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# A CRITERION FOR UNIRULEDNESS IN POSITIVE CHARACTERISTIC

Dedicated to Professor Joji Kajiwara on his sixtieth birthday

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**Abstract.** The author shows the existence of a smooth projective uniruled but not separably uniruled variety in positive characteristic satisfying the numerical condition of Miyaoka and Mori. It is a counterexample to a problem which Miyaoka and Mori posed.

In this paper we make a remark on a criterion for a projective variety to be (separably) uniruled in positive characteristic $p$.

Let $X$ be an $n$-dimensional variety defined over an algebraically closed field $k$ which is uncountable. $X$ is said to be uniruled if there exist an $(n-1)$-dimensional $k$-variety $W$ and a dominant rational map $f: \boldsymbol{P}^1 \times W \dashrightarrow X$. $X$ is said to be separably uniruled if the morphism $f$ can be chosen to be separable.

Miyaoka and Mori [Mi-Mo] gave the following numerical criterion for uniruledness:

THEOREM (Miyaoka and Mori [Mi-Mo]). *Let $X$ be a smooth projective variety over the complex number field $\boldsymbol{C}$. Then the following two conditions are equivalent*:

(UR) *$X$ is uniruled.*

(NC) *There exists a non-empty open subset $U \subset X$ such that for every $x \in U$, there is an irreducible curve $C$ through $x$ with $(K_X \cdot C) < 0$.*

In positive characteristic, the implication (NC) $\Rightarrow$ (UR) holds good but the converse (UR) $\Rightarrow$ (NC) does not hold as was pointed out by [Mi-Mo]. They asked if (NC) is equivalent to

(SUR) $X$ is separably uniruled.

In this paper we first consider a criterion for a variety to be separably uniruled and show in Corollary 1.2 that a separably uniruled and smooth projective variety has the property (NC). Next we study a counterexample to the implication (NC) $\Rightarrow$ (SUR) in positive characteristic. Without assuming the smoothness of the variety $X$ in question, we can construct $X$ very easily by using $\boldsymbol{P}(T_{\boldsymbol{P}^n})$ with two projective space bundle structures and the Frobenius morphism of the base projective space $\boldsymbol{P}^n$. Since the resolution of

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singularities is not known yet in positive characteristic, however, it does not seem to be easy to get a smooth variety. Thus to overcome the obstacle we must make several preparations as in §2 and §4. Moreover by virtue of the theory of toroidal embedding, we have:

THEOREM (positive characteristic). *For an odd integer $m$ ($\geq 3$) there is an $m$-dimensional smooth projective variety $X$ enjoying the following properties*:

(1) *$X$ is defined over an algebraically closed field $k$ whose characteristic $p$ ($>0$) is less than* $(m+3)/2$.

(2) *$X$ is uniruled but not separably uniruled.*

(3) *$X$ has the above property* (NC).

Note that a smooth projective surface $X$ with $\kappa(X) = -\infty$ is ruled by the classification theory of algebraic surfaces. Therefore, the implication (NC) $\Rightarrow$ (SUR) holds good for such surfaces.

Basically we use the customary terminology in algebraic geometry. For a smooth variety $X$, $T_X$ denotes the tangent bundle of $X$. When $E$ is a vector bundle on a variety, $\boldsymbol{P}(E)$ (resp. $\boldsymbol{V}(E)$) means Proj (resp. Spec) of the symmetric algebra of $E$. $\check{E}$ means the dual vector bundle of $E$. For a variety $Y$ and a closed subscheme $Z$ in $Y$ which is locally a complete intersection, $N_{Z/Y}$ denotes the normal bundle of $Z$ in $Y$. $\mathcal{O}_{\boldsymbol{P}^n}(1)$ means the line bundle corresponding to hyperplanes of the projective space $\boldsymbol{P}^n$. When $F$ is a coherent sheaf on $\boldsymbol{P}^n$, $F(a)$ denotes $F \otimes \mathcal{O}_{\boldsymbol{P}^n}(1)^{\otimes a}$. $k^\times$ denotes $k - \{0\}$.

From now on the characteristic $p$ of the ground field $k$ is assumed to be positive.

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**1. Separable uniruledness.** First we give a criterion for a projective variety to be separably uniruled.

PROPOSITION 1.1. *Let $X$ be an $n$-dimensional projective variety and consider the following two conditions*:

(1) *$X$ is separably uniruled.*

(2) *There is a smooth open subset $U$ in $X$ and a non-constant morphism $f: \boldsymbol{P}^1 \to U \subset X$ such that $f^* T_U$ is generated by global sections.*

*Then* (2) *implies* (1).

