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A CLASS OF NON-KÄHLER COMPLEX MANIFOLDS

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Introduction. In this paper we shall describe a class of non-algebraic manifolds, which extends several previously known classes. In [15] I constructed a class of manifolds Y of dimension n having $\pi_1(Y) \cong \boldsymbol{Z}^{n-1}$. Tsuchihashi ([18]) describes a class with $\pi_1(Y) \cong \boldsymbol{Z}$, one of which is an earlier example of Kato ([12]). All these reduce in the surface case to the well-known Inoue-Hirzebruch (hyperbolic Inoue) surfaces ([10]). The manifolds we describe in this paper have free abelian fundamental group with $1 \leq \mathrm{rk}\, \pi_1(Y) \leq n-1$. Up to bimeromorphic equivalence they are, in the surface case, precisely those occurring in [10] and, in the 3-fold case, precisely those occurring in the union of [15] and [18]. We are really interested in bimeromorphic equivalence classes, but what we shall do is take a convenient representative and do our calculations with that.

The paper is divided into six sections. Section 1 covers the basic construction of what we call the indecomposable manifolds. Section 2 is devoted to establishing that the objects used in the construction really do exist. In Section 3 we construct one-parameter degenerations and in Section 4 we explain what happens if the condition of indecomposability is dropped. Section 5 is short and easy and calculates the Kodaira dimension and the algebraic dimension. In Section 6 we calculate all the Hodge and Betti numbers. The calculation of $H^i(Y, \mathcal{O}_Y)$ is the main difficulty here (Theorem 6.3): our method is quite different from the method used in [18] to prove the result in the case $\mathrm{rk}\, \pi_1(Y) = 1$. The method we do use may be considered as an extension of an idea of Freitag [6]. The patience and alertness of the referee in detecting mistakes in the proof of Theorem 6.3 deserve much praise.

An earlier, incomplete, version of this paper dealt only with the case $\mathrm{rk}\, \pi_1(Y) = n-2$, but receipt of a preprint of [18] alerted me to the arbitrariness of this restriction. I should like to thank Professor Tsuchihashi for the preprint, Eduard Looijenga for listening to me, and the Science and Engineering Research Council of the United Kingdom for financial support.

1. Construction of the manifolds, indecomposable case. The outline of the construction is similar to those in the papers mentioned above. We fix a free abelian group M of rank n and another W of rank $b \leq n-1$, together with an action of W on M given by a faithful representation $\rho\colon W \to GL(M)$, so that M becomes a W-module. Then we construct an (infinite) fan Σ which fills up a suitable part of $M \otimes \boldsymbol{R}$ and is W-

invariant. We form the (non-Noetherian) torus embedding $X_{\Sigma,M}$, which has an action of W induced by ρ, and show that a certain open (in the Hausdorff topology) subset X of $X_{\Sigma,M}$ is such that $Y=X/W$ is a compact complex manifold. For the toric geometry, we refer to [4] and to [13].

We mention two notational conventions. We shall use multiplicative notation for W but additive notation for M, and if we write $G<G'$ we mean that G is a *proper* subgroup of G' (otherwise we write $G\leq G'$).

To begin with we make the following assumption, which is less restrictive than it may appear.

ASSUMPTION A. M is a rationally irreducible W-module, i.e., $M\otimes\boldsymbol{Q}$ is irreducible as a $\boldsymbol{Q}W$-module.

Manifolds constructed under Assumption A will be called indecomposable.

We also make, this time permanently, another assumption about the W-module structure of M; it follows from Assumption A but we shall need it later in the absence of Assumption A.

ASSUMPTION B. $M\otimes\boldsymbol{C}$ has a basis consisting of eigenvectors of W.

Of course this makes sense because W is abelian. The construction of parabolic Inoue surfaces ([9]) can be described by toric methods using an action of W ($\cong\boldsymbol{Z}$) for which Assumption B fails, and this is done in [13]. With Assumption B, W can be thought of as a subgroup of $GL(M)\cong GL(n,\boldsymbol{Z})$, diagonalizable over $\boldsymbol{C}$. By passing from W to W^2, which amounts to taking an unramified double cover of Y, we can assume $W\leq SL(n,\boldsymbol{Z})$.

Assumption A enables us to apply the following standard result ([16, p. 23]).

PROPOSITION 1.1. *Under Assumption* A *there exist a number field K and embeddings $\phi\colon M\hookrightarrow\mathcal{O}_K$, $\phi'\colon W\hookrightarrow\mathcal{O}_K^*$ such that $\phi'(\eta)\phi(m)=\rho(\eta)(m)$ for any $\eta\in W$, $m\in M$. Moreover, $|\mathcal{O}_K\colon\phi(M)|$ is finite.*

Here $\mathcal{O}_K$ denotes the ring of integers of K and $\mathcal{O}_K^*$ the group of units.

In view of Proposition 1.1 we can consider M as an ideal in a number field and W as a subgroup of the group of units. In particular $b\leq n-1$, since the last clause of Proposition 1.1 implies that $|K\colon\boldsymbol{Q}|=n$.

Next we fix an isomorphism $M\otimes\boldsymbol{R}\cong\boldsymbol{R}^n$ as follows. Suppose K has r real and s complex places, so $r+2s=n$. Take these in some fixed order, the real ones first, and let the corresponding embeddings be $\sigma_i\colon K\hookrightarrow\boldsymbol{R}$, $1\leq i\leq r$ and $\tau_j, \bar{\tau}_j\colon K\hookrightarrow\boldsymbol{C}$, $1\leq j\leq s$. Then $x\longmapsto(\sigma_1(x),\cdots,\sigma_r(x),\tau_1(x),\cdots,\tau_s(x))$ gives an embedding $\sigma\colon K\hookrightarrow\boldsymbol{R}^r\times\boldsymbol{C}^s=\boldsymbol{R}^n$ (identifying $\boldsymbol{C}$ with $\boldsymbol{R}^2$), and σ gives an isomorphism of $M\otimes\boldsymbol{R}$ with $\boldsymbol{R}^n$. σ is somewhat arbitrary (we chose the order of the places and which complex embedding to call τ and which $\bar{\tau}$) but it respects the action of W on M.

In order to proceed with the construction we require W to preserve a degenerate convex open cone Ω in $V=M\otimes \boldsymbol{R}$. Recall that a cone Ω is degenerate if its closure $\bar{\Omega}$ contains a line. Let H be the largest linear subspace of V contained in $\bar{\Omega}$, $\dim H=h>0$.

LEMMA 1.2. *$H=\{x \mid \sigma_{i_1}(x)=\cdots=\sigma_{i_{n-h}}(x)=0\}$. In particular $h\geq 2s$.*

PROOF. Clearly H must be preserved by W. The linear subspaces preserved by W are those given by the vanishing of some of the σ_i's and τ_j's. But if $\tau_j(x)=0$ on H then $\tau_j(\Omega)$ is a convex cone in $\tau_j(V)=\boldsymbol{C}$ preserved by W. On $\tau_j(V)$ an element $\eta\in W$ acts by multiplication by $\tau_j(\eta)\in \boldsymbol{C}$. If in fact $\tau_j(\eta)\in\boldsymbol{R}$ for all $\eta\in W$ then $\{m\in M \mid \tau_j(m)=\bar{\tau}_j(m)\}$ is a proper $\boldsymbol{Q}W$-submodule of M, contradicting Assumption A. So for some $\eta\in W$, $\tau_j(\eta)\in \boldsymbol{C}\backslash\boldsymbol{R}$: but then $\tau_j(\Omega)=\boldsymbol{C}$ because multiplication by a non-real complex number preserves no proper cone in $\boldsymbol{C}=\boldsymbol{R}^2$. In particular, $0\in\tau_j(\Omega)$, so $H\cap\Omega\neq\varnothing$. But this, as is easily seen, contradicts the convexity of Ω. ■

We relabel the σ_i's so that H is given by $\sigma_1(x)=\cdots=\sigma_{n-h}(x)=0$.

LEMMA 1.3. *$\Omega=H\times L_+$, where $L=\{\sigma_{n-h+1}(x)=\cdots=\sigma_r(x)=\tau_1(x)=\cdots=\tau_s(x)=0\}$ and L_+ is one of the orthants $\{x\in L \mid \pm\sigma_i(x)>0,\ 1\leq i\leq n-h\}$.*

PROOF. Obvious. ■

We may as well suppose that L_+ is the first orthant ($\sigma_i(x)>0$ for all i). If it is not, we simply multiply M by a suitable element of $\mathcal{O}_K$ and start again.

Next, we must have $b=n-h$, so that L_+/W is (or could be) a compact real manifold.

REMARK. If $b>n-h$ the construction breaks down totally, but if $b<n-h$ it gives a non-compact Y. The case $b=1$, $h=0$ gives the resolution of a Hilbert modular variety cusp.

REMARK. $h\geq 2s$, so $b\leq n-2s$, an inequality sharper than the $b\leq n-s-1$ of the Dirichlet units theorem. Thus, in general we cannot take $W=\mathcal{O}_K^*$.

We fix notation $\eta_i=\sigma_i(\eta)$ if $i\leq r$ and $\eta_i=|\tau_{i-r}(\eta)|$ if $r<i\leq r+s$. We make one further assumption (and we shall need a genericity condition to ensure the truth of 1.4, below).

ASSUMPTION C. If $1\neq\eta\in W$, either there is an $i\leq b$ such that $\eta_j>\eta_i$ for all $j>b$ or there is an $i\leq b$ such that $\eta_j<\eta_i$ for all $j>b$.

Clearly Assumption C is not true for a general $W\leq\mathcal{O}_K^*$ of rank b. We shall see in § 2, below, that such W do exist for any K. For the present we shall make do with a few remarks.

REMARK. If W is of finite index in $\mathcal{O}_K^*$, so that $b=r+s-1$, Assumption C is automatically true. However the bound $b\leq n-2s$ in the remark above shows that then $s=0$ (as in [15]) or $s=1$.

REMARK. If $b=1$ Assumption C implies that the condition of Definition 1.1 of [18] is satisfied by either η or η^{-1}.

