closure). Hence, for each $\chi \in \hat{G}$,

$$\begin{aligned} \pi_\phi(A^\alpha(-\chi)q) &\subseteq \pi_\phi(A^\alpha(-\chi))\pi_\phi(A^\alpha(\chi))^- \\ &\subseteq \pi_\phi(A^\alpha(-\chi)A^\alpha(\chi))^- \subseteq \pi_\phi(A^\alpha)^- . \end{aligned}$$

Since the closed linear span of $\bigcup_{\chi \in \hat{G}} A^\alpha(-\chi)$ is equal to A, and $\pi_\phi(A)$ is irreducible, this shows that $\pi_\phi(qA^\alpha q)$ is irreducible on $\pi_\phi(q)H_\phi$. Now we have $\pi_\phi(q) \in \pi_\phi(A^\alpha)''$ and

$$\pi_\phi(A^\alpha)'\pi_\phi(q) = \pi_\phi(qA^\alpha q)'\pi_\phi(q) = \mathbf{C}\pi_\phi(q) .$$

Since $\pi_\phi(q)\Phi = \Phi$ it follows that $\pi_\phi(A^\alpha)'\Phi = \mathbf{C}\Phi$.

Ad $11 \Rightarrow 10$. This is evident.

Ad $10 \Rightarrow 12$. This is Corollary 5.4, below.

REMARK. It is interesting to inquire whether the implications $9 \Rightarrow 11$ and $11 \Rightarrow 9$ (which are now established, since $12 \Rightarrow 1$ is evident) can be proved directly. Certainly, our proof of $8 \Rightarrow 11$ yields an alternative proof of $8 \Rightarrow 9$, since the pure states ϕ constructed in the proof of $8 \Rightarrow 11$ constitute a direct integral decomposition of the invariant state ω constructed in our proof of $8 \Rightarrow 9$ above. Given a state ω as in 9, and a direct integral decomposition of π_ω as $\int^{\oplus} \pi(\zeta)d\mu(\zeta)$, must almost every $\pi(\zeta)$ verify 11?

Ad $11 \Leftrightarrow 11'$. If π is an irreducible representation of A, then, as we shall show, π verifies 11 if, and only if, it verifies $11'$. (See also [20].)

$11 \Rightarrow 11'$, for π, is just Lemma 3.5 of [15]. (The proof in [15] does not require that π be faithful.)

Assume $11'$. Set $\int_G^{\oplus} \pi\alpha_g dg = \rho$. ρ is α-covariant, and α is implemented by the right regular representation of G on $H_\pi \otimes L^2(G)$ (we do not need here that G is abelian). Since G is compact,

$$\rho(A^\alpha)'' = (\rho(A)'')^\alpha = (\pi(A)'' \otimes L^\infty(G))^\alpha = \pi(A)'' \otimes 1 .$$

But $\rho(A^\alpha) = \pi(A^\alpha) \otimes 1$, so

$$\pi(A)'' \otimes 1 = \rho(A^\alpha)'' = \pi(A^\alpha)'' \otimes 1 .$$

(We have not used here that π is irreducible.)

Ad $12 \Rightarrow 13$. Assume 12. Let β be an automorphism of $M^\infty(A)$ such that $\beta \,|\, A^\alpha = 1$. Let us prove that $\beta = \alpha_g$ for some $g \in G$.

By Proposition 3.3, below, $M^\infty(A)^\alpha \subseteq M^\infty(A^\alpha)$, and it follows that $\beta \,|\, M^\infty(A)^\alpha = 1$. We now note that, except for continuity of α from G into Aut $M^\infty(A)$, all the hypotheses of Theorem 2.1 of [3] are fulfilled, with $M^\infty(A)$ in place of A, and $(U(M^\infty(A)^\alpha), \mathrm{Ad})$ in place of (H, τ). The proof of Theorem 2.1 of [3], which is valid without continuity of α until the very last line—provided that $M^\infty(A)_F$ is defined as the set of all $x \in M^\infty(A)$ such that the linear span of $\alpha_G(x)$ is finite-dimensional—, yields that, for some $g \in G$,

$$\beta(x) = \alpha_g(x) \qquad \text{for all} \quad x \in M^\infty(A)_F .$$

In particular, this holds for all $x \in A_F$, and since A_F is dense in A (continuity is known for $\alpha\colon G \to \mathrm{Aut}\, A$), this shows that $\beta = \alpha_g$. (Here we have not used that G is abelian. A proof in the case that G is abelian can also be obtained by modifying, in a somewhat less trivial way, the proof of Theorem 3.1 of [21].)

Ad $13 \Rightarrow 14$. Assume 13. Then for each unitary $u \in (A^\alpha)' \cap M^\infty(A)$, $\mathrm{Ad}\, u = \alpha_g$ for some $g \in G$. By Proposition 3.1, as A is prime, Centre $M^\infty(A) = \boldsymbol{C}$. By commutativity of G, it follows that $\alpha_g(u) \in \boldsymbol{T}u$ for every $u \in (A^\alpha)' \cap M^\infty(A)$ and every $g \in G$. But, for fixed such u, and fixed $g \in G$, it follows from the fact that $\alpha_g(v) \in \boldsymbol{T}v$ for every unitary v in the C^*-algebra generated by u, that $\alpha_g(u) = u$. Since g is arbitrary, it follows that u is in $M^\infty(A)^\alpha$, which by Proposition 3.3 is contained in $M^\infty(A^\alpha)$. But $u \in (A^\alpha)'$, so u belongs to Centre $M^\infty(A^\alpha)$. By Proposition 3.1, as A^α is prime, this is equal to $\boldsymbol{C}$. Since any C^*-algebra is spanned linearly by its unitary elements (Proposition 3, page 4 of [8]), we have $(A^\alpha)' \cap M^\infty(A) = \boldsymbol{C}$.

We should like to point out that if G is not abelian, then the implication 13 $\rightarrow$ 14 may fail. For example, it fails if $A = M_n$, $n = 2, 3, \cdots$, and $G = \operatorname{Aut} A$. However, this may be essentially the only case in which the implication fails.

Ad 14 $\rightarrow$ 15. Assume 14. Let us first show that A is prime. Clearly, Centre $M^\infty(A) = \boldsymbol{C}$; by Proposition 3.1, this just says that A is prime.

Let $g \in G$, and suppose that α_g is not properly outer. By Proposition 3.2, there is a unitary u in $M^\infty(A)$ such that $\alpha_g = \operatorname{Ad} u$. Since $u \in (A^\alpha)' \cap M^\infty(A)$, by 14 we have $\alpha_g = 1$. Since α is faithful, $g = 0$.

Ad 15 $\rightarrow$ 4. Assume 15. In particular, A^α is prime. Hence by Lemma 2.1 (as in the proof of 4 $\rightarrow$ 5), with $H = G$, also $A \rtimes_\alpha G$ is prime.

Let $\gamma \in \hat{G}$, and suppose that $\hat{\alpha}_\gamma$ is not properly outer. We must prove that $\gamma = 0$. By Proposition 3.2, as $A \rtimes_\alpha G$ is prime, there exists a unitary $u \in M^\infty(A \rtimes_\alpha G)$ such that $\hat{\alpha}_\gamma = \operatorname{Ad} u$. By Proposition 3.1, Centre $M^\infty(A \rtimes_\alpha G) = \boldsymbol{C}$. Therefore u is unique up to a scalar multiple. By commutativity of $\hat{G}$, it follows that $u^{-1}\hat{\alpha}_\xi(u) \in \boldsymbol{T}$ for every $\xi \in \hat{G}$. Therefore the map $\xi \longmapsto u^{-1}\hat{\alpha}_\xi(u)$ is a character of $\hat{G}$, and so there exists $g \in G = \hat{\hat{G}}$ with

$$\hat{\alpha}_\xi(u) = \langle \xi, g \rangle u, \qquad \xi \in \hat{G}.$$

Since also

$$\hat{\alpha}_\xi(\lambda(g)) = \langle \xi, g \rangle \lambda(g), \qquad \xi \in \hat{G},$$

where $\lambda(g)$ is the canonical unitary multiplier of $A \rtimes_\alpha G$ corresponding to g, it follows that, with $v = \lambda(g)u^*$,

$$v \in M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}.$$

By the choice of u, $\lambda(t)u\lambda(t)^{-1} = \langle -\gamma, t \rangle u$, $t \in G$, and hence as G is abelian, $\lambda(t)v\lambda(t)^{-1} = \langle \gamma, t \rangle v$, $t \in G$. Since $uAu^{-1} = A$, also $vAv^{-1} = A$. Hence by Proposition 3.4,

$$v \in M^\infty(A) \subseteq M^\infty(A \rtimes_\alpha G).$$

Since $(\operatorname{Ad} v) \mid A = \alpha_g$, it follows by Proposition 3.2 that α_g is not properly outer. Therefore, by 15, $g = 0$. This shows that

$$u \in M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}.$$

Recall that $\lambda(t)u\lambda(t)^{-1} = \langle -\gamma, t \rangle u$, $t \in G$, and $uAu^{-1} = A$. Hence by Proposition 3.4,

$$u \in M^\infty(A) \subseteq M^\infty(A \rtimes_\alpha G).$$

By the choice of u, $\operatorname{Ad} u = \hat{\alpha}_\gamma$, and in particular, $(\operatorname{Ad} u) \mid A = 1$. Therefore, $u \in$ Centre $M^\infty(A)$. As A is prime, by Proposition 3.1 Centre $M^\infty(A) = \boldsymbol{C}$, and so $u \in \boldsymbol{T}$. Hence $\hat{\alpha}_\gamma = \operatorname{Ad} u = 1$, and $\gamma = 0$, as desired.