*Moreover, assume that $X$ is smooth. Then the two conditions are equivalent to each other.*

PROOF. Let us show that (2) implies (1). Let $\Gamma \subset \boldsymbol{P}^1 \times X$ be the graph of $f$. Obviously, $\Gamma$ is contained in the smooth part $\bar{X}$ in $\boldsymbol{P}^1 \times X$ and the normal bundle ($=N$) of $\Gamma$ in $\bar{X}$ is isomorphic to $f^* T_U$. Now take the Hilbert scheme $H'$ in $P^1 \times X$ of $\Gamma$. By the condition (2), the first cohomology group of $f^* T_U$ vanishes, from which it follows that $H'$ is smooth at the point $\gamma$ corresponding to the curve $\Gamma$. Thus we can take the irreducible component $H$ of $H'$ containing the point $\gamma$. Let $W$ be the universal scheme corresponding to the Hilbert scheme $H$ and $s$, $t$ be the second and third projections from $W \subset \boldsymbol{P}^1 \times X \times H$ to $X$ and $H$, respectively. Then since $t^{-1}(\gamma) = \Gamma \times \gamma$ $(\simeq \boldsymbol{P}^1)$ and $H$ is smooth at $\gamma$, we infer that the morphism $t$ is a smooth morphism around the fiber $t^{-1}(\gamma)$, that is, there is a smooth open set $H_0$ $(\ni \gamma)$ in $H$ so that $t^{-1}(H_0) \to H_0$ is a $\boldsymbol{P}^1$-bundle over $H_0$. Since $s(\Gamma \times \gamma) \subset U$ and therefore $s^{-1}(X-U) \cap t^{-1}(\gamma) = \varnothing$, there is an open subset $H_0'$ $(\ni \gamma)$ in $H_0$ such that $s(W) \subset U$ with $W = t^{-1}(H_0')$. Thus for the natural homomorphism $\bar{s}: T_W \to s^* T_U$, the restriction of the homomorphism $\bar{s}$ to $t^{-1}(\gamma)$ is the one induced by the canonical isomorphism from $H^0(\boldsymbol{P}^1, f^* T_X)$ to the Zariski tangent space $T_{H_0,\gamma}$ of $H_0$ at $\gamma$. Therefore we see easily that $\bar{s}$ is generically surjective. This implies that the morphism $s: W \to X$ is dominant and separable. The first property of $s$ says that $X$ is uniruled.

Secondly by virtue of [G, Theorem 8.2], there is a variety $V$ and an étale morphism $h: V \to H_0'$ with $\gamma \in h(V)$ which induces a $V$-isomorphism $i: W \times_{H_0'} V$ $(=\bar{W}) \simeq \boldsymbol{P}^1 \times V$. Therefore letting $\bar{h}: \bar{W} \to W$ to be the morphism induced by the morphism $h: V \to H_0'$, we infer that $\bar{h} \cdot s: \boldsymbol{P}^1 \times V \to X$ is separable. Thus we see easily that there exists an $(n-1)$-dimensional closed subvariety $V'$ in $V$ such that the induced map $\boldsymbol{P}^1 \times V' \to X$ is dominant and separable. Therefore $X$ is separably uniruled.

As for the latter statement in Proposition 1.1, we have only to show the following:

Claim. Let $Y$ be a variety and $f: \boldsymbol{P}^1 \times Y \dashrightarrow Z$ a dominant rational map with $\dim Y + 1 = \dim Z$. Assume that $f$ is separable and there is a smooth open subset $Y_0$ in $Y$ and a smooth open set $Z_0$ in $Z$ with $f(\boldsymbol{P}^1 \times Y_0) \subset Z_0$. Namely, let $V \subset \boldsymbol{P}^1 \times Y \times Z$ be the graph of the map $f$ with $s: V \to \boldsymbol{P}^1 \times Y$ the natural projection which is a birational morphism, and $t: V \to Z$ the third projection. Then $t(s^{-1}(\boldsymbol{P}^1 \times Y_0))$ is contained in $Z_0$. Hence there is a projective rational curve $C \subset Z_0$ so that $\varphi^* T_{Z_0}$ is generated by global sections, where $\varphi: \boldsymbol{P}^1 \to C$ is the normalization.

Indeed, shrinking $Y_0$ to an open subset $Y_1$ in $Y$, we see that $(su)^{-1} Y_1 \to Y_1$ is a $\boldsymbol{P}^1$-bundle over $Y_1$ with the natural projection $u: \boldsymbol{P}^1 \times Y \to Y$. Since, $(su)^{-1} Y_1$ $(=V_0)$ is smooth, the morphism $t$ gives a homomorphism $\bar{t}: T_{V_0} \to t^* T_{Z_0}$ between the tangent bundles of $V_0$ and $Z_0$. On the other hand, we can easily check that for every point $y$ in $Y_1$, $T_{V_0|su^{-1}(y)}$ is isomorphic to $\mathcal{O}_{\boldsymbol{P}^1}(2) \oplus (\mathcal{O}_{\boldsymbol{P}^1})^{\oplus a}$ with $a = \dim Y$, and hence is generated by global sections. The assumption that $f$ is separable means that $\bar{t}$ is generically surjective, which implies that for a general point $y$ in $Y_1$, $t^* T_{Z_0|su^{-1}(y)}$ is generated by global sections. Letting $C = t(su^{-1}(y))$ and $\varphi$ the normalization $\boldsymbol{P}^1 \to C$, we see that $\varphi^* T_{Z_0}$ is generated by global sections as required.