PROPOSITION 1.4. *For general W, the action of W on Ω is free and properly discontinuous.*

PROOF. The precise meaning of the term "general" in this context will be explained below. It is enough to show that the action of W on L_+ is free and properly discontinuous, since the projection $\pi_L: \Omega=H\times L_+\to L_+$ commutes with the action of W. If $\eta\in W$ has a fixed point $(p_1,\cdots,p_b)\in L_+$ then $\eta_1=1$, since $\eta(p_1,\cdots,p_b)=(\eta_1 p_1,\cdots,\eta_b p_b)$; but $\sigma_1: K\to \boldsymbol{R}$ is an embedding, so $\eta=1$. To prove discontinuity we go to the logarithmic space $\mathscr{L}$. Here (see [2, Ch. II, § 3]) $\mathscr{L}\subseteq \boldsymbol{R}^{r+s}$ is the linear subspace defined by $x_1+\cdots+x_{r+s}=0$ and $\mathcal{O}_K^*$ (modulo torsion) is embedded as a full lattice in $\mathscr{L}$ by $\ell: \eta\to(\ell_1(\eta),\cdots,\ell_{r+s}(\eta))$, where $\ell_i(\eta)=\log\eta_i$ for $i\leq r$ and $\ell_i(\eta)=2\log|\eta_i|$ for $i>r$. Let $\pi': \boldsymbol{R}^{r+s}\to\boldsymbol{R}^b$ be the projection on the first b coordinates. We must show that $\pi'(\ell(W))$ is a lattice in $\boldsymbol{R}^b$. For a general real linear subspace $\mathscr{L}'\subseteq\mathscr{L}$ of dimension b, $\pi': \mathscr{L}'\to\boldsymbol{R}^b$ is an isomorphism: in particular the image of any lattice in $\mathscr{L}'$ is a lattice in $\boldsymbol{R}^b$. So if the real span of $\ell(W)$ is such a "general" linear subspace we are done. The subspaces $\mathscr{L}'$ are parametrized by the Grassmannian $\mathscr{G}(b, r+s-1)$ of real b-planes in $\boldsymbol{R}^{r+s-1}$ and the subgroups W by the rational points of $\mathscr{G}(b, r+s-1)$. The general $\mathscr{L}'$ and W are parametrized by a Zariski-open subset. ■

Suppose Σ is a fan which is W-invariant (that is $\eta\sigma\in\Sigma$ if $\sigma\in\Sigma$ and $\eta\in W$) and such that Σ/W is finite and $|\Sigma|=(\Omega\setminus L_+)\cup\{0\}=\Omega'$ (see Figure 1).

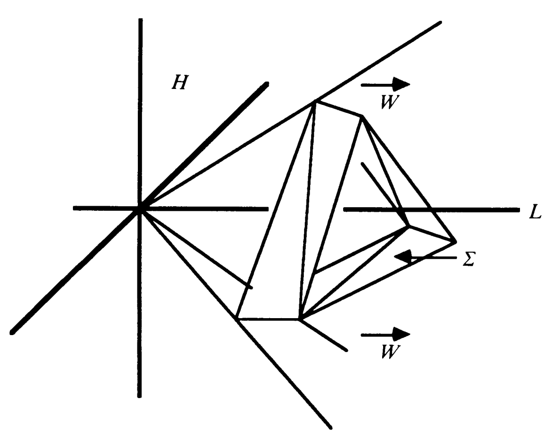

FIGURE 1

Let X_Σ be the torus embedding corresponding to the pair (Σ, M). (I avoid calling it a variety since Σ is infinite and X_Σ is therefore non-Noetherian.) X_Σ is a partial compactification of the big torus $T_M=\operatorname{Spec}\boldsymbol{C}[M^\vee]$ (with the usual toric notation). T_M may also be described as follows: we complexify $V=M\otimes\boldsymbol{R}$ so that V is the imaginary part of $\tilde{V}\cong\boldsymbol{C}^n$. Then M acts on $\tilde{V}$ by translation, and T_M is $\tilde{V}/M$. Write $\tilde{F}=X_\Sigma\setminus T_M$ and denote by $q\colon \boldsymbol{C}^n=\tilde{V}\to T_M=\tilde{V}/M$ the quotient map. $\boldsymbol{H}^b\times\boldsymbol{C}^h$, where $\boldsymbol{H}$ denotes the upper half-plane, is the preimage in $\tilde{V}$ of Ω; $X=q(\boldsymbol{H}^b\times\boldsymbol{C}^h)\cup\tilde{F}$ is a connected subset of X_Σ, open in the complex topology. W acts on X because of the W-invariance of Σ and Ω and W-equivariance of q. By a standard process ([1]) if a Σ satisfying the above conditions exists at all it can be chosen so that X_Σ is smooth and the action of W on X is free and properly discontinuous. Then $Y=X/W$ is a complex manifold, containing a distinguished analytic closed subset $F=\tilde{F}/W$ of codimension one.

THEOREM 1.5. *The complex manifold Y is compact.*

PROOF. Assume that $b<n-1$; for the case $b=n-1$ we refer to [15]. Denote by $G=G(M, W)$ the split extension of M by W determined by ρ. From the discussion above it is clear that Y is the disjoint union of F with $Y_0=(\boldsymbol{H}^b\times\boldsymbol{C}^h)/G$. F has simple normal crossings and is therefore a deformation retract of a tubular neighbourhood $\mathcal{N}_F$. Write $\mathcal{N}_F^*$ for $\mathcal{N}_F\setminus F$. The first few terms of the Mayer-Vietoris sequence associated to the couple $\{Y_0, \mathcal{N}_F\}$ are

$$0\to H_{2n}(Y)\to H_{2n-1}(\mathcal{N}_F^*)\to H_{2n-1}(Y_0)\oplus H_{2n-1}(\mathcal{N}_F)\to\cdots$$

since $Y_0\cap\mathcal{N}_F=\mathcal{N}_F^*$. $H_{2n-1}(\mathcal{N}_F)=H_{2n-1}(F)=0$ since the real dimension of F is $2n-2$. Y_0 is a $K(G, 1)$ so it is homotopy equivalent to a real n-torus bundle over a real b-torus. So the homological dimension of Y_0 is $n+b<2n-1$, so $H_{2n-1}(Y_0)=0$, too. On the other hand $H_{2n-1}(\mathcal{N}_F^*)\neq 0$, from the Gysin sequence ([5, VIII.12.1]) and the fact, which follows from the finiteness of Σ/W, that F is compact. Thus $H_{2n}(Y)\neq 0$, so Y must be a compact real $2n$-manifold, by [5, VIII.3.4] (Y is orientable because it is a complex manifold). ■

A more geometric idea of why Y is compact, and of the structure of Y, can be got from the manifold with corners $\mathrm{Mc}(M, \Sigma)$ discussed in [13]. It is a real manifold with boundary which is a partial compactification at infinity of V, whose structure accurately reflects that of X_Σ as a partial compactification of T_M. Specifically $\mathrm{Mc}(M, \Sigma)=X_\Sigma/CT_M$ (in the notation of [13]) and the quotient map is proper and (in this case) W-invariant. Consequently it would be sufficient to write down a fundamental domain for the action of W on $\Omega\subset V$ whose closure in $\mathrm{Mc}(M, \Sigma)$ is a compact fundamental domain for the action of W on $X/CT_M\subset\mathrm{Mc}(M, \Sigma)$: see Figure 2. Unfortunately this does not seem to be practical in general, and we must have recourse to the topological argument above.

2. Existence theorems. In this section we shall prove the existence of W satisfying Assumption C for any field K, and (using Assumption B and Assumption C only) the existence of a fan Σ satisfying the conditions specified in the previous section.

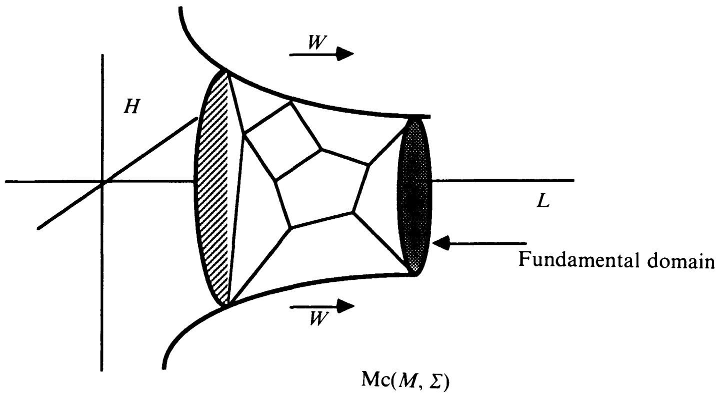

FIGURE 2

THEOREM 2.1. *With the notation of the previous section, for given K, b and H there exists a* $W \leq \mathcal{O}_K^*$ *verifying Assumption* C.

PROOF. Write $g = r + s - b$. To each $\eta \in \mathcal{O}_K^*$ we associate a point of $\boldsymbol{R}^{bg}$ by $\eta \longmapsto (t_{ij})$, $1 \leq i \leq b$, $1 \leq j \leq g$, where

$$t_{ij} = \log(\eta_i/\eta_{b+j}).$$

We let $\pi_i \colon \boldsymbol{R}^{bg} \to \boldsymbol{R}^g$, $1 \leq i \leq b$, be the projection $(t_{ij})_{1 \leq i \leq b, 1 \leq j \leq g} \longmapsto (t_{ij})_{1 \leq j \leq g}$. The statement of Assumption C is that for any $\eta \in W$ ($\eta \neq 1$) there is an i, $1 \leq i \leq b$, such that $\pi_i(\eta) \in Q$, where Q is $\pm$ the open first orthant in $\boldsymbol{R}^g$ (i.e., where the t_{ij} all have the same sign). By abuse of notation we shall continue to denote by W and $\mathcal{O}_K^*$ the images of W and $\mathcal{O}_K^*$ in $\boldsymbol{R}^{bg}$.

If $\zeta \in \mathcal{O}_K^*$ then obviously

$$(\zeta_1/\zeta_{b+1})^{g-1}(\zeta_{b+2}/\zeta_1) \cdot \cdots \cdot (\zeta_{b+g}/\zeta_1) = (\zeta_2/\zeta_{b+1})^{g-1}(\zeta_{b+2}/\zeta_2) \cdot \cdots \cdot (\zeta_{b+g}/\zeta_2).$$

So $-(g-1)t_{11} + t_{12} + \cdots + t_{1g} = -(g-1)t_{21} + t_{22} + \cdots + t_{2g}$: similarly $t_{11} - (g-1)t_{12} + \cdots + t_{1g} = t_{21} - (g-1)t_{22} + \cdots + t_{2g}$, and so on, so we get g equations relating (t_{1j}) and (t_{ij}), $i \neq 1$. It is easy to see that $g-1$ of these relations are independent, and we can therefore write down $(g-1)(b-1)$ independent linear relations satisfied by the linear subspace $\mathcal{O}_K^* \otimes \boldsymbol{R}$ in $\boldsymbol{R}^{bg}$. But $\dim(\mathcal{O}_K^* \otimes \boldsymbol{R}) = b + g - 1 = bg - (b-1)(g-1)$, so these equations define $\mathcal{O}_K^* \otimes \boldsymbol{R}$. From the equations it is clear that we do not get any relations involving (t_{ij}) for only one i: that is to say, $\pi_i(\mathcal{O}_K^* \otimes \boldsymbol{R})$ is the whole of $\boldsymbol{R}^g$.

We shall have shown the existence of W if we can exhibit a b-plane A in $\mathcal{O}_K^* \otimes \boldsymbol{R}$ which is rational (generated as a linear space by points of $\mathcal{O}_K^*$) and such that for any $p(\neq 0) \in A$, $\pi_i(p) \in Q$ for some i; then $W = A \cap \mathcal{O}_K^*$.

Let $\mathcal{G}(k, n)$ be the Grassmannian of real k-planes in $\boldsymbol{R}^n$, with the real topology.