Ad 4 $\rightarrow$ 1. This follows from 4 $\rightarrow$ 12, established above, since 12 is clearly stronger than 1. Let us, however, give a more direct proof of $4 \Rightarrow 1$.

Assume 4. Let x and y be nonzero elements of A, and let us find $a \in A^\alpha$ such that $xay \neq 0$. Replacing x and y by x^*x and yy^*, we may suppose that x and y are positive.

As in the proof of Lemma 2.1, above, we shall identify the systems $((A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}, \hat{\hat{\alpha}})$ and $(A \otimes K(L^2(G)), \alpha \otimes \lambda)$, and the subalgebras $A \rtimes_\alpha G \subseteq (A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ and $(A \otimes K)^{\alpha \otimes \lambda} \subseteq A \otimes K$. Recall that

$$(1 \otimes e_{00})(A \otimes K)^{\alpha \otimes \lambda}(1 \otimes e_{00}) = A^\alpha \otimes e_{00} ,$$

where e_{00} is the projection onto the one-dimensional subspace of $L^2(G)$ generated by the trivial character.

Since x and y are positive and nonzero, so also are $x_0 = \int_G \alpha_g(x)dg$ and $y_0 = \int_G \alpha_g(y)dg$. We shall prove the following inequality, which is stronger than what is needed:

$$\sup_{a \in A^\alpha, \|a\| \leqslant 1} \|xay\| \geqslant \|x_0\| \, \|y_0\| .$$

Let us identify x, y with $x \otimes e_{00}$, $y \otimes e_{00} \in A \otimes e_{00}$. Recalling that $A \otimes K = (A \rtimes_\alpha G) \rtimes_\alpha \hat{G}$, we may replace x and y by finite approximating sums

$$\sum_{\gamma \in F} x_\gamma u(\gamma) , \qquad \sum_{\gamma \in F} y_\gamma u(\gamma) ,$$

where x_γ, $y_\gamma \in A \rtimes_\alpha G$ for each γ in the finite set $F \subseteq \hat{G}$. Here we cannot insist that all x_γ and y_γ be the canonical (Fourier) coefficients of (the original) x and y, since the Fourier series only converges in the Cesàro mean in general. However, we may certainly assume that x_0 and y_0 are as defined above (the zeroth Fourier coefficients). We may also assume that all x_γ and y_γ lie in $A^\alpha \otimes e_{00}$, the cutdown of $A \rtimes_\alpha G$ by $1 \otimes e_{00}$. From now on we shall suppose that x and y are equal to such finite sums. We shall also suppose that $\|x\| = \|y\| = 1$.

We shall now use Proposition 6.4 of [25], as extended in Lemma 7.1 of [25]. (We could equally well use Lemma 3.2 of [17].) Let us apply Lemma 7.1 of [25], with $(A \otimes K)^{\alpha \otimes \lambda} = A \rtimes_\alpha G$ in place of A, $(\hat{\alpha}_\gamma)_{\gamma \in F \setminus \{0\}}$ in place of $\alpha_1, \cdots, \alpha_n$, and, successively, $(x_\gamma)_{\gamma \in F}$ and $(y_\gamma)_{\gamma \in F}$ in place of $a_0, a_1, \cdots, a_n$. This yields, for each $\varepsilon > 0$, elements w and z of $A \rtimes_\alpha G$, of norm one, such that

$$\|wx_0w\| \geqslant \|x_0\| - \varepsilon , \qquad \|zy_0z\| \geqslant \|y_0\| - \varepsilon , \quad \text{and}$$

$$\|wx_\gamma \hat{\alpha}_\gamma(w)\| \leqslant \varepsilon , \quad \|zy_\gamma \hat{\alpha}_\gamma(z)\| \leqslant \varepsilon , \quad \gamma \in F \setminus \{0\} .$$

The proof of Lemma 7.1 of [25] in fact produces w and z that belong to the hereditary sub-C^*-algebras generated by x_0 and y_0, which are contained in $A^\alpha \otimes e_{00}$, and so we may suppose that $w, z \in A^\alpha \otimes e_{00}$.

Since $A^\alpha \otimes e_{00}$ is prime, there exists $b \in A^\alpha \otimes e_{00}$ such that $\|b\| = 1$ and

$$\|wx_0wbzy_0z\| \geqslant \|wx_0w\| \, \|zy_0z\| - \varepsilon .$$

Hence, with $a=wbz$, we have $\|a\|\leqslant 1$, $a\in A^\alpha\otimes e_{00}$, and

$$\begin{aligned}\|xay\| &\geqslant \|wxwbzyz\|\\ &\geqslant \|wx_0wbzy_0z\| - \sum_{\gamma,\gamma'\in F\setminus\{0\}} \|wx_\gamma u(\gamma)wbzy_{\gamma'}u(\gamma')z\|\\ &\geqslant \|wx_0w\|\,\|zy_0z\| - \varepsilon - \sum_{\gamma,\gamma'\in F\setminus\{0\}} \|wx_\gamma\hat{\alpha}_\gamma(w)\|\,\|zy_{\gamma'}\hat{\alpha}_{\gamma'}(z)\|\\ &\geqslant \|x_0\|\,\|y_0\| - 3\varepsilon - n^2\varepsilon^2\,,\end{aligned}$$

where $n=\mathrm{card}(F\setminus\{0\})$. Since $\varepsilon>0$ is arbitrary, the desired inequality is proved.

Finally, suppose that $G=\boldsymbol{T}$ or $G=\boldsymbol{Z}/p\boldsymbol{Z}$ with p prime, and let us show that 16 is equivalent to 1 to 15.

Ad $3\Rightarrow 16$. This is evident.

Ad $16\Rightarrow 3$. In the case $G=\boldsymbol{Z}/p\boldsymbol{Z}$ with p prime, this is evident, as G is simple.

In the case $G=\boldsymbol{T}$, there are nontrivial proper closed subgroups, but these are all finite. Assume 16. Let H be a closed subgroup of G, where now $G=\boldsymbol{T}$. If $G/H\cong\boldsymbol{Z}/n\boldsymbol{Z}$ for some $n=1, 2, \cdots$, then necessarily $n=1$ and $H=G$, and so A^H is equal to A^α, which is prime by 16. If $G/H\cong\boldsymbol{T}$, then H is finite and cyclic; choose an element h generating H. We must show that A^H is prime.

By 16, A is prime. Therefore, by Theorem 1 of [26], it is equivalent to show that, if β denotes the restriction of α to the subgroup $H\subseteq G$, then the Connes spectrum of β is equal to the Arveson spectrum of β—i.e., $\hat{H}(\beta)=\mathrm{Sp}\,\beta$. In terms of the automorphism α_h, this says that, for every nonzero hereditary sub-C^*-algebra B of A which is invariant under α_h,

$$(*)\qquad \mathrm{Sp}(\alpha_h\,|\,B)=\mathrm{Sp}\,\alpha_h\,.$$

By 16, A^α is prime, and so by Theorem 1 of [26], $\hat{G}(\alpha)=\mathrm{Sp}\,\alpha$. In particular, $(*)$ holds if B is α-invariant. By Proposition 5.1 of [23], applied to α_h, and as simplified using that A is prime, there exists a canonical nonzero closed two-sided ideal J of A, invariant under α_h, such that $(*)$ holds when both sides are restricted to J, i.e.

$$\mathrm{Sp}(\alpha_h\,|\,B\cap J)=\mathrm{Sp}(\alpha_h\,|\,J)\,.$$

That J is canonical entails that J is invariant under α. (J is in fact constructed to contain all other such ideals. See also Proposition 3.1 of [18].) Therefore (as $\hat{G}(\alpha)=\mathrm{Sp}\,\alpha$), $(*)$ holds for J. Hence, for any α_h-invariant B,

$$\mathrm{Sp}\,\alpha_h\supseteq\mathrm{Sp}(\alpha_h\,|\,B)\supseteq\mathrm{Sp}(\alpha_h\,|\,B\cap J)=\mathrm{Sp}(\alpha_h\,|\,J)=\mathrm{Sp}\,\alpha_h\,,$$

i.e. $(*)$ holds for B, as desired.