Thus we complete the proof of this proposition.

The proof of the latter part in Proposition 1.1 shows:

Corollary 1.2. *A separably uniruled and smooth projective variety has the property* (NC).

**2. Non-uniruledness of a hypersurface.** Let $\{c_j\}_{j\in \boldsymbol{N}}$ be an infinite sequence of elements $c_j$ in $k$ where for each pair $i$ and $j$ $(i\neq j)$, $c_i$ is not equal to $c_j$.

For each $i$ let $F_i$ be a homogeneous polynomial $\prod_{j=1}^{d_i}(Y_i-c_jY_0)$ with a positive integer $d_i$.

Now for each $n$ consider the following homogeneous polynomial equation of degree $d[n]:=\sum_{i=1}^{n-1}(d_i\prod_{j\geq i+1}^{n}e_j)+d_n$

$$Y_0^{s_n}W^{e_1e_2\cdots e_n}=F_1^{e_2e_3\cdots e_n}F_2^{e_3\cdots e_n}\times\cdots\times F_{n-1}^{e_n}F_n\,, \tag{2.1}$$

where $s_1=d_1-e_1$ and for $n\geq 2$, $s_n=d[n]-\prod_{i=1}^{n}e_i$.

From now on we assume that

$$d_i=(p+1)^{a_i+1} \text{ and } e_i=(p+1)^{a_i} \text{ with a positive integer } a_i\geq 2\,. \tag{2.2}$$

Let $W$, $Y_0$, $Y_1$, $\cdots$, $Y_n$ be homogeneous coordinates of $\boldsymbol{P}^{n+1}$ and let us consider a hypersurface $A_n$ in $\boldsymbol{P}^{n+1}$ defined by the equation (2.1) with the property (2.2).

Then we show:

PROPOSITION 2.3. *For each $n$, $A_n$ is not uniruled.*

PROOF. We prove this lemma by induction $n$. When $n=1$, the normal compactification $\bar{A}_1$ of $A_1$ is a branched covering over $\boldsymbol{P}^1$ with at least $d_1$ branch points. Moreover, the covering is separable, because $p$ does not divide $e_1$. Thus Hurwitz's theorem yields an inequality $2g(\bar{A}_1)-2\geq e_1(2\times 0-2)+d_1>0$, which says that $\bar{A}_1$ is of genus $\geq 1$.

Next on the affine open subset $\{Y_0\neq 0\}$ let us consider a rational map $\varphi: \boldsymbol{A}^{n+1}\dashrightarrow \boldsymbol{A}^2$ from the $(n+1)$-dimensional affine space to the 2-dimensional affine space given by sending a general point $(y_1,\cdots,y_n,z)$ in $\boldsymbol{A}^{n+1}$ to the point $(\bar{y}_n,\bar{z})=(y_n, z^{\bar{e}_0}/\prod_{i=1}^{n-1}F_i(1,Y_i)^{\bar{e}_i})$ in $\boldsymbol{A}^2$ where $\bar{e}_i=\prod_{j\geq i+1}^{n-1}e_j$ and $\bar{e}_{n-1}=1$. Then we remark that $\varphi(A_n)$ is contained in the curve defined by the equation $\bar{z}^{e_n}=F_n(1,\bar{y}_n)$, which is not a rational curve. By the induction assumption, the fiber of $\varphi$ is not uniruled. Thus $A_n$ is not uniruled either. q.e.d.

The above proposition yields:

COROLLARY 2.4. *A generic hypersurface of degree $d[n]$ $(\geq n+1)$ in $\boldsymbol{P}^{n+1}$ is not uniruled. In particular if $d[n]$ linear forms $L_1,\cdots,L_{d[n]}$ with respect to variables $Y_0,\cdots,Y_n$ are in general position (which means that each $n+1$ forms $L_{i_1},\cdots,L_{i_{n+1}}$ of $\{L_i\}_{1\leq i\leq d[n]}$ are linearly independent over the field $k$), then the hypersurface defined by the equation $Y_0^{s_n}W^{d[n]-s_n}=\prod_{i=1}^{d[n]}L_i$ is not uniruled.*

PROOF. Since the uniruledness is an open condition, the former part is obvious. As for the latter, consider a hypersurface in $\boldsymbol{P}^{d[n](n+1)}\times\boldsymbol{P}^{n+1}$ defined by the following multi-homogeneous polynomial with indeterminates $A$, $A_j^i$'s, $Y_j$'s, $W$, $V$:

$$A^{d[n]}Y_0^{s_n}W^{d[n]-s_n}=\prod_{i=1}^{d[n]}\left(\sum_{j=0}^{n}A_j^iY_j\right),$$

where $A, A_j^i$ $(1\leq i\leq d[n], 0\leq j\leq n)$ are homogeneous coordinates of $\boldsymbol{P}^{d[n](n+1)}$ and $W, Y_0, \cdots, Y_n$ are those of $\boldsymbol{P}^{n+1}$. Now suppose that the conclusion does not hold. In order to get a contradiction it suffices to show the following:

SUBLEMMA. *Let $G$ and $H$ be projective varieties and $f: G\to H$ a surjective morphism. Assume that a fiber $f^{-1}(h)_{\text{red}}$ is uniruled for a general point in $H$. Then every fiber $f^{-1}(h)$ is uniruled.*

PROOF. By the countability of the Hilbert polynomial, we have only to prove that $f^{-1}(h)$ is covered by rational curves. Now considering the Hilbert scheme of rational curves on $G$, we see that there are a Hilbert polynomial $P$ and a component $J$ of $\text{Hilb}_{P(t)}X$ satisfying the following: When $h: M\to J$ is the universal scheme of $J$, a generic fiber of $h$ corresponds to a rational curve in some fiber of $f$, which implies that the support of every fiber of $h$ consists of rational curves and that the canonical projection $g: M\to G$ is surjective.

Therefore we infer that for every point $x$ in $G$, there is a rational curve $C$ in $X$ and a point $a$ in $J$ so that $C$ is the image of a component of $h^{-1}(a)$ via the morphism $g$ and $C$ passes through the point $x$. Thus by continuity, such rational curves are contained in some fiber of $h$. Thus we complete the proof of the sublemma.

**3. The structure of $\boldsymbol{P}(\boldsymbol{Fr}^{(r)*}(T_{\boldsymbol{P}^n}(-1)))$.** Let

$$X=\left\{(x_0:,\cdots,:x_n)\times(y_0:,\cdots,:y_n)\in\boldsymbol{P}_1^n\times\boldsymbol{P}_2^n\;\middle|\;\sum_{i=0}^{n}x_iy_i=0\right\},$$

where $\boldsymbol{P}_i^n\simeq\boldsymbol{P}^n$ with $i=1,2$ and $a$ (resp. $b$) is the first (resp. second) projection. Let

$$X(q)=\left\{(z_0:,\cdots,:z_n)\times(y_0:,\cdots,:y_n)\in\boldsymbol{P}_3^n\times\boldsymbol{P}_2^n\;\middle|\;\sum_{i=0}^{n}z_i^qy_i=0\right\}$$

with $q=p^r$, where $p$ is the characteristic of the base field $k$ and $\boldsymbol{P}_3^n\simeq\boldsymbol{P}^n$.

REMARK 3.1. (1) $X$ is isomorphic to $\boldsymbol{P}(T_{\boldsymbol{P}^n}(-1))$.

(2) Regarding the above projection $a$ as the canonical projection $\boldsymbol{P}(T_{\boldsymbol{P}^n}(-1))\to\boldsymbol{P}^n$ induced by the natural projection $T_{\boldsymbol{P}^n}(-1)\to\boldsymbol{P}^n$, we can say that the projection $b$ is the morphism induced by the tautological line bundle of $T_{\boldsymbol{P}^n}(-1)$.

(3) $X(q)$ is isomorphic to $\boldsymbol{P}(\boldsymbol{Fr}^{(r)*}(T_{\boldsymbol{P}^n}(-1)))$ where $q=p^r$, $\boldsymbol{Fr}: \boldsymbol{P}_3^n\to\boldsymbol{P}_1^n$ is the Frobenius map and $\boldsymbol{Fr}^{(r)}=\boldsymbol{Fr}\circ\cdots\circ\boldsymbol{Fr}$ ($r$-times). We identify $\boldsymbol{Fr}^{(0)}$ with the identity morphism. Then we have the following diagram:

$$\begin{array}{ccc} P_2^n & & \\ {\scriptstyle b'}\uparrow & \nwarrow{\scriptstyle b} & \\ X(q) & \longrightarrow & X \\ {\scriptstyle a'}\downarrow & & \downarrow{\scriptstyle a} \\ P_3^n & \xrightarrow{Fr^{(r)}} & P_1^n . \end{array}$$

Let $H$ be a hyperplane in $P_1^n$ and $H'$ the reduced part of $(Fr^{(r)})^{-1}(H)$, which is a hyperplane in $P_3^n$ as a set. For a vector bundle $E$ on $P_1^n$, $E^{(q)}$ denotes the pull-back $Fr^{(r)*}(E)$ of $E$ via the morphism $Fr^{(r)}$ where $q=p^r$. Then we get:

PROPOSITION 3.2. *In the above notation, we have a splitting on the hyperplane $H$*:

(1) (#)
$$T_{P^n}(-1)_{|H}=T_{P^{n-1}}(-1)\oplus \mathcal{O}_{P^{n-1}} .$$

*Consequently, we get a splitting on $H'$*

(##)
$$(T_{P^n}(-1))^{(q)}_{|H'}=(T_{P^{n-1}}(-1))^{(q)}\oplus \mathcal{O}_{P^{n-1}} .$$

*Let the quotient trivial line bundle of* (#) *and* (##) *correspond to some fiber $Z$ of $b$ and the reduced part $Z'$ of a fiber of $b'$, respectively. Then $Z$ as well as $Z'$ are isomorphic to $P^{n-1}$.*

(2) *The normal bundle $N_{Z'/\bar{H}}$ is $(\Omega_{P^{n-1}}(1))^{(q)}$ with $\bar{H}=(a')^{-1}(H')$.*

PROOF. (1) is trivial. Remark 3.1.3 and (1) yield (2).