Write $P=\boldsymbol{R}^g\setminus Q$ and $\tilde{P}=(\mathcal{O}_K^*\otimes\boldsymbol{R})\cap\bigcap_{i=1}^b \pi_i^{-1}(P)$. We require A to intersect $\tilde{P}$ only at 0, or, projectively, that the projective hyperplane of dimension $b-1$ in $\boldsymbol{RP}^{b+g-2}$ miss the projectivization of $\tilde{P}$. $\tilde{P}$ is closed, and it is easy to see that the condition that A miss it in this sense is an open one on $\mathcal{G}(b, b+g-1)$. Since the rational points of $\mathcal{G}(b, b+g-1)$ are dense it will be sufficient to exhibit an A missing $\tilde{P}$, as A can then be perturbed to make it rational and to satisfy the genericity condition of 1.4.

The existence of A will be proved by induction on b. For $b=1$ we can take for A the line through the origin and any point of Q. Write U for $\mathcal{O}_K^*\otimes\boldsymbol{R}\cong\boldsymbol{R}^{b+g-1}$. Let T be a codimension one linear subspace of $\boldsymbol{R}^g$ contained in P and let $U'=\pi_b^{-1}(T)$. For generic T, U' will have the property $\pi_i(U')=\boldsymbol{R}^g$, $1\le i\le b-1$, so we can assume this. Then, by induction, there is a hyperplane A' of dimension $b-1$ in the $(b+g-2)$-dimensional space U' meeting $U'\cap\bigcap_{i=1}^{b-1}\pi_i^{-1}(P_0)$ only at the origin. Let $\pi_{A'}\colon U\to\boldsymbol{R}^g$ be the projection along A'. $\pi_{A'}\pi_b^{-1}\colon \boldsymbol{R}^g\to\boldsymbol{R}^g$ is an element of $GL(g,\boldsymbol{R})$ and $Q_{A'}=\pi_{A'}\pi_b^{-1}(Q)$ an open double cone in $R_{A'}$. Let l be a line in $Q_{A'}$, and take $A=\pi_{A'}^{-1}(l)$. It is easy to check that $A\cap\tilde{P}=\{0\}$, and this completes the proof of Theorem 2.1. ■

The point of Assumption C is that it ensures that every infinite cyclic subgroup of W collapses nondegenerate closed cones in Ω onto L, and, as we shall see, ensures the existence of Σ.

PROPOSITION 2.2. *If $C\subset\Omega\cup\{0\}$ is a nondegenerate closed cone and $\eta\in W$, then $\bigcap_{a\in\boldsymbol{Z}}\eta^a C\subseteq L_+$.*

PROOF. $C\subseteq C_\delta=\{v\in\Omega\,|\,\sum_{i=1}^b v_i^2\ge\delta\sum_{j>b}v_j^2\}$ for some $\delta>0$, so it is enough to show that $\bigcap_{a\in\boldsymbol{Z}}\eta^a C_\delta\subseteq L_+$. Without loss of generality, we may suppose that $\eta_1/\eta_j>N>1$ for all $j>b$. For $v\in C_\delta$, $(\sum_{i=1}^b v_i^2)/\delta(\sum_{j>b}v_j^2)\ge 1$, so

$$\sum_{i=1}^b(\eta^a v)_i^2\bigg/\Big(N^a\delta\sum_{j>b}(\eta^a v)_j^2\Big)\ge(\eta^a v)_1^2\bigg/\Big(N^a\delta\sum_{j>b}(\eta^a v)_j^2\Big)\ge N^a v_1^2\Big/\delta\sum_{j>b}v_j^2\ge 1\qquad\text{for}\quad a\gg 0.$$

So $\eta^a v\in C_{N^a\delta}$ for $a\gg 0$. This proves the result. ■

COROLLARY 2.3. *W acts discontinuously and without fixed points on $\hat{P}=\boldsymbol{P}(\Omega')$.*

Here and later we use $\boldsymbol{P}(A)$ to denote the image of a subset A of V in $\boldsymbol{P}(V)=\boldsymbol{RP}^{n-1}$.

Corollary 2.3 is not by itself enough for our purposes: we need a compact fundamental domain and for this to exist Assumption C is essential.

THEOREM 2.4. *The action of W on $\hat{P}$ admits a compact fundamental domain Z, and the quotient $\hat{P}/W$ is homeomorphic with $S^{h-1}\times(S^1)^b$.*

PROOF. The idea is to show that the choice of N and a in the proof of Proposition 2.2 can be made uniformly for all η and to use this to determine a subgroup of W of finite index for which we can write down a fundamental domain.

With notation as for Theorem 2.1, it is clear that Assumption C is equivalent to "for all $\eta(\neq 0)\in W\otimes\boldsymbol{R}$ there is an i $(1\leq i\leq b)$ such that $\pi_i(\eta)\in Q$". Let Δ be the ball of radius $1/2$ in $W\otimes\boldsymbol{R}$ and $\partial\Delta$ its boundary: for $\varepsilon>0$ define

$$Q(\varepsilon)=\{t=(t_{i1},\cdots,t_{ig})\in\boldsymbol{R}^g\,|\,t_{ij}>\varepsilon,\ \forall j \text{ or } t_{ij}<\varepsilon,\ \forall j\}\,.$$

For $0\neq\eta\in W\otimes\boldsymbol{R}$ write (uniquely) $\eta=\lambda\eta_0$, $\lambda>0$, $\eta_0\in\partial\Delta$. For each i there is a unique $\varepsilon_i(\eta_0)$ given by $\varepsilon_i(\eta_0)=\sup\{\varepsilon\,|\,\eta_0\in Q(\varepsilon)\}$ or $\varepsilon_i(\eta_0)=0$ if $\eta_0\notin Q$. Put $e(\eta_0)=\max\varepsilon_i(\eta_0)$. By Assumption C, $e(\eta_0)>0$. Clearly $e\colon\partial\Delta\to\boldsymbol{R}$ is continuous, so it is bounded below by $e_0>0$. If $\eta\in W$ then $\lambda>0$ so Assumption C also implies that for all $\eta\in W$ there is an i $(1\leq i\leq b)$ such that $\pi_i(\eta)\in Q(e_0)$.

So in the proof of Proposition 2.2 we can choose N uniformly for all η such that $\eta_i>\eta_j$ for some $i\leq b$. Explicitly, we require that, for all such η, there be an $i\leq b$ such that $\eta_i/\eta_j>N>1$ for all $j>b$, and this can be arranged by choosing $N=\exp(e_0)$. We may suppose that $\eta_1\geq\eta_i$ for $i\leq b$; then

$$1\leq\sum_{i\leq b}\left(v_i^2/\delta\sum_{j>b}v_j^2\right)\leq b\left(v_1^2/\delta\sum_{j>b}v_j^2\right)$$

so $v_1^2/(\delta\sum_{j>b}v_j^2)\geq 1/b$ and $N^a v_1^2/(\delta\sum_{j>b}v_j^2)>1$ as long as $a>b/\log N$. Thus a can also be chosen uniformly, so that $\eta^a C_\delta\subset C_{N^a\delta}$ for all η such that $\eta_i>\eta_j$ for some $i\leq b$.

Consider now the group $W^a=\{\eta^a\,|\,\eta\in W\}$. It is of finite index in W and acts on $\boldsymbol{P}(L_+)$. Let $U<W^a$ be any group such that $W^a/U\cong\boldsymbol{Z}$. The action of U on $\boldsymbol{P}(L_+)$ is fixed-point free (W has no eigenvectors in L_+) and discontinuous and is therefore equivalent to the action of a full lattice in $\boldsymbol{R}^{b-1}\simeq\boldsymbol{P}(L_+)$, so it has a fundamental domain Z_0.

Let W^a_+ be the set of $\eta\in W^a$ for which $\eta_i>\eta_j$ for some $i\leq b$ and all $j>b$. Put $Z'=\boldsymbol{P}(\bar{C}_\delta\setminus\bigcup_{\eta\in W^a_+}\eta C_\delta)\cap\pi_L^{-1}(Z_0)$. Z' is closed by its construction, clearly bounded and hence compact. From the choice of W^a and of Z_0 it follows that Z' is a fundamental domain for W^a. Since $|W\colon W^a|<\infty$ it is easy to see that there must also exist a compact fundamental domain Z for W, namely $Z=Z'\setminus\bigcup_\eta\eta Z'$ where η runs through those η for which $\eta Z'\cap Z'\neq\varnothing$ or $\eta^{-1}Z'\cap Z'\neq\varnothing$ but not both. The last part of the theorem is clear. ■

THEOREM 2.5. *There exists a W-invariant fan Σ with Σ/W finite and such that $|\Sigma|=\Omega'=(\Omega\setminus L_+)\cup\{0\}$.*

PROOF. Let Z be a compact fundamental domain for W in $\hat{P}$, as in Theorem 2.4, so $\hat{P}=\bigcup_{\eta\in W}\eta Z$. Because the rational points of $\hat{P}$ are dense and the action of W is discontinuous, for each point $p\in Z$ there is an open rational polyhedron S such that $p\in S$ and $\eta S\cap S=\varnothing$ for $\eta(\neq 1)\in W$. Let $S_1,\cdots,S_k$ be a finite cover of Z by such polyhedra. Put

$$S_1^{(1)}=S_1\,,\qquad S_i^{(1)}=S_i\setminus\bigcup_{\eta\in W}\eta S_1\,,\qquad 1<i\leq k\,.$$

$\eta S_1 \cap S_i = \varnothing$ for all but finitely many η so $S_i^{(1)}$ is a rational polyhedron (neither open nor closed). Inductively, we suppose that $S_i^{(j)}$ are polyhedra such that

$$S_i^{(j)} \subseteq S_i\,, \qquad \hat{P} = \bigcup_{\eta \in W} \bigcup_{i=1}^{k} \eta S_i^{(j)} \quad \text{and} \quad \eta S_i^{(j)} \cap S_l^{(j)} = \varnothing$$

for all $\eta \in W$ if $i<l$ and $i \neq j$. Then if we put

$$S_i^{(j+1)} = S_i^{(j)}, \qquad i \leq j+1\,, \quad \text{and} \quad S_i^{(j+1)} = S_i^{(j)} \setminus \bigcup_{\eta \in W} \eta S_{j+1}^{(j)}\,, \qquad i > j+1\,,$$

$\{S_i^{(j+1)}\}$ satisfy the same conditions with $i+1$ in place of i. Consequently $\hat{P} = \bigcup_{\eta \in W} \bigcup_{i=1}^{k} \eta S_i^{(k-1)}$ and the $S_i^{(k-1)}$ are disjoint.

$S_0 = \bigcup_{i=1}^{k} S_i^{(k-1)}$ is thus a polyhedral fundamental domain for W in $\hat{P}$. The corresponding cone in Ω can then be subdivided in such a way that it, together with its translates, gives a fan Σ as described. If we want Σ to be basic, simplicial or whatever, that can also be arranged. An explicit description of the subdivision process can be found in [14, Theorem 3] (the proof of that theorem, as was pointed out to me by C. T. C. Wall, is incomplete, but the correction is easy). ■

REMARK. The proof of Theorem 2.5 follows the argument of [17, Proposition 2], with minor modifications to adapt it to the present context.