3. Auxiliary results concerning the limit multiplier algebra.

3.1. PROPOSITION. *Let A be a C^*-algebra. The following four properties are equivalent.*

(i) *A is prime.*

(ii) *$M^\infty(A)$ is prime.*

(iii) Centre $M^\infty(A)=\boldsymbol{C}$.

(iv) Centre $M(I)=\boldsymbol{C}$ *for every nonzero closed two-sided ideal I of A.*

PROOF. *Ad* (i) $\rightarrow$ (ii). As pointed out on page 303 of [29], this follows from the fact that each nonzero closed two-sided ideal of $M^\infty(A)$ has a nonzero intersection with A.

Ad (ii) $\rightarrow$ (iii). This is evident.

Ad (iii) $\rightarrow$ (iv). Assume (iii). Let I be a nonzero closed two-sided ideal of A. To show that Centre $M(I)=\boldsymbol{C}$ it is sufficient to do this with I replaced by $I+J$ where $IJ=0$. Therefore, we may suppose that I is essential, so that $M(I)\subseteq M^\infty(A)$. If J is any essential closed two-sided ideal of I, then Centre $M(I)\subseteq$ Centre $M(J)$, as follows by considering a faithful representation of I which is nondegenerate on J. Hence

$$\text{Centre } M(I)\subseteq \text{Centre } M^\infty(A)\,.$$

In particular, from (iii) follows Centre $M(I)=\boldsymbol{C}$.

A natural question arises here: is Centre $M^\infty(A)$ the inductive limit of Centre $M(I)$ (I an essential ideal)?

Ad (iv) $\rightarrow$ (i). If A is not prime, then there exist nonzero closed two-sided ideals I_1 and I_2 of A with $I_1I_2=0$. Set $I_1+I_2=I$. Then I is nonzero and Centre $M(I)\neq\boldsymbol{C}$.

3.2. PROPOSITION. *Let A be a separable prime C^*-algebra, and let α be an automorphism of A. The following three properties are equivalent.*

(i) *α is not properly outer.*

(ii) *α is inner in $M^\infty(A)$.*

(iii) *α is weakly inner in every faithful factor representation of A.*

PROOF. *Ad* (i) $\rightarrow$ (ii). Assume (i). By definition, there is a nonzero invariant closed two-sided ideal I of A such that for some unitary u in $M(I)$, $\|\alpha\,|\,I-(\mathrm{Ad}\,u)\,|\,I\|<2$. By the Kadison-Ringrose theorem ([16]), there exists a derivation δ of I such that

$$\alpha\,\big|\,I=(\mathrm{Ad}\,u)\exp\delta\,.$$

By Proposition 2 of [29], as A is separable, δ is inner in $M^\infty(I)$. Since A is prime, $M^\infty(I)=M^\infty(A)$, so α is inner in $M^\infty(A)$, as desired.

Ad (ii) $\rightarrow$ (iii). This follows from the fact, stated on page 303 of [29], that any faithful factor representation of A extends to a representation of $M^\infty(A)$. (Here we do not need A to be separable. Also, the implication (ii) $\rightarrow$ (i) holds for any C^*-algebra.)

Ad (iii) $\Rightarrow$ (i). Assume that α is properly outer. By the proof of Theorem 2.1 of [17], with Lemma 1.1 of [17] replaced by Proposition 6.4 of [25] (see also Proposition 6.5 of [25]), there exists a pure state ϕ of A such that $\phi\alpha$ is disjoint from ϕ. A modification of the proof of Theorem 2.1 of [17], using that A is separable and prime in the same way as in the proof of 4 $\Rightarrow$ 5 of Theorem 1, above, shows that ϕ may be chosen so that π_ϕ is faithful. Thus, π_ϕ is a faithful factor representation in which α is not weakly inner.

3.3. PROPOSITION. *Let A be a C^*-algebra, let G be a compact group, and let α be an action of G on A. Then $M^\infty(A)^\alpha \subseteq M^\infty(A^\alpha)$.*

PROOF. As shown in the proof of 6 $\Rightarrow$ 7 of Theorem 1, above, if I is a nonzero closed two-sided ideal of A, then (as G is compact) I contains a nonzero α-invariant closed two-sided ideal; the largest such is of course $\bigcap_{g\in G}\alpha_g(I)$. It follows easily that if I is essential, then also $\bigcap_{g\in G}\alpha_g(I)$ is essential.

This shows that, in the definition of $M^\infty(A)$, as the inductive limit of multiplier algebras $M(I)$ over all essential closed two-sided ideals I, we may restrict I to be α-invariant without changing the definition (or, at least, without changing the resulting algebra). Thus,

$$M^\infty(A) = \lim_{I \text{ invariant}} M(I)\,.$$

Hence, using a second time that G is compact, we have

$$M^\infty(A)^\alpha = \lim_{I \text{ invariant}} M(I)^\alpha\,.$$

Next, let us show that for invariant I, $M(I)^\alpha = M(I^\alpha)$. We have

$$M(I)^\alpha \subseteq M(I^\alpha) \subseteq M(I)\,,$$

where the first inclusion is evident, and the second holds as I^α contains an approximate unit for I. (This uses again that G is compact: If (e_i) is an approximate unit for I, then so also is $(\int_G \alpha_g(e_i)dg)$. To see this just note that $e_i\alpha_g^{-1}(a)\to\alpha_g^{-1}(a)$ uniformly in g since G is compact, or, in other words, $\alpha_g(e_i)a\to a$ uniformly in g.) Hence, immediately, $M(I^\alpha) = M(I)^\alpha$.

We now have

$$M^\infty(A)^\alpha = \lim_{I \text{ invariant}} M(I^\alpha) \subseteq M^\infty(A^\alpha)\,.$$

3.4. PROPOSITION. *Let A be a C^*-algebra, let G be a compact abelian group, and let α be an action of G on A. It follows that*

$$M^\infty(A) \subseteq M^\infty(A \rtimes_\alpha G)^{\hat\alpha}\,.$$

Assume that A is separable and prime, that A^α is prime, and that G is separable, and let u be a unitary element of $M^\infty(A \rtimes_\alpha G)^{\hat\alpha}$ such that $uAu^{-1} = A$, and, for some $\gamma \in \hat G$,

$$\lambda(g)u\lambda(g)^{-1} = \langle \gamma, g\rangle u\,, \qquad g \in G\,.$$

It follows that $u \in M^{\infty}(A)$.

PROOF. First, using only that G is compact, let us show that $M^{\infty}(A) \subseteq M^{\infty}(A \rtimes_{\alpha} G)$.

If I is an α-invariant closed two-sided ideal of A, then $I \subseteq M(I \rtimes_{\alpha} G)$, and since an approximate unit for I acts also as one on $I \rtimes_{\alpha} G$, also $M(I) \subseteq M(I \rtimes_{\alpha} G)$. (This uses only that G is locally compact.)

Since G is amenable, for each α-invariant essential closed two-sided ideal I of A, the crossed product ideal $I \rtimes_{\alpha} G$ is essential in $A \rtimes_{\alpha} G$. (By 7.7.8 of [30], for any faithful representation π of A, the representation of $A \rtimes_{\alpha} G$ on $H_{\pi} \otimes L^2(G)$ induced by π is faithful. If π is chosen to be nondegenerate on I, so that I and A have the same weak closure in the representation π, then $I \rtimes_{\alpha} G$ and $A \rtimes_{\alpha} G$ have the same weak closure in the induced representation, and since this is faithful it follows that $I \rtimes_{\alpha} G$ is essential.)