Now for the inclusions $Z'\subset\bar{H}\subset X(q)$ of subvarieties, we have an exact sequence of normal bundles

$$0\to N_{Z'/\bar{H}}\to N_{Z'/X(q)}\to N_{\bar{H}/X(q)|Z'}\to 0 . \tag{3.3}$$

In this sequence, $(N_{\bar{H}/X(q)})_{|Z'}$ is isomorphic to $\mathcal{O}_{P^{n-1}}(1)$.

Let $L$ be a line in $Z'$ and let us calculate the intersection number $(K_{X(q)}\cdot L)$. The inclusion $Z'\subset X(q)$ yields an exact sequence

$$0\to T_{Z'}\to(T_{X(q)})_{|Z'}\to N_{Z'/X(q)}\to 0 . \tag{3.4}$$

Thus we see that $-(K_{X(q)}\cdot L)=\deg(T_{X(q)})_{|L}=\deg(T_{Z'|L})+\deg(N_{Z'/X(q)})_{|L}$. Note that $(T_{Z'})_{|L}\simeq\mathcal{O}_{P^1}(2)\oplus(\mathcal{O}_{P^1}(1))^{\oplus n-2}$.

Moreover by Proposition 3.2, (2),

$$(N_{Z'/\bar{H}})_{|L}\simeq\Omega_{P^{n-1}}(1)^{(q)}_{|L}\simeq(\Omega_{P^{n-1}}(1)_{|L})^{(q)}\simeq(\mathcal{O}_{P^1})^{\oplus(n-2)}\oplus\mathcal{O}_{P^1}(-q) .$$

Hence we infer that $-(K_{X(q)}\cdot L)=n-q+1$ by (3.3).

Hence we have:

PROPOSITION 3.5. *In $X(q)$ let $L$ be a line on a fiber of $b'$ ($\simeq P^{n-1}$ as a set). Then*

$-(K_{X(q)}\cdot L)=n-p^r+1$.

Let $C$ be a rational curve in $Z'$ and $\varphi: P^1 \to C$ ($\subset Z'$) the normalization of $C$. Then by (3.4) we have an exact sequence

$$(3.6) \qquad 0 \to \varphi^* T_{Z'} \to \varphi^* T_{X(q)} \to \varphi^* N_{Z'/X(q)} \to 0 .$$

PROPOSITION 3.7. *Let a morphism $\varphi: P^1 \to C$ be as in* (3.6). *Assume that $\varphi^* T_{X(q)}$ is generated by global sections. Then $q=1$, that is, the characteristic of the base field is zero.*

PROOF. By assumption and the exact sequence (3.6) we have a generic isomorphism on $P^1$

$$(\sharp) \qquad (\mathcal{O}_{P^1})^{\oplus n} \to \varphi^* N_{Z'/X(q)} \quad (=N) .$$

By taking the $n$-th exterior product of ($\sharp$), we get an injective homomorphism $\mathcal{O}_{P^1} \to \bigwedge^n N$. On the other hand, since $\bigwedge^{n-1}(\Omega_{P^{n-1}}(1))^{(q)} = \mathcal{O}_{P^{n-1}}(-q)$ and therefore $\bigwedge^n N_{Z'/X(q)} = \mathcal{O}_{P^{n-1}}(1-q)$ by (3.3), we see that $\bigwedge^n N = \mathcal{O}_{P^1}((1-q)\deg C)$, which yields $q=1$. q.e.d.

**4. The desingularization of some divisor in $P(T_{P^n})$.** Let $\mathscr{P} := P^n \times P^n$, where $(u_0, \dots, u_n)$ is the homogeneous coordinate of the first $P^n$, while $(v_0, \dots, v_n)$ is the homogeneous coordinate of the second $P^n$. Let

$$X(p) := \left\{ (u_0, \dots, u_n) \times (v_0, \dots, v_n) \in \mathscr{P} \,\middle|\, \sum_{j=0}^{n} u_j v_j^p = 0 \right\}$$

as stated in §3 and let $\{H_i\}_{0 \le i \le m}$ be a collection of hyperplanes in the first $P^n$ with $m \ge n-1$.

Moreover, let us consider the following condition:

(4.0) For all $n$-tuples $i_0, i_1, \dots, i_{n-1}$ of elements in $\{0, 1, \dots, m\}$, $\dim \bigcap_{j=0}^{n-1} H_{J_i} = 0$ holds in the first $P^n$.

Let $D_i := d^{-1}(H_i) \cap X(p)$, where $d: \mathscr{P} \to P^n$ is the first projection.