3. One-parameter degenerations. By a method similar to that of Makio, who gave one-parameter degenerations of Inoue-Hirzebruch surfaces (described in [13]), we construct one-parameter degenerations of the manifolds described above.

Put $\tilde{M} = M \oplus \boldsymbol{Z}e$, $\tilde{V} = \tilde{M} \otimes \boldsymbol{R}$, $\tilde{H} = H \oplus \boldsymbol{R}e$, $\tilde{L} = L \subset V \subset \tilde{V}$, $\tilde{L}_+ = L_+$, $\tilde{\Omega} = \tilde{H} \times \tilde{L}_+$ and $\tilde{\Omega}' = (\tilde{\Omega} \setminus \tilde{L}_+) \cup \{0\}$. W acts on $\tilde{M}$ by $\eta : e \longmapsto e$ for all $\eta \in W$ (so the representation $\tilde{\rho} : W \to SL(\tilde{M})$ is $\rho \oplus 1$). According to Theorem 2.5 we can choose a rational polyhedral fundamental domain S in $\hat{P}$ and a corresponding fan Σ in V. Let Λ be the fan in $\boldsymbol{R}e$ whose cones are $\boldsymbol{R}_+ e$, $\boldsymbol{R}_+(-e)$ and $\{0\}$.

THEOREM 3.1. *There is a W-invariant fan $\tilde{\Sigma}$ with $|\tilde{\Sigma}| = \tilde{\Omega}'$, such that $\{\sigma \in \tilde{\Sigma} \mid \sigma \subset V\} = \Sigma$ and the projection $\tilde{M} \to \boldsymbol{Z}e$ induces a morphism $(\tilde{\Sigma}, \tilde{M}) \to (\Lambda, \boldsymbol{Z}e)$ of fans, respecting the action of W.*

PROOF. Let $E = \{\eta \in W \mid \bar{S} \cap \eta \bar{S} \neq \varnothing\}$. E is a finite set, by the choice of S, and E generates W. For if not, let W' be the subgroup generated by E: each coset of W' determines a connected component of $\hat{P}$, but $\hat{P}$ is connected unless $h=1$ when it has two connected components which are not equivalent under $SL(M)$.

Let $\tau \in \Sigma$ be a cone contained in exactly q translates of $\bar{S}$, say $\tau \subseteq \bigcap_{j=1}^{q} \xi_j \bar{S}$. Pick an $m \in M$ and define $\tilde{\tau}_{\pm} = \mathrm{Span}(\tau, \{\xi_j m \pm e\})$. Let $\tilde{\Sigma} = \{\text{faces of } \tilde{\tau}_{\pm} \mid \tau \in \Sigma\}$. It is clear that $\tilde{\Sigma}$ is W-invariant, that $\{\sigma \in \tilde{\Sigma} \mid \sigma \subset V\} = \Sigma$, and that the projection induces a morphism of fans, as long as $\tilde{\Sigma}$ actually is a fan. It remains to show that $\tilde{\Sigma}$ is a fan and that $|\tilde{\Sigma}| = \tilde{\Omega}'$.

The map $\tau \longmapsto E_\tau = \{\eta \in W \mid \tau \subseteq \eta \bar{S}\}$ is inclusion-reversing and defines a subdivision of $W \otimes \boldsymbol{R}$ into polyhedral cells (the dual complex arising from the decomposition $\hat{P} = \bigcup_{\eta \in W} \eta S$). The cells are Γ_τ, where the faces of Γ_τ are the Γ_σ for σ a face of τ. We must check that for $\sigma_1, \sigma_2 \in \tilde{\Sigma}$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. We may suppose that $\sigma_1 = \bar{\tau}_{1+}$ and $\sigma_2 = \eta \bar{\tau}_{2+}$ for some $\tau_1, \tau_2 \subset \bar{S}$ and some $\eta \in W$. Then $\sigma_1 \cap \sigma_2 = \mathrm{Span}(\tau_1 \cap \eta\tau_2, (E_{\tau_1} \cap E_{\eta\tau_2})m + e)$. But since Σ is a fan, $\tau_1 \cap \eta\tau_2$ is a face of both τ_1 and τ_2. $(E_{\tau_1} \cap E_{\eta\tau_2})m + e = \{E_\sigma m + e \mid \tau_1$ and τ_2 both faces of $\sigma\}$, which is a face of Γ_{τ_i}, and the result follows. Finally, $|\Sigma| = \tilde{\Omega}'$, as E generates W (so that $\{\Gamma_\tau\}$ defines a subdivision of the whole of $W \otimes \boldsymbol{R}$). The construction is illustrated in Figure 3.

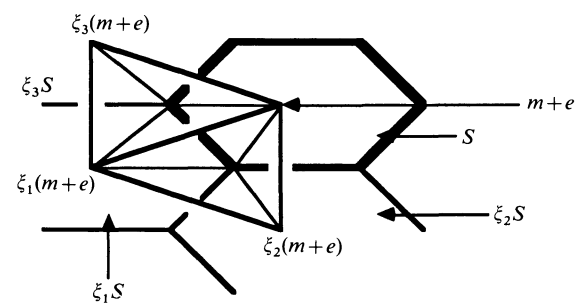

FIGURE 3

COROLLARY 3.2. *There is a complex analytic space $\mathscr{Y}$ and a map $\phi : \mathscr{Y} \to \boldsymbol{P}^1$ such that*

(i) *ϕ is proper and flat*

(ii) *$\phi^{-1}(t) \cong Y$ for $t \neq 0, \infty$*

(iii) *$\phi^{-1}(0)$ and $\phi^{-1}(\infty)$ are complete toric varieties with some pairs of disjoint torus-invariant codimension one subvarieties identified.*

PROOF. $\mathscr{Y}$ is constructed from $\tilde{\Sigma}$ and $\tilde{M}$ just as Y is from Σ and M. ϕ is induced by $\tilde{M} \to \boldsymbol{Z}e$ and (ii) and (iii) follow from the construction of $\tilde{\Sigma}$. ∎

A more precise description of $\phi^{-1}(0)$ is this: let $N = \tilde{M}/\boldsymbol{Z}(m+e)$. E can be listed as $\{\xi_{\pm i} \mid 0 \le i \le a\}$ with $\xi_{-i} = \xi_i^{-1}$. Π is a fan whose 1-skeleton is $\{e_{\pm i}\} \cup \{f_j\}$, where $e_{\pm i}$ is the class of $\xi_{\pm i}(m+e)$ in N and f_j is the class in N of the generator of $M \cap \tau_j$, with $\tau_j \in \Sigma^{(1)}$ and $\tau_j \subset \bar{S}$; the larger cones in Π are similarly described. Then $\phi^{-1}(0)$ is got by identifying $\overline{\mathrm{orb}(e_i)}$ with $\overline{\mathrm{orb}(e_{-i})}$ in $X_{\Pi, N}$.

In general $\mathscr{Y}$ and $X_{\Pi, N}$ are not smooth. It can be arranged that they be smooth by subdividing $\tilde{\Sigma}$ (in a W-invariant way): if this is done without altering Σ (which is possible) the effect is of a bimeromorphic modification of $\mathscr{Y}$ which is biholomorphic away from the boundary $\phi^{-1}(0) \cup \phi^{-1}(\infty)$, so we get a smooth family in which the special fibres have normal crossings.

$\mathscr{Y}$ is the simplest example of a decomposable construction, i.e., without

Assumption A. If we are interested only in degenerating Y we can of course look at $\tilde{\tau}_+$ only and get a family over the affine line.

4. The decomposable case. We now drop Assumption A and consider what happens if ρ is reducible over $\boldsymbol{Q}$. We retain Assumption B: according to this, $\rho_{\boldsymbol{C}}$ splits over $\boldsymbol{C}$ into a sum of characters. Therefore $\rho_{\boldsymbol{Q}}$ is completely reducible over $\boldsymbol{Q}$, so suppose $\rho_{\boldsymbol{Q}}=\rho_0\oplus\rho_1$. For simplicity, we assume throughout this section that ρ_0 and ρ_1 are irreducible over $\boldsymbol{Q}$ — the case where $\rho_{\boldsymbol{Q}}$ has more than two irreducible factors is essentially the same. So $\rho_{\boldsymbol{C}}\colon W\to SL(M\otimes\boldsymbol{C})\cong SL(n,\boldsymbol{C})$ is the direct sum of not necessarily distinct characters χ_i indexed by $i\in\mathscr{R}=\{1,\cdots,n\}$. $\mathscr{R}$ is partitioned as $\mathscr{R}=\mathscr{R}_0\amalg\mathscr{R}_1$ with $\rho_\nu=\bigoplus_{i\in\mathscr{R}_\nu}\chi_i$. $\mathscr{R}$ is also partitioned as $\mathscr{R}=\mathscr{B}\amalg\mathscr{H}$, where $\{\chi_i\mid i\in\mathscr{B}\}$ is a set of real characters and $\operatorname{Card}\mathscr{B}=b$. Put $\mathscr{B}_\nu=\mathscr{B}\cap\mathscr{R}_\nu$, $\mathscr{H}_\nu=\mathscr{H}\cap\mathscr{R}_\nu$, $b_\nu=\operatorname{Card}\mathscr{B}_\nu$, etc. As in § 1 we take $V=M\otimes\boldsymbol{R}$ and define W (the image of the faithful representation ρ) and Ω, L_+ and Ω' from $\mathscr{B}$ and $\mathscr{H}$. More precisely, $V=\bigoplus_{i\in\mathscr{R}}V_i$, where V_i is a subspace of V to which $\rho_{\boldsymbol{R}}$ restricts and such that $\rho_{\boldsymbol{R}}|_{V_i}=\chi_i$ when χ_i is real (and $\dim V_i=1$) or

$$\rho_{\boldsymbol{R}}|_{V_i}=\begin{pmatrix}\operatorname{Re}\chi_i & -\operatorname{Im}\chi_i\\ \operatorname{Im}\chi_i & \operatorname{Re}\chi_i\end{pmatrix}$$

if χ_i is not real (and $\dim V_i=2$). $L=\bigoplus_{i\in\mathscr{B}}V_i$, L_+ is an orthant and $\Omega=\pi_L^{-1}(L_+)$.

With the obvious notation, Assumption C asserts that for every $\eta\in W$ either there is an $i\in\mathscr{B}$ such that $\eta_i>|\eta_j|$ for all $j\in\mathscr{H}$ or there is an $i\in\mathscr{B}$ such that $\eta_i<|\eta_j|$ for all $j\in\mathscr{H}$. From this it follows that W acts on $\hat{P}=\boldsymbol{P}(\Omega')$ freely and properly discontinuously with a compact polyhedral fundamental domain, and the construction of Σ and Y goes through exactly as in §§ 1 and 2.

Put $W_\nu=\rho_\nu(\operatorname{Ker}\rho_{1-\nu})$, the subgroup of W which acts trivially on $\bigoplus_{i\in\mathscr{R}_{1-\nu}}V_i$. Since $\rho_{\boldsymbol{Q}}=\rho_0\oplus\rho_1$, $W\otimes\boldsymbol{Q}=(W_0\otimes\boldsymbol{Q})\oplus(W_1\otimes\boldsymbol{Q})$.