Hence by compactness of G, as in the proof of Proposition 3.3,

$$\begin{aligned} M^{\infty}(A) &= \lim\nolimits_{I \text{ essential}} M(I) = \lim\nolimits_{I \text{ essential and invariant}} M(I) \\ &\subseteq \lim\nolimits_{I \text{ essential and invariant}} M(I \rtimes_{\alpha} G) \subseteq M^{\infty}(A \rtimes_{\alpha} G) . \end{aligned}$$

It of course follows, as G is abelian, that

$$M^{\infty}(A) \subseteq M^{\infty}(A \rtimes_{\alpha} G)^{\hat{\alpha}} .$$

Now, assume that A is separable and prime, and let $u \in M^{\infty}(A \rtimes_{\alpha} G)^{\hat{\alpha}}$ and $\gamma \in \hat{G}$ be such that u is unitary, $uAu^{-1} = A$, and $\lambda(g) u \lambda(g)^{-1} = \langle \gamma, g \rangle u$, $g \in G$. Let us show that $u \in M^{\infty}(A)$. (If, in addition, $u \in M(A \rtimes_{\alpha} G)$, then it follows from 7.8.9 of [30] that $u \in M(A)$. What we are establishing is a very limited generalization of 7.8.9 of [30] to the limit multiplier algebra. In particular, the assumption that A and $A \rtimes_{\alpha} G$ are separable and prime may be superfluous.)

First, let us show that there exists $v \in M^{\infty}(A)$ such that $\operatorname{Ad} v$ agrees on A with $\operatorname{Ad} u$. By Proposition 3.2, for this it is sufficient to show that the automorphism $\beta = (\operatorname{Ad} u) \mid A$ is weakly inner in every faithful factor representation of A. Let π be a faithful factor representation of A, and denote by ρ the representation of $A \rtimes_{\alpha} G$ induced by π on $H_{\pi} \otimes L^2(G)$. Note that $\hat{\alpha}$, which extends to $\rho(A \rtimes_{\alpha} G)''$, acts ergodically on the centre of $\rho(A \rtimes_{\alpha} G)''$ (as π is factorial). It follows, as we shall show below, that ρ can be extended from A to $M^{\infty}(A \rtimes_{\alpha} G)^{\hat{\alpha}}$, mapping this algebra into $(\rho(A \rtimes_{\alpha} G)'')^{\hat{\alpha}}$, and commuting with $\operatorname{Ad} \lambda(g)$ for each $g \in G$. Since $\lambda(g) \rho(u) \lambda(g)^{-1} = \langle \gamma, g \rangle \rho(u)$, $g \in G$, it follows that $\rho(u) = V \otimes \gamma$ with $V \in \pi(A)''$. ($\rho(u)(1 \otimes \gamma)^{-1}$ commutes with $1 \otimes \xi$ and $1 \otimes \lambda(g)$ for all $\xi \in \hat{G}$ and $g \in G$, and therefore with $1 \otimes B(L^2(G))$. By construction, $\rho(A \rtimes_{\alpha} G)''$ is contained in $\pi(A)'' \otimes B(L^2(G))$.) We now have, for each $a \in A$,

$$\rho(\beta(a)) = \rho(uau^{-1}) = \rho(u)\rho(a)\rho(u)^{-1} = (V \otimes \gamma)\rho(a)(V \otimes \gamma)^{-1} ,$$

and since $\rho(a)$ is just the function $t \longmapsto \pi(\alpha_t(a))$, evaluating at $t = 0$ we get

$$\pi\beta = (\mathrm{Ad}\, V)\pi\,,$$

and so β is weakly inner in π, as desired.

Before proceeding to modify u using v, let us show as announced that if ρ is a faithful representation of $A \rtimes_\alpha G$ such that the restriction of ρ to any $\hat{\alpha}$-invariant essential closed two-sided ideal of $A \rtimes_\alpha G$ is nondegenerate, then $\rho \,|\, A$ can be extended to $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$. (It was pointed out earlier in the proof of this theorem that ρ as defined in the preceding paragraph is faithful; the second property also holds for that ρ, since $\hat{\alpha}$ extends to an action on $\rho(A \rtimes_\alpha G)''$ which is ergodic on the centre.) Let $x \in M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$, and let (J_n) be a sequence of essential closed two-sided ideals of $A \rtimes_\alpha G$ such that there exists $x_n \in M(J_n)$ with $\|x - x_n\| = \varepsilon_n \to 0$. Then, for any m and n, and any $\xi, \eta \in \hat{G}$,

$$\|\hat{\alpha}_\xi(x_m) - \hat{\alpha}_\eta(x_n)\| \leqslant \varepsilon_m + \varepsilon_n$$

(this uses the triangle inequality and $\hat{\alpha}_\xi(x) = x = \hat{\alpha}_\eta(x)$). Denote by e^n the unit of $\rho(J_n)''$, a central projection in $\rho(A \rtimes_\alpha G)''$. For each n, the representation $\rho \,|\, J_n$, on the Hilbert space $e^n H_\rho$, has a unique extension to a representation of $M(J_n)$, which we could denote by $(\rho \,|\, J_n)^{**}$, but will denote by ρ^n for brevity. Let $\xi_1, \xi_2, \cdots$ be an enumeration of $\hat{G}$, which is countable since G is compact and separable. Fix n, and define projections $p_1^n, p_2^n, \cdots$ in Centre $\rho(A \rtimes_\alpha G)''$ by orthogonalizing the units of $\rho(\hat{\alpha}_{\xi_1}(J_n))''$, $\rho(\hat{\alpha}_{\xi_2}(J_n))''$, $\cdots$, which we shall denote by $e_1^n, e_2^n, \cdots$. Thus,

$$p_1^n = e_1^n,\, p_2^n = (1 - p_1^n)e_2^n,\, p_3^n = (1 - p_1^n \vee p_2^n)e_3^n,\, \cdots\,.$$

Then $\bigvee_k p_k^n = \bigvee_k e_k^n = 1$, since $\bigvee_k e_k^n$ is the unit of $\rho(I_n)''$ where I_n is the smallest closed two-sided ideal of $A \rtimes_\alpha G$ containing $\hat{\alpha}_{\xi_1}(J_n), \hat{\alpha}_{\xi_2}(J_n), \cdots$, and I_n is $\hat{\alpha}$-invariant and essential (so by hypothesis ρ is nondegenerate on I_n). For each k denote by ρ_k^n the unique extension of $\rho \,|\, \hat{\alpha}_{\xi_k}(J_n)$ to a representation of $M(\hat{\alpha}_{\xi_k}(J_n)) = \hat{\alpha}_{\xi_k}(M(J_n))$ on the Hilbert space $e_k^n H_\rho$. (Thus, $\rho_k^n = (\rho \,|\, \hat{\alpha}_{\xi_k}(J_n))^{**}$.) Set

$$\textstyle\sum_k p_k^n \rho_k^n(\hat{\alpha}_{\xi_k}(x_n)) = y_n\,.$$

Then $y_n \in \rho(A \rtimes_\alpha G)''$. Furthermore, the sequence (y_n) is Cauchy:

$$\begin{aligned}
\|y_m - y_n\| &= \sup_{k,l} \|p_k^m p_l^n (y_m - y_n)\| \\
&= \sup_{k,l} \|p_k^m p_l^n (\rho_k^m(\hat{\alpha}_{\xi_k}(x_m)) - \rho_l^n(\hat{\alpha}_{\xi_l}(x_n)))\| \\
&\leqslant \sup_{k,l} \|(\rho \,|\, \hat{\alpha}_{\xi_k}(J_m) \cap \hat{\alpha}_{\xi_l}(J_n))^{**}(\hat{\alpha}_{\xi_k}(x_m) - \hat{\alpha}_{\xi_l}(x_n))\| \\
&= \sup_{k,l} \|\hat{\alpha}_{\xi_k}(x_m) - \hat{\alpha}_{\xi_l}(x_n)\| \leqslant \varepsilon_m + \varepsilon_n\,.
\end{aligned}$$

Here we have used that $p_k^m p_l^n \leqslant e_k^m e_l^n$, and that $e_k^m e_l^n$ is the unit of $\rho(\hat{\alpha}_{\xi_k}(J_m) \cap \hat{\alpha}_{\xi_l}(J_n))''$. Set

$$\lim y_n = \rho(x)\,.$$

From what we have shown, namely, that

$$\|y_m - y_n\| \leqslant \|x_m - x\| + \|x_n - x\| ,$$

it is clear that $\rho(x)$ is independent of any choices made in the construction. Hence, in particular, $\rho(x)$ depends additively and multiplicatively on x, and $\rho(x^*) = \rho(x)^*$. Furthermore, ρ defined on $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$ in this way agrees with the unique extension of ρ to a representation of $M(A \rtimes_\alpha G)$ (or to $M(J)$ for any closed two-sided ideal J of $A \rtimes_\alpha G$ on which ρ is nondegenerate). Finally, for use at the end of this proof, let us note that, by construction, ρ is isometric on $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$.