Then our aim in this section is to study the following:

(4.1) Under the condition (4.0), what kind of modification $\sigma: \bar{\mathscr{P}} \to \mathscr{P}$ should one take for the reduced structure of the closed subscheme $\sigma^{-1}[X(p)] \cap \sigma^{-1}(\bigcup_{i=0}^{m} D_i)$ to have only normal crossing in the proper transform $\sigma^{-1}[X(p)]$ of $X(p)$ via $\sigma$?

For the study of the above problem, it suffices to consider the case $m=n-1$. By a suitable linear transformation of the base space we can express each $H_{i_j}$ of the above $n$ hyperplanes as $\{u_j=0\}$.

Hence without loss of generality, we assume that $D_i = \{u_i = 0\} \cap X(p)$ for $i = 0, \dots, n-1$.

(4.2) In this section we fix the coordinates $u_i$'s and $v_j$'s used above. $U_{ij}$ denotes the affine open subset $\{u_i v_j \ne 0\}$ in $\mathscr{P}$ which is isomorphic to $A^{2n}$. Thus $\mathscr{P}$ is covered with affine open set $U_{ij}$ which are affine spaces.

REMARK 4.2.1. The canonical coordinate of the above affine space is determined by the original homogeneous coordinates of $\mathscr{P}$.

(4.2.2) As for the canonical coordinate of $U_{ij}$, we should, strictly speaking, use $u_\alpha/u_i$ and $v_\alpha/v_j$ ($\alpha=0, 1, \dots, \check{i}, \dots, n$) or denoted simply $\bar{u}_{\alpha i}$ and $\bar{v}_{\alpha j}$, respectively. As the notation for the coordinates for $U_{ij}$ in this section, however, we adopt the same notation $u_\alpha$'s and $v_\alpha$'s without the suffices $i$ and $j$, since no confusion would arise.

(4.3) We introduce a terminology: Let $A^s$ be an $s$-dimensional affine space with coordinates $X_1, \dots, X_s$. When $M$ is a closed subscheme defined by a part of the coordinates, i.e. $X_{i_1}, \dots, X_{i_r}$, $M$ is said to be of type $L$ "with respect to the coordinate system $X_1, \dots, X_s$". Usually, we omit the part in quotation.

(4.3.1) We remark that the variety obtained by blowing up $A^s$ along the above $M$ is covered by finitely many affine spaces obtained canonically.

Let us return to $\mathscr{P}$.

(4.4) Let $S$ be a smooth, closed subscheme in $\mathscr{P}$ and $B=\{0, \dots, n\}$.

Assume that for any $i, j \in B$, $S \cap U_{ij}$ is of type $L$ with respect to the canonical coordinate (Remark 4.2). Then take the blowing-up $\sigma: \mathscr{P}^1 \to \mathscr{P}$ of $\mathscr{P}$ along $S$. We see easily that

(4.4.1) $\mathscr{P}^1$ is covered by the affine open sets which are affine spaces in a canonical way and the coordinate system for each affine space induced by the coordinates of $\mathscr{P}$ is the natural one for the affine space (see Remarks 4.2 and 4.3.1).

Moreover, given a smooth subscheme $S^1$ of $\mathscr{P}^1$ so that on each affine open set (which is an affine space) the restriction of $S^1$ is of type $L$, we make the same modification of $\mathscr{P}^1$ along $S^1$ to get $\mathscr{P}^2$. Then we infer that $\mathscr{P}^2$ has the same property as above (4.4.1). We then repeat the same procedure.

(4.4.2) Thus the word "the affine open set" in this section is used in the sense of the restricted affine spaces by which each of the ambient spaces $\mathscr{P}, \mathscr{P}^1, \dots, \mathscr{P}^i$ defined below is covered. As in 4.2.1, we do not explicitly describe the difference between the coordinates of affine open sets in $\mathscr{P}^i$ and those in $\mathscr{P}^{i+1}$ obtained by blowing up $\mathscr{P}^i$ along $S^{i-1}$ if there is no fear of confusion.

Before considering blowing-ups of $\mathscr{P}$ we first study those locally. Let us begin with the following:

(4.5) Let $A^{2m}$ be a $2m$-dimensional affine space with the canonical coordinates $u_1, \dots, u_m, v_1, \dots, v_m$. For $\sharp=\alpha, \beta, \gamma$ let $V_\sharp$ be the closed subvariety in $A^{2m}$ defined by polynomials $u_{i_1}, \dots, u_{i_s}$ ($s \le m-1$) and $F_\sharp$, where $1<i_1<\cdots<i_s$, $a \le m$ and $F_\sharp$ is as follows:

($\alpha$) $F_\alpha := 1 + \sum_{i \ge 2}^{a} v_i^p u_i$.

($\beta$) $F_\beta := v_1^p + u_2 + \sum_{i \ge 3}^{a} v_i^p u_i$.

($\gamma$) $F_\gamma := v_1^p + \sum_{i\geq 2}^{a} v_i^p u_i$.

Here $V_\sharp$ for $\sharp = \alpha, \beta, \gamma$ is said to be of type $\sharp$.