LEMMA 4.1. *W_ν has rank b_ν.*

PROOF. If $h>h_\nu>0$ then W_ν, which clearly satisfies Assumption C, has a compact fundamental domain on $\hat{P}_\nu$, so $\operatorname{rk}W_\nu=b_\nu$. If $h_\nu=0$, or if $h_\nu=h$ and $b_\nu=0$, this breaks down since $\hat{P}_\nu$ is empty. If $h_\nu=0$ and $b_\nu<b$ the above argument works for $W_{1-\nu}$; if $h_\nu=0$ and $b_\nu=b$, which is the remaining alternative, $W_{1-\nu}$ must be zero since otherwise it fixes L_ν pointwise and W does not act freely on Ω. ■

LEMMA 4.2. *The case $h_\nu=0$ does not occur.*

PROOF. If $h_\nu=0$ then ρ_ν embeds W_ν in $SL(b_\nu,\boldsymbol{Z})$, since $b_\nu=r_\nu$. Hence, by Proposition 1.1, $W_\nu\le\mathcal{O}_{K'}^*$ with $|K':\boldsymbol{Q}|=b_\nu$. But then $\operatorname{rk}W_\nu<b_\nu$, contradicting Lemma 4.1. ■

THEOREM 4.3. *If $Y=Y(W)$ is decomposable then there exist a manifold $\hat{Y}$, a generically finite proper morphism $\phi\colon\hat{Y}\to Y$, and a morphism $\psi\colon\hat{Y}\to Y_1$ with connected* *fibres such that*

(i) Y_1 *is an indecomposable manifold arising from* W_1, *or* $Y_1 = \boldsymbol{P}^1$ *if* $b_1 = 0$

(ii) *if* $p \in Y$ *does not lie on the exceptional divisor* F_1 *of* Y_1 ($F_1 = \{0, \infty\}$ *if* $Y_1 = \boldsymbol{P}^1$), *then* $\psi^{-1}(p) = Y_0$, *an indecomposable manifold arising from* W_0

(iii) *if* $p \in F_1$, $\psi^{-1}(p)$ *consists of toric varieties crossing normally.*

PROOF. We assume that $b_0 \neq 0$, interchanging the suffices if necessary. Write W' for $W_0 \oplus W_1$. W' is of finite index in W, and we can take a finite étale cover $Y' \to Y$ ($Y' = X/W'$ in the notation of § 1) and work with W' instead of W. We define $V_\nu = \bigoplus_{i \in \mathscr{R}_\nu} V_i$ for $\nu = 0, 1$, and look at $\Omega_0 = \Omega \cap V_0$ and $L_0 = L \cap V_0$.

In general $\{\sigma \in \Sigma \mid \sigma \subset \Omega_0\}$ will not be a fan, if Σ is a W'-invariant basic fan in Ω, but it will be W'-invariant. Therefore we can subdivide Σ in a W'-invariant way to Σ' so that $\Sigma_0 = \{\sigma \in \Sigma' \mid \sigma \subset \Omega_0\}$ is a fan in V_0, basic with respect to $M_0 = M \cap V_0$; note that M_0 is a full lattice in V_0 since V_0 is defined over $\boldsymbol{Q}$.

Next we want the projection π_{V_1} to induce a W_1-equivariant morphism of fans $\Sigma' \to \Sigma_1$ for some suitable fan Σ_1 in V_1. In general it will not and we shall have to subdivide Σ' again: the rest of the proof consists of constructing a suitable subdivision.

Let S' be a cone in V such that $\boldsymbol{P}(S')$ is a compact fundamental domain for the action of W' on $\hat{P}$. Let $\mathscr{P} = \boldsymbol{P}(\pi_{V_1}(S'))$: $\mathscr{P}$ is a compact domain in $\tilde{P}^1 = \boldsymbol{P}(\Omega^{1'})$, where $\Omega^1 = \pi_{V_1}(\Omega)$. Furthermore, $\bigcup_{\xi \in W_1} \xi\mathscr{P} = \hat{P}^1$, since if $\eta = \zeta + \xi$, $\eta \in W'$, $\zeta \in W_0$, $\xi \in W_1$, then $\eta\pi_{V_1}(S') = \xi\pi_{V_1}(S')$ so

$$\bigcup_{\xi \in W_1} \xi\pi_{V_1}(S') = \bigcup_{\eta \in W'} \eta\pi_{V_1}(S') = \pi_{V_1}\left(\bigcup_{\eta \in W'} \eta S'\right) = \pi_{V_1}(\Omega') = \Omega^{1'}.$$

We can also find an open fundamental domain $\mathscr{S}$ for the action of W_1 on $\hat{P}^1$, by Theorem 2.4. Since $\mathscr{P}$ is compact we can write $\mathscr{P} \subseteq \bigcup_{i=1}^{k} \xi_i\mathscr{S}$ for some finite set $\{\xi_1, \cdots, \xi_k\} \subset W_1$. Now for $\xi \in W_1$

$$\xi\mathscr{P} \cap \mathscr{P} \subseteq \xi\left(\bigcup_{i=1}^{k} \xi_i\mathscr{S}\right) \cap \bigcup_{i=1}^{k} \xi_i\mathscr{S} = \bigcup_{i=1}^{k} \bigcup_{i'=1}^{k} \xi\xi_i\mathscr{S} \cap \xi_{i'}\mathscr{S} = \varnothing$$

which is empty for all but finitely many ξ. So $\xi\mathscr{P} \cap \mathscr{P} = \varnothing$ unless $\xi \in \mathscr{W}$ for some fixed finite $\mathscr{W} \subset W_1$.

Because of the way Σ is constructed (Theorem 2.5) we can choose S' to be a rational polyhedral cone such that if $\sigma \in \Sigma'$ and $\sigma \cap S' \neq \{0\}$ (σ relatively open) then $\sigma \subset S'$. Obviously this property will not have been harmed when we subdivided Σ. The point is that we can now work with just the set $\mathscr{C}$ of cones in V_1 given by

$$\begin{aligned}\mathscr{C} = \{\pi_{V_1}(\sigma) \mid \boldsymbol{P}(\pi_{V_1}(\sigma)) \cap \mathscr{P} \neq \varnothing\} &= \{\pi_{V_1}(\sigma) \mid \sigma \cap \xi S' \neq \{0\}, \text{ some } \xi \in \mathscr{W}\} \\ &= \{\pi_{V_1}(\sigma) \mid \sigma \subseteq \xi S', \text{ some } \xi \in \mathscr{W}\}\end{aligned}$$

which is a finite set, indexed by I, say.

We need a fan Σ_1 in V_1 with the obvious properties and a W'-invariant subdivision

Σ'' of Σ' such that for all $\sigma\in\Sigma''$ there exists a $\tau\in\Sigma_1$ such that $\pi_{V_1}(\sigma)\subseteq\tau$; see the definition of morphism of fans in [4] or [13]. We first construct Σ_1 so that it is compatible with the structure of Σ' as far as possible, and then go back and amend Σ'. Obviously $\bigcup_{\xi\in W_1}\bigcup_{j\in I}\xi\tau_j=\Omega^{1'}$. To build Σ_1 we start with

$$\mathscr{T}_1=\left\{\bigcap_{j\in J}\tau_j \,\middle|\, J\subseteq I\right\}$$

which is a set of closed polyhedral cones covering $\pi_{V_1}(S')$. Then we take

$$\mathscr{T}_2=\mathscr{T}_1\cup\{\tau\,|\,\tau \text{ is a face of } \tau'\in\mathscr{T}_1\}$$

and

$$\mathscr{T}_3=\{\tau\cap\pi_{V_1}(S')\,|\,\tau\in\mathscr{T}_2\}\,.$$

These cones need not be convex and we shall have to subdivide them further to ensure that they are: we must also ensure compatibility on $\mathscr{P}\cap\xi\mathscr{P}$ when this is non-empty. We also want Σ_1 to be basic with respect to the lattice $\pi_{V_1}(M)$ (not just M_1, which is in general a proper sublattice of finite index). All this is just routine subdivision (see [4]). Figure 4 shows a simple case.

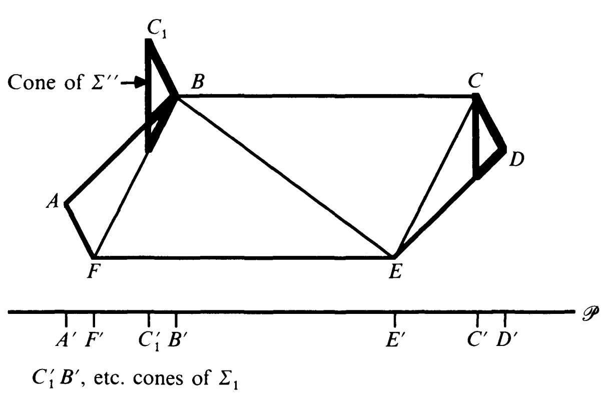

FIGURE 4

To construct Σ'', subdivide Σ' by replacing each σ such that $\pi_{V_1}(\sigma)\in\mathscr{C}$ by $\{\sigma\cap\pi_{V_1}^{-1}(\tau)\,|\,\tau\in\Sigma_1\}$, which is a finite set by the construction of Σ_1. Actually it is sufficient to do this for finitely many such σ, namely a set of representatives for the W'-equivalence classes. Let Σ'' be the set of all W'-translates of such cones (or, what is the same thing $\{\sigma\cap\pi_{V_1}^{-1}(\tau)\,|\,\sigma\in\Sigma',\ \tau\in\Sigma_1\}$) suitably subdivided so that Σ'' is a basic W'-invariant fan. Notice that the subdivision will not affect Σ_0, nor will it damage the compatibility of π_{V_1} with the fans since it will only make the σ's smaller.

The proof of the theorem is completed by taking $\hat{Y}$ to be the decomposable manifold constructed from Σ'', M and W'; Y_1 to be the indecomposable manifold constructed from Σ_1, $\pi_{V_1}(M)$ and W_1; and Y_0 to be the indecomposable manifold coming from Σ_0, M_0 and W_0, with ϕ the quotient by W_1 of the toric morphism $X_{\Sigma'',W'} \to X_{\Sigma_1,W_1}$ induced by π_{V_1}. The conclusions of the theorem ((iii) may call for a little more subdivision of Σ'') are clear: see the diagram. ∎

$$\begin{tikzcd}
 & Y_0 = Y(W_0, \Sigma_0, M_0) \arrow[d] \arrow[r, phantom, "="] & Y_0 = Y(W_0, \Sigma_0, M_0) \arrow[d] \\
Y(W', \Sigma, M) \arrow[d, "\text{finite}"] & Y(W', \Sigma', M) \arrow[l, "\text{bimer}"'] & \hat{Y} = Y(W', \Sigma'', M) \arrow[l, "\text{bimer}"'] \arrow[d, "\phi"] \\
Y(W, \Sigma, M) & & Y_1 = Y(W_1, \Sigma_1, \pi_{V_1}(M))
\end{tikzcd}$$

REMARKS. 1. If neither b_0 nor b_1 is zero the suffices are interchangeable and we can arrange for the fibre to be Y_1 and the base Y_0. But if $b_0 = 0$ then $\Omega_0 = \varnothing$ and the construction breaks down.