Now let us return to the proof that $u \in M^\infty(A)$. As we have shown, there exists a unitary $v \in M^\infty(A)$ such that $(\mathrm{Ad}\, v) \,|\, A = (\mathrm{Ad}\, u) \,|\, A$. Since $(\mathrm{Ad}\, u) \,|\, A$ commutes with $(\mathrm{Ad}\, \lambda(g)) \,|\, A = \alpha_g$ for each $g \in G$, it follows that $v^{-1}\alpha_g(v)$ belongs to Centre $M^\infty(A)$ for each $g \in G$. By Proposition 3.1, as A is prime, Centre $M^\infty(A) = \boldsymbol{C}$. Hence, by Proposition 3.5, below, the map $g \longmapsto v^{-1}\alpha_g(v)$ is continuous. This map is clearly multiplicative. Therefore, there exists $\xi \in \hat{G}$ such that

$$\lambda(g) v \lambda(g)^{-1} = \langle \xi, g \rangle v , \qquad g \in G .$$

Replacing u by uv^*, and γ by $\gamma - \xi$, we then have that u fulfills the hypotheses of the proposition and, in addition, $uau^{-1} = a$ for all $a \in A$. In other words, we now have that $u \in M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$ and, replacing γ by $-\gamma$,

$$\hat{\alpha}_\gamma = \mathrm{Ad}\, u .$$

Using only the hypothesis that A and A^α have faithful irreducible representations, we shall now deduce that $\gamma = 0$, and hence that u is a scalar multiple of 1.

First, let us show that $\gamma = 0$. Since A is prime, by Theorem 3.4 of [24] we have $G(\hat{\alpha}) = G$. To show that $\gamma = 0$, therefore, it is sufficient to show that $\gamma \in G(\hat{\alpha})^\perp$. By Proposition 4.2 of [25], for this it is sufficient to find a nonzero $\hat{\alpha}$-invariant hereditary sub-C^*-algebra B of $A \rtimes_\alpha G$ such that $\hat{\alpha}_\gamma \,|\, B = \exp \delta$ for some $\hat{\alpha}$-invariant derivation of B. If B is $\hat{\alpha}$-invariant and $|\,\mathrm{Sp}(\hat{\alpha}_\gamma \,|\, B) - 1\,| \leqslant 1$ then this of course holds, with $\delta = \log(\hat{\alpha}_\gamma \,|\, B)$.

Since $\hat{\alpha}_\gamma = \mathrm{Ad}\, u$ with $u \in M^\infty(A \rtimes_\alpha G)$, $\hat{\alpha}_\gamma$ is not properly outer. (The implication (ii) $\Rightarrow$ (i) of Proposition 3.2 holds for any C^*-algebra; just note that if an automorphism β of a C^*-algebra is, when restricted to a not necessarily invariant closed two-sided ideal I, strictly within distance two of an automorphism of I, then β leaves I invariant.) Hence, by (viii) $\Rightarrow$ (i) of Theorem 6.6 of [25] (this implication does not use separability), there exists a nonzero $\hat{\alpha}_\gamma$-invariant hereditary sub-C^*-algebra B_0 of $A \rtimes_\alpha G$ such that $|\,\mathrm{Sp}(\hat{\alpha}_\gamma \,|\, B_0) - 1\,| \leqslant 1$. Using that u is $\hat{\alpha}$-invariant, we shall show that if B denotes the $\hat{\alpha}$-invariant hereditary sub-C^*-algebra of $A \rtimes_\alpha G$ generated by B_0, then also $|\,\mathrm{Sp}(\hat{\alpha}_\gamma \,|\, B) - 1\,| \leqslant 1$, as desired.

We shall in fact show that $\mathrm{Sp}(\hat{\alpha}_\gamma \,|\, B) = \mathrm{Sp}(\hat{\alpha}_\gamma \,|\, B_0)$. To do this we shall proceed in two steps, using a faithful irreducible representation π of $A \rtimes_\alpha G$. (Recall that by Lemma 2.1, with $H = G$, the hypothesis that A^α is prime implies that $A \rtimes_\alpha G$ is prime.) Since π extends to $M^\infty(A \rtimes_\alpha G)$ (being both faithful and factorial), and $\hat{\alpha}_\gamma = \mathrm{Ad}\, u$, we may extend

$\hat{\alpha}_\gamma$ to $\pi(A \rtimes_\alpha G)''$, writing $\hat{\alpha}_\gamma = \mathrm{Ad}\,\pi(u)$.

We shall prove first that

$$\mathrm{Sp}(\hat{\alpha}_\gamma \mid B_0) = \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B_0)''),$$

and second that

$$\mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B_0)'') = \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B)'').$$

Since, for single automorphisms, spectrum and Arveson spectrum, and therefore also point spectrum, coincide—see [30], 8.1.14—, we have

$$\mathrm{Sp}(\hat{\alpha}_\gamma \mid B_0) \subseteq \mathrm{Sp}(\hat{\alpha}_\gamma \mid B) \subseteq \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B)''),$$

and the desired equality,

$$\mathrm{Sp}(\hat{\alpha}_\gamma \mid B_0) = \mathrm{Sp}(\hat{\alpha}_\gamma \mid B),$$

follows.

Let us show that $\mathrm{Sp}(\hat{\alpha}_\gamma \mid B_0) = \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B_0)'')$. By 8.1.9 of [30], $\lambda \in \mathrm{Sp}\,\beta$, where β is an automorphism of a C^*-algebra or a von Neumann algebra, if and only if, for each $f \in l^1(\boldsymbol{Z})$ with $\hat{f}(\lambda) \neq 0$, $\sum f(n)\beta^n \neq 0$. Applying this first with $\beta = \hat{\alpha}_\gamma \mid B_0$ and then with $\beta = \hat{\alpha}_\gamma \mid \pi(B_0)''$, we see that $\mathrm{Sp}(\hat{\alpha}_\gamma \mid B_0) = \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B_0)'')$, as desired.

Let us show that $\mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B_0)'') = \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B)'')$. As above, the inclusion of the spectrum on the smaller domain in the spectrum on the larger domain holds since the spectrum is point spectrum. Conversely, let $\lambda \in \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B)'')$, and let us show that $\lambda \in \mathrm{Sp}(\hat{\alpha}_\gamma \mid \pi(B_0)'')$. By Lemma 2.3.10 of [7], $\lambda = \lambda_1 \lambda_2^{-1}$ with $\lambda_1, \lambda_2 \in \mathrm{Sp}\,\pi(u)$. We shall show that $\lambda_1, \lambda_2 \in \mathrm{Sp}\,e_0\pi(u)$ where e_0 is the unit of $\pi(B_0)''$, using that u is $\hat{\alpha}$-invariant. Since B is the smallest $\hat{\alpha}$-invariant hereditary sub-C^*-algebra of $A \rtimes_\alpha G$ containing B_0, the unit of $\pi(B)''$, say e, is the smallest projection containing the unit of $\pi(\hat{\alpha}_\xi(B_0))''$, say e_ξ, for every $\xi \in \hat{G}$. For each $\xi \in \hat{G}$, since $\hat{\alpha}_\xi(u) = u$, we have $u\hat{\alpha}_\xi(B_0)u^{-1} = \hat{\alpha}_\xi(B_0)$, and hence $e_\xi \pi(u) = \pi(u)e_\xi$.