Let $J = \{i_1, \dots, i_s\}$. For a scheme $W$, $SW$ means the singular locus of $W$. Then we have:

LEMMA 4.5.1. *In the situation as above, assume that $i_1 \geq 2$. Then we have:*

(1) *$V_\alpha$ is smooth.*

(2) *If $i_1 > 2$, then $V_\beta$ is smooth. If $i_1 = 2$, then $V_\beta$ is singular and the defining ideal of $SV_\beta$ is generated by the $u_i$'s with $i \in J$ and by the $v_j$'s with $j \in \{2, 3, 4, \dots, a\} - J$.*

(3) *$V_\gamma$ is singular and $SV_\gamma$ is defined by the same ideal as in* (2).

From now on, we restrict ourselves to the case $m = n$, and consider the $2n$-dimensional affine space $A(\check{u}_a, \check{v}_b)$ with the canonical coordinate $u_i$'s and $v_j$'s. Here $i$ runs through $0 \leq i \leq n$ with $i \neq a$ and $j$ through $0 \leq j \leq n$ with $j \neq b$.

Let $r$ be a non-negative integer. Then we let

$$F_\alpha(r) := 1 + \sum_{i\geq r, i\neq j} v_i^p u_i \qquad \text{for } \ j\ (r \leq j \leq n)\,,$$

$$F_\beta(r) := v_j^p + u_h + \sum_{i \geq r \neq j,h} v_i^p u_i \qquad \text{for } \ h, j\ (r \leq h, j \leq n)\,.$$

For $\sharp = \alpha$ or $\beta$, $D_\sharp(0, \dots, r-1)$ denotes the subvariety in the $2n$-dimensional affine space $A(\check{u}_j, \check{v}_h)$ defined by $u_0, u_1, \dots, u_{r-1}$ and $F_\sharp(r)$, where $F_\sharp(r)$ is $F_\alpha(r)$ if $j = h$ and $F_\beta(r)$ if $j \neq h$. Here if $r = 0$, $D_\sharp(0, \dots, r-1)$ is defined by $F_\sharp(0)$. Note that $D_\sharp(0, \dots, r-1)$ is smooth and $(2n-r-1)$-dimensional.

(4.5.2) For $k \geq r$, let $\bar{D}_\sharp(k) := D_\sharp(0, \dots, r-1) \cap \{u_k = 0\}$.

(1) $D_\alpha(k)$ is of type $\alpha$.

(2) If $k \neq h$, $\bar{D}_\beta(k)$ is of type $\beta$. If $k = h$, $\bar{D}_\beta(k)$ is of type $\gamma$.

Thus in view of the suffices of the coordinates, we easily get:

PROPOSITION 4.6. *Let the notation be as above. Then we have:*

(1) *$\bar{D}_\alpha(k)$ is smooth (in $A(\check{u}_j, \check{v}_j)$) for any $k$ ($\geq r$).*

(2) *$\bar{D}_\beta(h)$ is singular in $A(\check{u}_j, \check{v}_h)$ and the ideal of $S\bar{D}_\beta(h)$ is generated by $n+1$ monomials $u_\alpha$'s $(\alpha = 0, 1, \dots, r-1, h)$ and $v_\beta$'s $(\beta = r, r+1, \dots, h-1, h+1, \dots, n)$. Therefore $S\bar{D}_\beta(h)$ is smooth and of dimension $n-1$.*

(3) *$\bar{D}_\beta(k)$ is smooth in $A(\check{u}_j, \check{v}_h)$ for $k \neq h$ and $k \geq r$. Hence $S\bar{D}_\beta(k) \cap S\bar{D}_\beta(\bar{k})$ is empty in $A(\check{u}_j, \check{v}_h)$ for $k, \bar{k} \geq r$ ($\neq h$).*

(4.7) Let us fix the affine space $A(\check{u}_j, \check{v}_h)$ for $r \leq h$, $j \leq n$ and $h \neq j$ and we let $\sigma : \bar{A} \to A(\check{u}_j, \check{v}_h)$ be the blowing-up along singular locus $S\bar{D}_\beta(h)$ in Proposition 4.6.

$\bar{A}$ is covered by $n+1$ affine open sets $A_a = \operatorname{Spec} R_a$ where $a$ runs through the set $\{0, 1, \dots, n\}$. Here each $R_a$ is a polynomial ring over $k$ with $2n$ variables:

$$k[w_a, u_0/w_a, \ldots, u_{r-1}/w_a, u_r, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{h-1}, u_h/w_a, u_{h+1}, \ldots, u_n,$$
$$v_0, \ldots, v_{r-1}, v_r/w_a, \ldots, v_{h-1}/w_a, v_{h+1}/w_a, \ldots, v_n/w_a]$$

where

$$w_a = \begin{cases} u_a & a \in \{0, 1, \ldots, r-1, h\}, \\ v_a & a \in \{r, r+1, \ldots, h-1, h+1, \ldots, n\}. \end{cases}$$

(4.8) We study the total transform $\sigma^{-1}(\bar{D}_\beta(h))$ of $\bar{D}_\beta(h)$ via $\sigma$ whose ideal $J$ in $R_a$ is $(u_0, u_1, \ldots, u_{r-1}, u_h, F_\beta(r))R_a$.