2. By an obvious induction argument the result can be adapted to the case of more than two irreducible factors in ρ_Q.

3. The degenerations described in § 3 are just the special case $b_1 = 0$ of Theorem 4.3. In that case W must always satisfy Assumption C, and $\Sigma_1 = \Lambda$.

4. It is not hard to check that decomposable ρ, neither of whose factors are trivial, satisfying Assumption C, do exist at least sometimes: for instance, if rk $W_0 =$ rk $W_1 = 1$ (except when both come from the same real quadratic field). I do not know necessary and sufficient conditions for such ρ to exist.

5. Kodaira dimension and algebraic dimension. We keep the notation above and assume Y to be smooth. We denote the canonical divisor of Y by K_Y, the Kodaira dimension by $\kappa(Y)$ and the algebraic dimension—the number of algebraically independent non-constant meromorphic functions on Y—by $a(Y)$.

PROPOSITION 5.1. *$K_Y = -F$, where F is the distinguished cycle coming from the non-dense torus orbits on X_Σ.*

PROOF. The top differential $dz_1 \wedge \cdots \wedge dz_n$, where $(z_1, \cdots, z_n)$ are the obvious coordinates on $\boldsymbol{H}^b \times \boldsymbol{C}^h$, is clearly M-invariant and therefore defines a global meromorphic n-form on X_Σ with, as is easily seen, simple poles along $\tilde{F}$. It is also W-invariant, since $\eta \in W$ acts on it by $\wedge^n \eta = \det \rho(\eta) = 1$. $dz_1 \wedge \cdots \wedge dz_n$ therefore descends to a meromorphic n-form on Y with simple poles along F. ∎

COROLLARY 5.2. *The Kodaira dimension $\kappa(Y)$ is $-\infty$.*

Denote, here and later, the dual lattice of M by $M^\vee$. W acts on $M^\vee$ via $\rho^\vee$, the transpose of ρ (which, in the indecomposable case, is obtained in the same way as ρ—see § 1—but with τ_j and $\bar{\tau}_j$ interchanged).

THEOREM 5.3. *If Y is indecomposable it has no non-constant meromorphic functions.*

PROOF. Suppose $f\colon Y\to \boldsymbol{P}^1$ is a meromorphic function. Then there is a function $\tilde{f}\colon \boldsymbol{H}^b\times \boldsymbol{C}^h\to \boldsymbol{P}^1$ which is invariant for $G(M, W)$, got by lifting to $\boldsymbol{H}^b\times \boldsymbol{C}^h$ the restriction of f to $Y_0=Y\setminus F$. Since $\tilde{f}$ is M-periodic we have

$$\tilde{f}(z)=\sum_{p\in M^\vee}\alpha_p \exp\{2\pi i p(z)\}$$

for $z\in \boldsymbol{H}^b\times \boldsymbol{C}^h\subset \boldsymbol{C}^n=M\otimes \boldsymbol{C}$, and the series converges almost everywhere on $\boldsymbol{H}^b\times \boldsymbol{C}^h$. Since we have chosen a basis for $M\otimes \boldsymbol{R}$, we can think of $M^\vee$ as a lattice in K and therefore embedded in $M\otimes \boldsymbol{C}=\boldsymbol{C}^n$. With respect to this basis, $p=(p_1,\cdots,p_n)$ and (assuming K is not totally real, when the proof requires a trivial modification) $\rho^\vee(\eta)(p_{n-1},p_n)=(\operatorname{Re}\eta p_{n-1}-\operatorname{Im}\eta p_n,\ \operatorname{Im}\eta p_{n-1}+\operatorname{Re}\eta p_n)$. But $\alpha_{\rho^\vee(\eta)(p)}=\alpha_p$ by the W-equivariance of $\tilde{f}$, and, by convergence, for any $\delta,\varepsilon>0$ there are only finitely many $p\in M^\vee$ with $|p_n|>\varepsilon$ and $|\alpha_p|>\delta$; similarly for $|p_{n-1}|$. Hence $\alpha_p=0$ unless $p_{n-1}=p_n=0$, but then $p\in K$ and $\tau_s(p)=0$ so $p=0$. Therefore $\tilde{f}$, and hence f, is a constant. ∎

COROLLARY 5.4. *For general (decomposable) Y, $a(Y)$ is equal to the multiplicity of the trivial representation in ρ. In particular, Y is not a Moishezon manifold.*

PROOF. This follows at once from Theorem 5.3 and Theorem 4.3. ∎

6. Cohomological invariants. In this section we shall investigate the cohomology of some of the more interesting sheaves on Y and the Betti numbers.

LEMMA 6.1. *The fundamental group is isomorphic to $\boldsymbol{Z}^b$.*

PROOF. X_Σ is simply connected by a result of Mumford ([13, Proposition 10.2]); so, by the same argument, is X. Therefore $\pi_1(Y)\cong W$, since $Y=X/W$ and W acts without fixed points. ∎

Let $\varpi\colon X\to Y=X/W$ be the quotient map. For a locally free sheaf $\tilde{\mathscr{F}}$ on X let $\mathscr{F}$ be the subsheaf of $\varpi_*\tilde{\mathscr{F}}$ consisting of the images of germs of W-invariant sections (this is a locally free sheaf on Y). According to [8] there is a spectral sequence

$$\text{(S1)}\qquad E_2^{pq}(\mathscr{F})=H^p(W;H^q(X,\tilde{\mathscr{F}}))\Rightarrow H^{p+q}(Y,\mathscr{F})\,.$$

We shall want to consider the cases $\mathscr{F}=\mathcal{O}_Y$, $\mathscr{F}=\Theta_Y(-\log F)$ and $\mathscr{F}=\Omega_Y^p(\log F)$. In these cases we have $\tilde{\mathscr{F}}=\mathcal{O}_X$, $\tilde{\mathscr{F}}=\Theta_X(-\log\tilde{F})$ and $\tilde{\mathscr{F}}=\Omega_X^p(\log\tilde{F})$ respectively.

THEOREM 6.3. $\dim H^i(Y,\mathcal{O}_Y)=\binom{b}{i}$ *where* $\binom{b}{i}$ *denotes the binomial coefficient.*

PROOF. We begin with some calculations about $H^i(X,\mathcal{O}_X)$, with a view to using (S1). The fan Σ determines an open cover $\{X_\sigma\}_{\sigma\in\Sigma}$ of X_Σ. For $\sigma\in\Sigma$, define $X'_\sigma=X_\sigma\cap X$ and an open cover $\mathscr{X}=\{X'_\sigma\}$ of X. One checks easily that X'_σ is Stein, so that $\mathscr{X}$ is acyclic for any coherent sheaf, in particular for $\mathcal{O}_X$. Moreover $\mathcal{X}$ is closed under intersections, so that Leray's Theorem applies even though $\mathcal{X}$ is not locally finite ([7, II.5.4.1, Corollaire]). So

$$H^{\cdot}(X, \mathcal{O}_X) = H^{\cdot}(C^{\cdot}(\mathcal{X}, \mathcal{O}))$$

where $C^{\cdot}(\mathcal{X}, \mathcal{O})$ is the usual Čech complex

$$\{\cdots \xrightarrow{\partial} \prod \Gamma(X'_{\sigma}, \mathcal{O}) \xrightarrow{\partial} \cdots\}.$$

Now if $f \in \Gamma(X'_{\sigma}, \mathcal{O})$ we can restrict it to $X'_{\{0\}}$ and lift it to $\tilde{f}$ on $\boldsymbol{H}^b \times \boldsymbol{C}^h$ as in 5.3 above. Again we get

$$\tilde{f}(z) = \sum_{p \in M^{\vee}} \alpha_p \exp\{2\pi i p(z)\}$$

with certain restrictions on the α_p's ($\alpha_p = 0$ if $p \notin \check{\sigma}$ and obvious conditions to ensure convergence). For a W-orbit $\mathfrak{p} \in M^{\vee}/W$, put

$$^{\mathfrak{p}}\Gamma(X'_{\sigma}, \mathcal{O}) = \{f \in \Gamma(X'_{\sigma}, \mathcal{O}) \,|\, \alpha_p = 0 \text{ for } p \notin \mathfrak{p}\}$$

so that

$$\Gamma(X'_{\sigma}, \mathcal{O}) \subset \prod_{\mathfrak{p} \in M^{\vee}/W} {}^{\mathfrak{p}}\Gamma(X'_{\sigma}, \mathcal{O}).$$

The ${}^{\mathfrak{p}}C^*(\mathcal{X}, \mathcal{O})$ which are thus defined form a complex ${}^{\mathfrak{p}}C^{\cdot}(\mathcal{X}, \mathcal{O})$: that is to say, the Čech differentials commute with the projection maps.

In view of (S1), $H^*(Y, \mathcal{O}_Y)$ is the homology of the complex

$$C^{\cdot} = C^{\cdot}(W; C^{\cdot}(\mathcal{X}, \mathcal{O}))$$

where $C^{\cdot}(W; -)$ is, say, the complex arising from the standard resolution ([3, III.1]). Consider also the complex

$$\hat{C}^{\cdot} = C^{\cdot}\left(W; \prod_{\mathfrak{p} \in M^{\vee}/W} {}^{\mathfrak{p}}C^{\cdot}(\mathcal{X}, \mathcal{O}_X)\right)$$

which is equal to $\prod_{\mathfrak{p} \in M^{\vee}/W} {}^{\mathfrak{p}}C^{\cdot}$ with the obvious notation ${}^{\mathfrak{p}}C^{\cdot} = C^{\cdot}(W, {}^{\mathfrak{p}}C^{\cdot}(\mathcal{X}, \mathcal{O}_X))$. The inclusion $C^{\cdot} \to \hat{C}^{\cdot}$ induces a map $H^{\cdot}(C^{\cdot}) \to H^{\cdot}(\hat{C}^{\cdot})$, because the differentials involve only Čech differentials and W-module operations in the coefficient modules. It is easy to see that $H^{\cdot}(\hat{C}^{\cdot}) = \prod_{\mathfrak{p} \in M^{\vee}/W} H^{\cdot}({}^{\mathfrak{p}}C^{\cdot})$. Let $K^{\cdot}$ be the kernel of $H^{\cdot}(C^{\cdot}) \to H^{\cdot}(\hat{C}^{\cdot})$.