Let us show that, for each $\xi \in \hat{G}$, $\mathrm{Sp}\,e_\xi\pi(u) = \mathrm{Sp}\,e_0\pi(u)$. Since $\hat{\alpha}_\gamma \mid B_0 = (\mathrm{Ad}\,u) \mid B_0$ with $u \in M^\infty(A \rtimes_\alpha G)$, and B_0 is a hereditary sub-C^*-algebra of $A \rtimes_\alpha G$, so that every faithful factor representation of B_0 extends to a faithful factor representation of $A \rtimes_\alpha G$ (and hence of $M^\infty(A \rtimes_\alpha G)$) on a larger Hilbert space, by Proposition 3.2 there exists $w \in M^\infty(B_0)$ such that $\hat{\alpha}_\gamma \mid B_0 = (\mathrm{Ad}\,w) \mid B_0$. Since π is irreducible, also the restriction of π to B_0 is irreducible on the Hilbert space $e_0 H_\pi$. It follows that $\pi(w)$ is a scalar multiple of $e_0\pi(u)$, and so we may modify w so that $\pi(w) = e_0\pi(u)$. Hence, for any $a, b \in B_0$, $awb = aub$. It follows that, for any $\xi \in \hat{G}$, on considering the irreducible representation $\pi\hat{\alpha}_\xi$ of $A \rtimes_\alpha G$, and its restriction to B_0, which is irreducible on the Hilbert space $e_\xi H_\pi$, we have $\pi\hat{\alpha}_\xi(w) = e_\xi \pi\hat{\alpha}_\xi(u) = e_\xi\pi(u)$. Since $\pi\hat{\alpha}_\xi$ is faithful on B_0 and therefore on $M^\infty(B_0)$, we have $\mathrm{Sp}\,\pi\hat{\alpha}_\xi(w) = \mathrm{Sp}\,w$. This shows that $\mathrm{Sp}\,e_\xi\pi(u)$ is independent of ξ, as desired.

Now let us show, as announced, that λ_1 and λ_2 belong to $\mathrm{Sp}\,e_0\pi(u)$. Note that, since $e = \bigvee_{\xi \in \hat{G}} e_\xi$, the homomorphism

$$C^*(e\pi(u)) \ni x \longmapsto (e_\xi x) \in \prod_{\xi \in \hat{G}} C^*(e_\xi \pi(u))$$

is injective, so that $\operatorname{Sp} e\pi(u) = (\bigcup_{\xi \in \hat{G}} \operatorname{Sp} e_\xi \pi(u))^-$. Since $\operatorname{Sp} e_\xi \pi(u) = \operatorname{Sp} e_0 \pi(u)$ for each ξ, this shows that $\operatorname{Sp} e\pi(u) = \operatorname{Sp} e_0 \pi(u)$. In particular, $\lambda_1, \lambda_2 \in \operatorname{Sp} e_0 \pi(u)$.

Since $\lambda = \lambda_1 \lambda_2^{-1}$, and $\hat{\alpha}_\gamma \,|\, \pi(B_0)'' = (\operatorname{Ad} e_0 \pi(u)) \,|\, \pi(B_0)''$, it follows by Lemma 2.3.10 of [7] that $\lambda \in \operatorname{Sp} \hat{\alpha}_\gamma \,|\, \pi(B_0)''$, as asserted.

This completes the proof that, after u is modified as above, $\gamma = 0$. Let us now show that u, thus modified, is a scalar multiple of 1. Let π be a faithful factor representation of A, so that, as noted above, the induced representation ρ of $A \rtimes_\alpha G$ is also faithful, and, moreover, extends from $A \subseteq M(A \rtimes_\alpha G)$ to $M^\infty(A \rtimes_\alpha G)^{\hat{\alpha}}$, and is faithful there. What we must show, then, is that $\rho(u)$ is a scalar multiple of 1. As shown above, $\rho(u) = V \otimes \gamma$ with $V \in \pi(A)''$. Since $\gamma = 0 \in \hat{G}$, by which we mean that γ is the trivial character 1, we have $u = V \otimes 1$. As shown above, $\pi((\operatorname{Ad} u) \,|\, A) = (\operatorname{Ad} V)\pi$. Since $(\operatorname{Ad} u) \,|\, A = 1$ and π is factorial, it follows that V is a scalar multiple of 1, and therefore also u is. In particular, $u \in M^\infty(A)$.

We do not know if all the assumptions made in the second half of the proposition are necessary.

3.5. PROPOSITION. *Let A be a prime C^*-algebra and let α be an action of a compact group G on A. Let ϕ be a pure state of A such that π_ϕ is faithful, so that ϕ extends uniquely to a pure state of $M^\infty(A)$. It follows that for any $a, b, c \in M^\infty(A)$ the map*

$$G \ni g \longmapsto b\phi c(\alpha_g(a)) := \phi(c\alpha_g(a)b)$$

is continuous.

PROOF. As shown in the proof of Proposition 3.3, we have $M^\infty(A) = \lim_{I \text{ invariant}} M(I)$. Therefore it is sufficient to consider the case that $a \in M(I)$, where I is a nonzero α-invariant closed two-sided ideal of A. Again as shown in the proof of Proposition 3.3, I^α contains an approximate unit (e_i) for I. Then $\|\phi - \phi e_i\| \to 0$, and the same holds with $b\phi c$ in place of ϕ. The conclusion follows as $g \longmapsto \alpha_g(e_i a)$ is continuous.

4. Duality for a partially inner action.

4.1. THEOREM. *Let A be a separable prime C^*-algebra, and let α be an action of a compact abelian group G on A. Set*

$$H = \{t \in G;\ \alpha_t \text{ is not properly outer}\},$$

and suppose that A^G and A^H are prime.

If β is an automorphism of A such that $\beta \,|\, A^G = 1$ and $\beta\alpha_t = \alpha_t\beta$, $t \in H$, then there exists $g \in G$ such that $\beta = \alpha_g$.

PROOF. We may suppose that α is faithful.

By Proposition 3.2, for each $t \in H$ there exists a unitary $u(t) \in M^\infty(A)$ such that $\alpha_t =$

$\mathrm{Ad}\, u(t)$; furthermore, this holds only for $t \in H$. By Proposition 3.1, Centre $M^\infty(A) = \boldsymbol{C}$, and so $u(t)$ is unique up to a phase factor.

It follows in particular that H is a subgroup of G. Let us equip H with the discrete topology. Since G is abelian we have $\alpha_t = \mathrm{Ad}\, \alpha_g(u(t))$ for each $t \in H$ and $g \in G$, and by uniqueness of $u(t)$ we have $u(t)^{-1}\alpha_g(u(t)) \in \boldsymbol{T}$. Hence, for each fixed $t \in H$, by Proposition 3.5, the map $g \longmapsto u(t)^{-1}\alpha_g(u(t))$ is continuous. This map is clearly multiplicative, and is therefore a character of G, say $\psi(t)$. Clearly, also, $\psi: H \to \hat{G}$ is a homomorphism. Denoting by $\chi: G \to \hat{H}$ the dual of ψ, we have

$$\alpha_g(u(t)) = \langle g, \psi(t) \rangle u(t) = \langle \chi(g), t \rangle u(t)\,,$$

$g \in G$, $t \in H$.

Let N denote $\mathrm{Ker}\, \chi = (\mathrm{Im}\, \psi)^\perp$. We shall establish the following five assertions.

1. $\chi(\bar{H}) = \hat{H}$.
2. $\chi \mid H$ is injective.
3. $N\bar{H} = G$.
4. A^N is prime.
5. $\beta(A^N) = A^N$.

PROOF OF 1. Since $\chi(\bar{H})$ is a compact subgroup of $\hat{H}$, it suffices to show that $\chi(H)$ is dense in $\hat{H}$. Let $t \in H$ be such that $\langle \chi(h), t \rangle = 1$ for all $h \in H$, i.e. $\alpha_h(u(t)) = u(t)$, $h \in H$. Hence, by continuity of $g \longmapsto u(t)^{-1}\alpha_g(u(t))$ (see above), $\alpha_h(u(t)) = u(t)$ for all $h \in \bar{H}$. Therefore, $u(t) \in (A^{\bar{H}})' \cap M^\infty(A)^{\bar{H}}$. By Proposition 3.3, then $u(t) \in (A^{\bar{H}})' \cap M^\infty(A^{\bar{H}})$, i.e., $u(t) \in$ Centre $M^\infty(A^{\bar{H}})$. Since $A^{\bar{H}}$ is prime, by Proposition 3.1, Centre $M^\infty(A^{\bar{H}}) = \boldsymbol{C}$. This shows that $\alpha_t = 1$, and so $t = 0$.

PROOF OF 2. If $h, t \in H$, then

$$\langle \chi(h), t \rangle = u(h)u(t)u(h)^{-1}u(t)^{-1} = \langle \chi(t), h \rangle^{-1}\,.$$

It follows that if $t \in H$ and $\chi(t) = 0$ then $t \in \chi(H)^\perp = 0$.

PROOF OF 3. Since $\chi(\bar{H}) = \hat{H}$ and $\mathrm{Ker}\, \chi = N$, we have $N\bar{H} = G$.

PROOF OF 4. By definition of $N = \mathrm{Ker}\, \chi$,

$$\alpha_s(u(t)) = u(t)\,, \qquad s \in N\,, \quad t \in H\,,$$

i.e. $u(t) \in M^\infty(A)^N$, $t \in H$. By Proposition 3.3, it follows that $u(t) \in M^\infty(A^N)$, $t \in H$. Since G is abelian, A^N is α-invariant. Suppose that A^N has nonzero closed two-sided ideals I and J such that $IJ = 0$, and let us deduce an absurdity. We may suppose that $I + J$ is essential, and then $M^\infty(A^N) = M^\infty(I) + M^\infty(J)$, where $M^\infty(I)M^\infty(J) = 0$. Since $\alpha_t = \mathrm{Ad}\, u(t)$, $t \in H$, it follows from $u(s)$, $u(t) \in M^\infty(A^N)$ for $s, t \in H$ that $\alpha_s(I)\alpha_t(J) = 0$, for any $s, t \in H$ and hence for any $s, t \in \bar{H}$. Denote by I_0 and J_0 the smallest closed two-sided ideals of A^N containing I and J and invariant under $\alpha_{\bar{H}}$. Then $I_0 J_0 = 0$, and since $(A^N)^{\bar{H}} = A^{N\bar{H}} = A^G$, $I_0 \cap A^G$ and $J_0 \cap A^G$ are orthogonal nonzero ideals of A^G. (Note that $I_0 \cap A^G = I_0^{\bar{H}}$, $J_0 \cap A^G = J_0^{\bar{H}}$.) This contravenes the hypothesis that A^G is prime.

PROOF OF 5. Denote by σ the action of $\bar{H}$ on A^N obtained by restricting α. Denote by ψ_1 the composition of $\psi: H \to \hat{G}$ and the restriction map $\hat{G} \to \hat{\bar{H}}$. For each fixed $t \in H$ we have, as shown in the proof of 4, $u(t) \in M^\infty(A)^N \subseteq M^\infty(A^N)$. Furthermore,

$$\alpha_h(u(t)) = \langle h, \psi_1(t) \rangle u(t), \qquad h \in \bar{H},$$

and it follows, as we shall now show, that $\psi_1(t) \in \operatorname{Sp} \sigma$. As shown in the proof of Proposition 3.3,

$$M^\infty(A^N) = \lim\nolimits_{I\,\sigma\text{-invariant}} M(I),$$

and so there exist sequences (I_n) and (a_n), I_n a nonzero σ-invariant closed two-sided ideal of A^N and $a_n \in M(I_n)$, such that a_n converges to $u(t)$. Then with $b_n = \int_{\bar{H}} dh \overline{\langle h, \psi_1(t) \rangle} \sigma_h(a_n)$, the integral converging in the strict topology of $M(I_n)$, we have

$$\sigma_h(b_n) = \langle h, \psi_1(t) \rangle b_n,$$

and, as we shall show, $b_n \to u(t)$, and in particular, $b_n \neq 0$, at least for large n. To see that $b_n \to u(t)$, note that for each n, and for each $c \in I_n$ invariant under σ,

$$(b_n - u(t))c = \int_{\bar{H}} dh \overline{\langle h, \psi_1(t) \rangle} \sigma_h((a_n - u(t))c),$$

the integral converging in norm, and hence, if $\|c\| \leq 1$,

$$\|(b_n - u(t))c\| \leq \|(a_n - u(t))c\| \leq \|a_n - u(t)\|.$$

Since A^N is prime (by 4), and is separable, there is a faithful irreducible representation of A^N, necessarily nondegenerate on I_n, and extending to a faithful representation of $M^\infty(A^N)$. Hence

$$\|b_n - u(t)\| = \sup\nolimits_{c \in I_n, \|c\| \leq 1} \|(b_n - u(t))c\| \leq \|a_n - u(t)\| \to 0,$$

as desired. This shows that, at least for large n, $b_n \neq 0$, whence, for some σ-invariant $c_n \in I_n$, $b_n c_n \neq 0$. As $b_n c_n$ belongs to the spectral subspace of I_n for the action σ of $\bar{H}$ corresponding to $\psi_1(t) \in \hat{\bar{H}}$, $I_n^\sigma(\psi_1(t))$, and therefore to the spectral subspace $(A^N)^\sigma(\psi_1(t))$ of A^N, this shows that $(A^N)^\sigma(\psi_1(t)) \neq 0$, i.e. $\psi_1(t) \in \operatorname{Sp} \sigma$, as asserted.

We have shown that $\psi_1(H) \subseteq \operatorname{Sp} \sigma$. Let us show that $\psi_1(H) = \operatorname{Sp} \sigma$. Let $h \in \bar{H}$ be an element of $\psi_1(H)^\perp$. Then $\alpha_h(u(t)) = u(t)$, $t \in H$, and so $\chi(h) = 0$, i.e. $h \in N$. This shows that $\bar{H} \cap N \supseteq \psi_1(H)^\perp$, or, in other words, $(\bar{H} \cap N)^\perp \cap \hat{\bar{H}} \subseteq \psi_1(H)$. Since $\sigma \mid \bar{H} \cap N$ is trivial, one has $\operatorname{Sp} \sigma \subseteq (\bar{H} \cap N)^\perp \cap \hat{\bar{H}}$. This shows that $\operatorname{Sp} \sigma \subseteq \psi_1(H)$, and so $\operatorname{Sp} \sigma = \psi_1(H)$.

Since $\beta\alpha_t = \alpha_t\beta$ for $t \in H$, and Centre $M^\infty(A) = \boldsymbol{C}$ (Proposition 3.1), there is a $p \in \hat{H}$ such that

$$\beta(u(t)) = \langle t, p \rangle u(t), \qquad t \in H.$$

For each $t \in H$, and each $a \in (A^N)^\sigma(-\psi_1(t))$, one has $au(t) \in M^\infty(A)^G$ (since, by 3, $N\bar{H} = G$, and since $\alpha_h(u(t)) = \langle h, \psi_1(t) \rangle u(t)$ for $h \in \bar{H}$ and $\alpha_s(u(t)) = u(t)$ for $s \in N$). By hypothesis, $\beta \mid A^G = 1$, and it follows that $\beta \mid M^\infty(A)^G = 1$. (This can be seen by examining the proof of

Proposition 3.3, which identifies $M^\infty(A)^G$ with a subalgebra of $M^\infty(A^G)$: since each α-invariant closed two-sided ideal I of A has an approximate unit consisting of elements that are α-invariant, and therefore β-invariant, I is also β-invariant; hence $\beta(M(I))=M(I)$, and therefore $\beta \mid M(I)^G=1$; it follows in the limit that $\beta \mid M^\infty(A)^G=1$.) From this we obtain

$$\beta(au(t))=au(t)\,, \qquad t\in H\,,$$

and as $\beta(u(t))=\langle t,p\rangle u(t)$ it follows that $\beta(a)=\overline{\langle t,p\rangle}a$. This shows that $\beta((A^N)^\sigma(-\psi_1(t)))=(A^N)^\sigma(-\psi_1(t))$ for each $t\in H$, and since $\bar{H}$ is compact and $\operatorname{Sp}\sigma=\psi_1(H)$, it follows that $\beta(A^N)=A^N$, as desired.

Now let us show that $\beta=\alpha_g$ for some $g\in G$. First, we shall show that $\beta \mid A^N=\sigma_h$ for some $h\in\bar{H}$, where, as in the proof of 5, σ denotes the action of $\bar{H}$ by α on A^N. What we showed in the proof of 5 is that there exists $p\in\hat{H}$ such that, for each $t\in H$,

$$\beta \mid (A^N)^\sigma(\psi_1(t))=\langle t,p\rangle\,.$$

In particular, $\langle t,p\rangle$ depends only on $\psi_1(t)$; that is, there exists a character h_0 of $\psi_1(H)\subseteq\hat{\bar{H}}$ such that

$$\langle h_0,\psi_1(t)\rangle=\langle t,p\rangle\,, \qquad t\in H\,.$$

Extending h_0 to a character on $\hat{\bar{H}}$, we have $h\in\bar{H}$ such that $\langle h,\psi_1(t)\rangle=\langle t,p\rangle$, $t\in H$. Then

$$\beta \mid (A^N)^\sigma(\psi_1(t))=\langle h,\psi_1(t)\rangle=\sigma_h \mid (A^N)^\sigma(\psi_1(t))\,,$$

$t\in H$, and since $\psi_1(H)=\operatorname{Sp}\sigma$ (this was shown in the proof of 5), it follows that

$$\beta \mid A^N=\sigma_h=\alpha_h \mid A^N\,.$$

Set $\alpha_h^{-1}\beta=\beta_1$. Then $\beta_1 \mid A^N=1$, and we wish to show that $\beta_1=\alpha_s$ for some $s\in N$. By 2, $N\cap H=0$. In other words, α_s is properly outer for every $s\in N\diagdown\{0\}$. Since A is separable and prime, and (by 4) also A^N is prime, this shows that Condition 15 of Theorem 1 is verified with N in place of G (and $\alpha \mid N$ in place of α). Hence Condition 13 of Theorem 1 is also verified, with β_1 in place of β, and so $\beta_1=\alpha_s$ for some $s\in N$. This shows that $\beta=\alpha_g$, with $g=hs\in G$, as desired.