Suppose that $\gamma \in H^{\cdot}(C^{\cdot})$ and ${}^{\mathfrak{p}}\gamma \neq 0$ for some $\mathfrak{p} \neq \{0\}$. Then we may define, for any $r \in \boldsymbol{N}$, $\gamma[r]$ by ${}^{r\mathfrak{p}}\gamma[r] = {}^{\mathfrak{p}}\gamma$ for all $\mathfrak{p}$, and ${}^{\mathfrak{p}'}\gamma[r] = 0$ if $\mathfrak{p}' \neq r\mathfrak{p}$ for any $\mathfrak{p} \in M^{\vee}/W$. At the level of $\Gamma(X'_{\sigma}, \mathcal{O})$ it is given by $\widetilde{f[r]}(z) = \tilde{f}(rz)$, so the convergence is assured and $\gamma[r] \in H^{\cdot}(C^{\cdot})$. Clearly the $\gamma[r]$'s are linearly independent. But then $H^*(Y, \mathcal{O}_Y)$ is infinite-dimensional, which is a contradiction since Y is compact.

I claim also that $K^{\cdot} = 0$. Suppose $\xi \in K^{\cdot}$. ξ is determined by a (by no means unique)

$\Xi \in C^{\cdot}$ which in turn is given by a collection $\{\xi_i\}_{i\in I}$ of sections $\xi_i \in \Gamma(X'_\sigma, \mathcal{O})$. The index set I is countable, because W and $\mathcal{X}$ are both countable. For notational convenience fix a bijection $N \to M^{\vee}/W$ and write of ${}^t\Gamma(X'_\sigma, \mathcal{O})$ with $t \in N$ instead of ${}^{\mathfrak{p}_t}\Gamma(X'_\sigma, \mathcal{O})$. The statement that $\Xi \in C^{\cdot}$ amounts to saying that the formal sums $\xi_i(z) = \sum_{t\in N} {}^t\xi_i(z)$ converge for all $z \in H^b \times C^h$ and all $i \in I$. Given a cocycle $\Phi \in \hat{C}^{\cdot}$ and a function $g: N \to R$ we define $g_! \Phi$ by ${}^t(g_! \Phi) = g(t)({}^t\Phi)$. $g_! \Phi$ is also a cocycle.

FACT. There exists $g_1: N \to R_+$ such that $g_1(t) \to \infty$ as $t \to \infty$ but $g_{1!}\, \Xi \in C^{\cdot}$; that is, $\sum_{t\in N} g_1(t)({}^t\xi_i(z))$ converges for all $z \in H^b \times C^h$ and all $i \in I$.

Granted this, we observe that $g_j\colon t \longmapsto (g_1(t))^{1/j}$ has the same property for any $j \in N$. Take $g_0 \equiv 1$, and write $g_{j!}\,\xi$ for the class of $g_{j!}\, \Xi$ in $H^{\cdot}(C^{\cdot})$ (which is abuse of notation, since $g_{j!}\,\xi$ depends on the choice of Ξ). Since $H^*(Y, \mathcal{O}_Y)$ is finite-dimensional, there is a relation

$$\sum_{j=0}^{k} \beta_j(g_{j!}\,\xi) = 0$$

in $H^{\cdot}(C^{\cdot})$, and therefore there exists a $\Psi \in C^{\cdot}$ such that

$$\partial \Psi = \sum_{j=0}^{k} \beta_j(g_{j!}\,\Xi) = \left(\sum_{j=0}^{k} \beta_j g_j\right)_! \Xi .$$

Choose T so that $|\sum_{j=0}^{k} \beta_j g_j(t)| > 1$ for $t > T$. This is possible (unless $k=0$, when $\xi = 0$ immediately) because $|\sum_{j=0}^{k} \beta_j g_j(t)| \to \infty$ as $t \to \infty$.

We may assume that ${}^t\Xi = 0$ and ${}^t\Psi = 0$ for all $t \leq T$. To arrange this, we replace Ξ by $\Xi - \sum_{t \leq T} {}^t\Xi$, and similarly for Ψ. Obviously this does not affect the convergence of $\sum_{t\in N} {}^t\xi_i(z)$, because we are changing only finitely many terms. It does not affect the cohomology class of Ξ or of $g_{j!}\, \Xi$ either. For we have already shown that ${}^{\mathfrak{p}}\xi = 0$ for $\mathfrak{p} \neq 0$, that is, ${}^t\xi = 0$ for $t > 0$, by considering ${}^t\xi[r]$; and the change in the cohomology class of $g_{j!}\, \Xi$ caused by changing Ξ as stated is $\sum_{t \leq T} g_j(t)({}^t\xi)$. But that is zero, because if $\partial({}^t\Phi) = {}^t\Xi$ then $\partial(\sum_{t\leq T} g_j(t)({}^t\Phi)) = \sum_{t \leq T} g_j(t)({}^t\Xi)$ and $\sum_{t\leq T} g_j(t)({}^t\Phi)$ is certainly in $C^{\cdot}$.

But now $\Psi' = ((\sum_{j=0}^{k} \beta_j g_j)^{-1})_! \Psi \in C^{\cdot}$, and $\partial \Psi' = \Xi$, so $\xi = 0$.

The proof of the existence of g_1 is elementary analysis. Since $\sum_{t\in N} {}^t\xi_i(z)$ converges uniformly and absolutely on any compact subset $B \subset H^b \times C^h$ there is for each i a function $g_{B,i}$ such that $g_{B,i}(t) \to \infty$ as $t \to \infty$ but $\sum_{t\in N} g_{B,i}(t)({}^t\xi_i(z))$ converges on B. For by uniform convergence we may choose R_n so that $\sum_{i=R_n}^{R} |{}^t\xi_i(z)| < 2^{-2n}$ for all $z \in B$ and all $R > R_n$; we may as well take $R_n < R_{n+1}$ for all $n \in N$. Then it is enough to take $g_{B,i}(t) = 2^n$ if $R_n \leq t < R_{n+1}$. For then, if n' is such that $R_{n'} > R$

$$\sum_{t=R_n}^{R} |g_{B,i}(t)({}^t\xi_i(z))| \leq \sum_{k=n}^{n'} \sum_{t=R_k}^{R_{k+1}-1} |g_{B,i}(t)({}^t\xi_i(z))| = \sum_{k=n}^{n'} \sum_{t=R_k}^{R_{k+1}-1} 2^k |{}^t\xi_i(z)| < \sum_{k=n}^{n'} 2^k 2^{-2k} < 2^{-n+1}$$

so $\sum_{t\in N} g_{B,i}(t)({}^t\xi_i(z))$ converges.

In general, if $(h_j)_j \in N$ is a countable family of functions each tending to infinity, there is an $h: \boldsymbol{N} \to \boldsymbol{R}_+$ such that $h(t) \to \infty$ and $h_j(t)/h(t) \to \infty$ for all j. (But not uniformly! There is no reason why $\inf h_j(t)$, let alone $\inf h_j(t)/h(t)$, should tend to infinity.) To see that such an h exists, choose, for each $n \in \boldsymbol{N}$, N_n such that $h_j(t) > n^2$ for all $j \leq n$ whenever $t \geq N_n$. Again we can assume that $N_n < N_{n+1}$, and it is enough to take $h(t) = n$ if $N_n < t < N_{n+1}$.

Since $\boldsymbol{H}^b \times \boldsymbol{C}^h$ can be covered by countably many compact sets B_k, and since I is also countable, we can get a suitable g_1 by applying this construction to the family $g_{B_k, i}$.

Thus $K^{\cdot} = 0$ and the only ${}^{p}H^{\cdot}(C^{\cdot})$ which can be non-zero is ${}^{\{0\}}H^{\cdot}(C^{\cdot})$. So $H^{\cdot}(C^{\cdot}) \cong {}^{\{0\}}H^{\cdot}(C^{\cdot})$, which amounts to saying that the $H^*(Y, \mathcal{O}_Y)$ can be calculated entirely from the weight $\{0\}$ pieces. ${}^{\{0\}}\Gamma(X'_\sigma, \mathcal{O}) \cong \boldsymbol{C}$, so in (S1) we have $E_2^{pq} = H^p(W; H^q(\mathscr{X}, \boldsymbol{C}_X))$. But it is easy to see that $H^q(\mathscr{X}, \boldsymbol{C}_X)$ also calculates $H^q(\mathrm{Mc}(M, \Sigma); \boldsymbol{C})$ via the open cover $\{\mathrm{Mc}(M, \{\text{faces of } \sigma\})\}_{\sigma \in \Sigma}$. $\mathrm{Mc}(M, \Sigma)$ is contractible, so $H^0(\mathscr{X}, \boldsymbol{C}_X) \cong \boldsymbol{C}$ and $H^q(\mathscr{X}, \boldsymbol{C}_X) = 0$ for $q > 0$ and the result follows from (S1). ■

The construction of $\gamma[r]$ from γ has a geometric origin: it arises from the r-fold cover of X_Σ defined by $rM^\vee \hookrightarrow M^\vee$.

[18, Proposition 4.1] is a version of Theorem 6.3 under the stronger conditions ($b = 1$) which prevail in that paper, but is proved by quite different methods, using degenerations (cf. Corollary 3.2, above). A closer relative of Theorem 6.3 is [6, Satz 6.1]. As the referee of the present paper pointed out, in the case $b = n - 1$ Theorem 6.3 can be proved fairly simply using Freitag's method, which makes use of the fact that (in our notation) $\bigoplus_{p \in M^\vee/W} {}^{p}\Gamma(X'_\sigma, \mathcal{O})$ is dense in the Fréchet space $\Gamma(X'_\sigma, \mathcal{O})$. This is not sufficient here: the extra complication comes from the use of the Leray Theorem, which is not involved in the calculations of local cohomology in [6].

COROLLARY 6.4. *For $q \geq 0$ and $p > 0$,*

$$H^q(Y, \Theta_Y(-\log F)) = H^q(W; M \otimes \boldsymbol{C})$$

and

$$H^q(Y, \Omega_Y^p(\log F)) = H^q(W; \wedge^p(M^\vee \otimes \boldsymbol{C})).$$

PROOF. These follow from 6.2, 6.3 and the universal coefficient theorem. ■

The term $H^q(W; \wedge^p(M^\vee \otimes \boldsymbol{C}))$ which occurs in Corollary 6.4 is dual to the E_2^{qp}-term of the Hochschild-Serre spectral sequence ([3])

$$\text{(S2)} \qquad E_2^{pq} = H^p(W; H^q(M; \boldsymbol{C})) \Rightarrow H^{p+q}(G(M, W); \boldsymbol{C})$$

associated to the extension

$$0 \to M \to G(M, W) \to W \to 0 .$$

Since $\boldsymbol{H}^b \times \boldsymbol{C}^h$ is contractible, $H^i(G(M, W); \boldsymbol{C}) \cong H^i(Y_0; \boldsymbol{C})$.

Henceforth we suppose that Y is indecomposable.

PROPOSITION 6.5. *For $q \geq 0$ and $0 < p < n$, $H^q(W; \wedge^p(M \otimes C)) = 0$.*

PROOF. Since $M \otimes C$ is completely reducible as a CW-module (by Assumption B), $\wedge^p(M \otimes C)$ is also: the representation of W in $GL(\wedge^p(M \otimes C))$ splits into a sum of $\binom{n}{p}$ characters given by the determinants of the $p \times p$ minors of $\rho(\eta)$. Non-zero cohomology could only occur when one of these is 1 for all $\eta \in W$, but if that were to happen for some $p \times p$ minor with $0 < p < n$ it would define an invariant subspace of $M \otimes C$ defined over Q, contradicting Assumption A. ■

Clearly the same argument shows that $H^q(W; \wedge^p(M^\vee \otimes C)) = 0$ as well.