4.2. Remark. If α is ergodic under the assumptions of the theorem, then by [1] (see also [27]) H is dense in G. In particular, in this case $A^H=A^G$, and so the assumption that A^H is prime follows from the assumption that A^G is prime.

In general, the hypothesis that A^H is prime does not follow from the other hypotheses, and is necessary for the conclusion of the theorem. This is seen from the following example.

4.3. Example. Let σ be an outer automorphism of the Glimm C^*-algebra $A=M_{2^\infty}$ with period two, and define an action α of $\boldsymbol{Z}/2\boldsymbol{Z}\times\boldsymbol{Z}/2\boldsymbol{Z}$ on $M_2\otimes A$ by

$$\alpha_{(1,0)} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 ,$$

$$\alpha_{(0,1)} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sigma .$$

Then $A^\alpha = \{(a, \sigma(a));\ a \in A\} \cong A$, $H = \boldsymbol{Z}/2\boldsymbol{Z} \times 0$, $A^H = A \times A$, and $\beta = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$ verifies the conditions $\beta \,|\, A^\alpha = 1$, $\beta\alpha_t = \alpha_t\beta$, $t \in H$.

5. Duality for an action on a von Neumann algebra.

5.1. One purpose of this section is to give a new proof of the following von Neumann algebra analogue of the Tannaka duality theorem, given in [2], [33], and [22; Theorem VII.3.8]:

THEOREM (Araki, Haag, Kastler, Takesaki). *Let M be a von Neumann algebra, and let α be an action of a compact group G on M. Let H be another group and τ an action of H on M such that* $[\alpha, \tau] = 0$ (*i.e.* $\alpha_g \tau_h = \tau_h \alpha_g$ *for all* $g \in G$, $h \in H$). *Suppose that τ is ergodic (i.e. $M^\tau = \boldsymbol{C}$, where M^τ denotes the fixed point subalgebra for τ). It follows that for any automorphism β of M such that $\beta \,|\, M^\alpha = 1$ and $[\beta, \tau] = 0$, there exists a $g \in G$ such that*

$$\beta = \alpha_g .$$

For example, if $(M^\alpha)' \cap M = \boldsymbol{C}$, then H could be taken to be the unitary group of M^α, and τ to be the adjoint mapping, $h \longmapsto (\mathrm{Ad}\, h) \,|\, M$.

We shall deduce this theorem from the C^*-algebra analogue, given later in [3], in which the hypothesis of ergodicity of τ is replaced by a stronger condition called strong topological transitivity. To do this, we shall show that for an action of a group on a von Neumann algebra, the two conditions are equivalent: ergodicity implies strong topological transitivity.

One way in which our proof is new is that it does not depend on the type of the von Neumann algebra. The original proof consists of first reducing to the infinite case, and then using Roberts's construction of Hilbert spaces in the algebra ([31]). Our proof does not use Hilbert spaces in the algebra.

5.2. THEOREM. *Let M be a von Neumann algebra, H a group, and τ an action of H on M.*

The following three conditions are equivalent.

1. *τ is ergodic, i.e. $M^\tau = \boldsymbol{C}$.*
2. *τ is topologically transitive, i.e.*

$$x\tau_H(y) = 0 \Rightarrow x \otimes y = 0 .$$

3. *τ is strongly topologically transitive, i.e.*

$$\sum_{\text{(finite)}} x_i\tau_h(y_i)=0 \qquad \forall\, h\in H \;\Rightarrow\; \sum x_i\otimes y_i=0\,.$$

PROOF. The implications $3\Rightarrow 2$ and $2\Rightarrow 1$ hold in any C^*-algebra, the first trivially, and the second by spectral theory. (If $M^\tau \neq \boldsymbol{C}$ then there exist (positive) nonzero $x, y\in M^\tau$ with $xy=0$, whence $x\tau_h(h)=xy=0$ for all $h\in H$.)

Ad $1\Rightarrow 3$. We may suppose that M is represented covariantly on a Hilbert space, for example by taking the crossed product by τ. In other words, we may suppose that τ is determined by a unitary representation U of H: $\tau_h=(\operatorname{Ad} U(h))\,|\,M$, $h\in H$.

Assume that τ is ergodic, and let (x_i), (y_i) be finite sequences in M such that, for each $h\in H$,

$$\sum x_i\tau_h(y_i)=0\,, \qquad \text{i.e.}\quad \sum x_iU(h)y_i=0\,.$$

It follows that

$$\sum x_iU(h)z'y_i=0\,, \qquad h\in H\,,\quad z'\in M'\,.$$

Hence,

$$(*) \qquad \sum x_iby_i=0$$

for any b in the weakly closed linear span of $U(H)M'$. But since $U(h)M'U(h)^*=M'$ for each $h\in H$, the linear span of $U(H)M'$ is a $*$-algebra, and so by the bicommutant theorem its weak closure is

$$(M'\cup U(H))''=(M\cap U(H)')'=(M^\tau)'=\boldsymbol{C}'\,,$$

i.e. the algebra of all bounded operators on the Hilbert space.

In particular, $(*)$ holds with b an operator of rank one, i.e. with $b=\xi\otimes\eta^*$: $\zeta\longmapsto(\zeta\,|\,\eta)\xi$, and from

$$\sum x_i(\xi\otimes\eta^*)y_i=0\,, \qquad \text{i.e.}\quad \sum x_i\xi\otimes(y_i^*\eta)^*=0\,,$$

follows

$$\sum x_i\xi\otimes y_i^*\eta=0\,, \qquad \text{i.e.}\quad \left(\sum x_i\otimes y_i^*\right)(\xi\otimes\eta)=0\,.$$

Since $\sum x_i\otimes y_i^*$ is a bounded linear operator and the vectors ξ and η are arbitrary, this shows that $\sum x_i\otimes y_i^*=0$. Therefore, $\sum x_i\otimes y_i=0$.

5.3. COROLLARY. *Let A be a C^*-algebra, and let τ be an action of a group H on A. Suppose that there exists a faithful τ-covariant representation π of A such that the extension of τ to $\pi(A)''$ is ergodic (i.e. $(\pi(A)'')^\tau=\boldsymbol{C}$).*

It follows that τ is strongly topologically transitive.

5.4. COROLLARY (special case of 5.3). *Let A be a C^*-algebra, and let B be a sub-C^*-algebra of A. Suppose that there exists a faithful representation π of A such that*

$\pi(B)' \cap \pi(A)'' = \boldsymbol{C}$.

It follows that the unitary group of B (with unit adjoined, if necessary) acts strongly topologically transitively on A. (Compare 10 $\rightarrow$ 12 of Theorem 1.)

5.5. PROOF OF 5.1 (using 5.2 and [3]). Let β be an automorphism of M such that $\beta \mid A^\alpha = 1$ and $[\beta, \tau] = 0$. All the hypotheses of Theorem 2.1 of [3] are now verified, except that the system (M, G, α) is assumed only to be a W^*-dynamical system, not a C^*-dynamical system. It is straightforward, however, to modify the proof of Theorem 2.1 of [3] by putting the ultraweak topology of M in place of the norm topology. The conclusion $\beta = \alpha_g$ for some $g \in G$ follows.

Alternatively, as in the proof of 12 $\rightarrow$ 13 of Theorem 1 above (in Section 2), we may note that the proof of Theorem 2.1 of [3] is valid without any assumption of continuity of α at all until the last line—provided that M_F is defined as the set of all $x \in M$ such that the linear span of $\alpha_G(x)$ is finite-dimensional. This yields that, for some $g \in G$, $\beta = \alpha_g$ on M_F. By the Peter-Weyl theorem generalized to boundedly complete locally convex spaces (including Banach space duals, and therefore W^*-algebras), M_F is ultraweakly dense in M, and hence $\beta = \alpha_g$.

5.6. We note, finally, that the condition that the relative commutant of the fixed point subalgebra be trivial appears in recent work of Doplicher and Roberts ([10], [11], [12]).

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