COROLLARY 6.6. *For $q \geq 0$ and $0 < p < n$,*

$$H^q(Y, \Theta_Y(-\log F)) = H^q(Y, \Omega_Y^p(\log F)) = 0 .$$

COROLLARY 6.7. *Y is non-Kähler.*

PROOF. $H^0(Y, \Omega_Y^1) \subseteq H^0(Y, \Omega_Y^1(\log F)) = 0$, but $b_1(Y) = b > 0$ by Lemma 6.1. ■

COROLLARY 6.8. *There is a spectral sequence*

$$\text{(S3)} \qquad E_1^{pq} = H^q(Y, \Omega_Y^p(\log F)) \Rightarrow H^{p+q}(Y_0; C)$$

which degenerates at the E_1-term.

Corollary 6.8 is a standard theorem in the case of divisors with normal crossings on a Kähler manifold.

The weight filtration on the sheaf $\Omega_Y^p(\log F)$ (which contains Ω_Y^p as a subsheaf) is given, as usual, by

$$W_k \Omega_Y^p(\log F) = \Omega_Y^{p-k} \wedge \Omega_Y^k(\log F) .$$

Write F as the union of its irreducible components, $F = F_1 \cup \cdots \cup F_N$, and for $I \subseteq \{1, \cdots, N\}$ denote $\bigcap_{i \in I} F_i$ by F_I. There are exact sequences

$$0 \to W_{k-1}\Omega_Y^p(\log F) \to W_k \Omega_Y^p(\log F) \xrightarrow{\text{P.R.}} \bigoplus_{\sharp I = k} \Omega_{F_I}^{p-k} \to 0$$

where P.R. denotes the Poincaré residue map. Since $\dim F_I = n - k \geq p - k$ and since we may choose Σ so that every F_I is a smooth complete toric variety, we have $H^q(F_I, \Omega_{F_I}^{p-k}) = 0$ for $q \neq p - k$ and $H^{p-k}(F_I, \Omega_{F_I}^{p-k}) \cong H^{2(p-k)}(F_I; C)$ by [4, Proposition 12.7]. (Actually we do not need F_I to be smooth: it is enough that the fan of F_I be simplicial, which is automatically true if Σ is simplicial). Also, by Corollary 6.6, $H^q(Y, W_p \Omega_Y^p(\log F)) = 0$ for $p \neq 0, n$.

THEOREM 6.9. *For $p \neq 0$ or n and $q \neq p$, $H^q(Y, \Omega_Y^p) = 0$.*

PROOF. Suppose first $q>p$. $H^q(Y, \Omega_Y^p)=H^q(Y, W_0\Omega_Y^p(\log F))$ and from the long exact sequences

$$\cdots \to H^q(Y, W_{k-1}\Omega_Y^p(\log F)) \to H^q(Y, W_k\Omega_Y^p(\log F)) \to \bigoplus_{\sharp I=k} H^q(F_I, \Omega_{F_I}^{p-k}) \to \cdots$$

and the above remarks it follows that

$$H^q(Y, W_{k-1}\Omega_Y^p(\log F)) \cong H^q(Y, W_k\Omega_Y^p(\log F)) \cong H^q(Y, W_p\Omega_Y^p(\log F))=0$$

for all $k\geq 1$.

The result follows in general by Serre duality, $H^q(Y, \Omega_Y^p)\cong H^{n-q}(Y, \Omega_Y^{n-p})^\vee$. ■

In fact it follows from this argument that $H^q(Y, W_k\Omega_Y^p(\log F))=0$ unless $p=0$ or $q=p-k$, using Theorem 6.9 to deal with the case $q<p-k$. We are therefore left with short exact sequences

$$0 \to H^{p-k}(Y, W_k\Omega_Y^p(\log F)) \to \bigoplus_{\sharp I=k} H^{2(p-k)}(F_I; \mathbf{C}) \to H^{p-k+1}(Y, W_{k-1}\Omega_Y^p(\log F)) \to 0$$

with $1\leq k\leq p$. Putting $\omega_k^{(p)}=\dim H^{p-k}(Y, W_k\Omega_Y^p(\log F))$, $\beta_k^{(p)}=\sum_{\sharp I=k} b_{2(p-k)}(F_I)$ (where b_i denotes the i-th Betti number) and taking the alternating sum, we get

$$\omega_0^{(p)}+(-1)^{p+1}\omega_p^{(p)}=\sum_{k=1}^{p}(-1)^{k+1}\beta_k^{(p)}.$$

But $\omega_0^{(p)}=\dim H^p(Y, \Omega_Y^p)$ and $\omega_p^{(p)}=\dim H^0(Y, \Omega_Y^p(\log F))=0$ unless $p=0$ or n by Proposition 6.5. This enables us to write down a formula for $H^p(Y, \Omega_Y^p)$.

THEOREM 6.10. *The dimension of $H^p(Y, \Omega_Y^p)$ is*

$$h^{p,p}=\sum_{k=1}^{p}\sum_{j=k}^{p}(-1)^{p-j+k-1}\cdot\binom{n-j}{n-p}\cdot\binom{j}{k}\cdot a_j$$

where a_j is the number of W-equivalence classes of j-cones in Σ.

PROOF. $\beta_k^{(p)}=\sum_{\sharp I=k} b_{2(p-k)}F_I=\sum_{\sigma\in\Sigma^{(k)}/W} b_{2(p-k)}F_\sigma$, where $F_\sigma=\overline{\mathrm{orb}(\sigma)}$, by the toric description of F. Since $\dim F_\sigma=n-k$, we may use Poincaré duality on F_σ to write $\beta_k^{(p)}=\sum_\sigma b_{2(n-p)}F_\sigma$ (this is optional and again it is sufficient that Σ be simplicial). Now

$$\sum_\sigma b_{2(n-p)}F_\sigma=\sum_\sigma\sum_{j=n-p}^{n-k}(-1)^{j-(n-p)}\cdot a_{n-k-j}(F_\sigma)\cdot\binom{j}{n-p}$$

by [4, 10.8]), where $a_i(F_\sigma)=\sharp\{i\text{-cones in the fan of } F_\sigma\}$. But from the description of the fan of F_σ, $a_i(F_\sigma)=\sharp\{\sigma'\in\Sigma^{(i+k)}/W \text{ with } \sigma \text{ a face of } \sigma'\}$ and we get

$$\beta_k^{(p)}=\sum_\sigma\sum_{j=n-p}^{n-k}(-1)^{j-n+p}\cdot\binom{j}{n-p}\cdot\sharp\{(n-j)\text{-cones } \sigma' \text{ of which } \sigma \text{ is a face}\}$$

$$= \sum_{j=n-p}^{n-k} (-1)^{j-n+p} \cdot \binom{j}{n-p} \sum_{\sigma' \in \Sigma^{(n-j)}/W} \#\{k\text{-faces of an } (n-j)\text{-cone}\}$$

$$= \sum_{j=n-p}^{n-k} (-1)^{j-n+p} \cdot \binom{j}{n-p} \cdot a_{n-j} \cdot \binom{n-j}{k} = \sum_{j=k}^{p} (-1)^{p-j} \binom{n-j}{n-p} \cdot \binom{j}{k} \cdot a_j,$$

so $h^{p,p} = \sum_{k=1}^{p} \sum_{j=k}^{p} (-1)^{p-j+k-1} \cdot \binom{n-j}{n-p} \cdot \binom{j}{k} \cdot a_j$, as claimed. ■

In particular, $h^{1,1} = a_1 =$ number of irreducible components of F, as is in any case obvious from the exact sequence

$$0 \to H^0(Y, \Omega_Y^1(\log F)) \to \bigoplus H^0(F_i; \boldsymbol{C}) \to H^1(Y, \Omega_Y^1) \to 0 .$$

In principle we could now calculate the Euler characteristic of Y from the Hodge spectral sequence, but there is a less laborious method.

THEOREM 6.11. *$\chi(Y) = a_n$, the number of ordinary n-fold points of F.*

PROOF. We use the Mayer-Vietoris sequence

$$\cdots \to H_r(\mathcal{N}_F^*) \to H_r(Y_0) \oplus H_r(F) \to H_r(Y) \to \cdots$$

which we saw in §1, from which it follows that $\chi(Y) = -\chi(\mathcal{N}_F^*) + \chi(Y_0) + \chi(F)$. $\chi(\mathcal{N}_F^*) = 0$ from the Gysin sequence. Y_0 is a $K(G(M, W), 1)$ so $\chi(Y_0) = \chi(G(M, W)) = \chi(M) \cdot \chi(W) = 0$. Finally $\chi(F) = a_n$ by the argument of [15, Proposition 3.3]. ■

REMARK. Theorem 6.11 applies to any Y, not just the indecomposable ones.

THEOREM 6.12. *The Hodge spectral sequence $E_1^{pq} = H^q(Y, \Omega_Y^p)$ degenerates at the E_1-term and abuts to $H^{p+q}(Y; \boldsymbol{C})$.*

PROOF. Standard Hodge theory says that E_1^{pq} abuts to $H^{p+q}(Y; \boldsymbol{C})$. Since $E_1^{pq} = 0$ unless $p = q$ or $p = 0$ and $q \leq b$ or $p = n$ and $q \geq n - b$, the only differentials that can possibly be non-zero are $d_r \colon E_1^{q0} \to E_1^{q-r+1,r}$ with $r = (q+1)/2$ and their duals. It is clearly enough to show that $s \colon H^q(Y; \boldsymbol{C}) \to H^q(Y, \mathcal{O}_Y)$ is surjective for $q \leq b$. By Theorem 6.3, $H^q(Y, \mathcal{O}_Y) \cong H^q(W; \boldsymbol{C})$. By Lemma 6.1, $H^1(Y; \boldsymbol{C}) \to H^1(W; \boldsymbol{C})$ is an isomorphism. But the cohomology ring $H^*(W; \boldsymbol{C})$ is generated by $H^1(W; \boldsymbol{C})$, so s is surjective in all degrees (it is an isomorphism on the part of the ring $H^*(Y; \boldsymbol{C})$ generated by $H^1(Y; \boldsymbol{C})$). ■

The Betti numbers of an indecomposable Y are entirely determined by 6.3, 6.9, 6.10 and 6.12.

Finally, the method of [18] gives an expression for $H^i(Y, \Theta_Y)$.

THEOREM 6.13. *For indecomposable Y,*

$$H^0(Y, \Theta_Y)=0 \quad and \quad H^i(Y, \Theta_Y)=\bigoplus_{j=1}^{a_1} H^i(F_j, \mathcal{O}_{F_j}(F_j)) \qquad for \quad i>0 .$$

PROOF. This follows exactly as in [18] from the exact sequence

$$0\to\Theta_Y(-\log F)\to\Theta_Y\to\bigoplus \mathcal{O}_{F_j}(F_j)\to 0$$

and Proposition 6.5. ■

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