When $0 \leq a \leq r-1$ or $a = h$, $J$ is

$$((u_0/u_a)u_a, \ldots, u_a, \ldots, (u_h/u_a)u_a, F_\beta(r))R_a = (u_a, F_\beta(r))R_a \text{ in } A(\check{u}_j, \check{v}_h).$$

When $a \geq r$ and $a \neq h$,

$$F_\beta(r) - u_h = v_a^p\left((v_j/v_a)^p + \sum_{i \geq r \neq h,j} (v_i/v_a)^p u_i\right)$$

and therefore

$$\begin{aligned} (4.8.0) \qquad J &= (u_0, u_1, \ldots, u_{r-1}, u_h, F_\beta(r))R_a \\ &= \left(u_0, u_1, \ldots, u_{r-1}, u_h, \left(v_j^p + \sum_{i \geq r \neq j,h} v_i^p u_i\right)\right)R_a \\ &= ((u_0/v_a)v_a, \ldots, (u_{r-1}/v_a)v_a, (u_h/v_a)v_a, Gv_a^p)R_a \\ &= v_a R_a \cap ((u_0/v_a), \ldots, (u_h/v_a), G)R_a \cap ((u_0/v_a), \ldots, (u_h/v_a), v_a^p)R_a \end{aligned}$$

in $R_a$ (written as $A(\check{u}_j, \check{v}_h)$ according to Remarks 4.4.1 and 4.4.2) where $G = (v_j/v_a)^p + \sum_{i \geq r \neq h,j} (v_i/v_a)^p u_i$ with $i \leq j \leq n$.

Thus we see that $\sigma^{-1}[\bar{D}_\beta(h)] \cap A_a$ is empty if $0 \leq a \leq r-1$ or $a = h$, and it is defined by the ideal $(u_0, u_1, \ldots, u_{r-1}, u_h, \bar{G})$ if $r \leq a \leq n$ $a \neq h$, where

$$\bar{G} = \begin{cases} 1 + \sum_{i \geq r \neq h,a} v_i^p u_i & \text{with } i \leq j \leq n \quad \text{if } a = j \\ v_j^p + u_a + \sum_{i \geq r \neq h,j,a} v_i^p u_i & \text{with } i \leq j \leq n \quad \text{if } a \neq j. \end{cases}$$

(Note that $A_a = A(\check{u}_j, \check{v}_h)$).

Summarizing the above results, we have:

PROPOSITION 4.8.1. *Let $\sigma: \bar{A} \to A(\check{u}_j, \check{v}_h)$ be the blowing-up along the singular locus $S\bar{D}_\beta(h)$. Then $\bar{A}$ is covered by $n+1$ affine open sets $A_a$. The proper transform $D(0, \ldots, r-1, h)$ $(h \geq r)$ of $\bar{D}_\beta(h)$ is a $(2n-r-2)$-dimensional smooth subvariety in $\bar{A}$. More precisely, it is defined, on each $A_a$, by a unit, or polynomials $u_0, u_1, \ldots, u_{r-1}, u_h$*

*and* $F_\sharp(r)$ *for* $\sharp=\alpha$ *or* $\beta$. *Finally the reduced structure of the total transform of* $\bar{D}_\beta(h)$ *via* $\sigma$ *is set-theoretically a union of* $D(0,\dots,r-1,h)$ *and a smooth exceptional divisor* $E$ *defined by* $v_a$, *and* $E$ *intersects* $D(0,\dots,r-1,h)$ *transversally.*

Pasting together the local description obtained so far, we now globally describe the blowing-up.

(4.9) We can inductively define the blowing-up $\sigma_i:\mathscr{P}^{i+1}\to\mathscr{P}^i$ of $\mathscr{P}^i$ along the subscheme $S^i$ of $\mathscr{P}^i$ as follows:

When $i=0$, we let $\mathscr{P}^0=\mathscr{P}$ and $S^0:=\bigcup_{i=0}^{n-1}SD_i$. Now let us assume that the blowing-up $\sigma_a:\mathscr{P}^{a+1}\to\mathscr{P}^a$ is defined for $a=0,\dots,i-1$ $(i\geq 1)$. Denote $\sigma(a-1):=\sigma_{a-1}\sigma_{i-2}\dots\sigma_0:\mathscr{P}^a\to\mathscr{P}^0$. For $0\leq k_0<\cdots<k_i\leq n-1$, $D(k_0,\dots,k_i):=\bigcap_{j=0}^{i}\sigma(i-1)^{-1}[D_{k_j}]$ is a closed subscheme of $\mathscr{P}^i$ of dimension $(2n-2-i)$, where $D(k)=D_k$ and $S^i$ denotes the disjoint union of $SD(k_0,\dots,k_i)$ (see (2) and (3) in Proposition 4.6) for $0\leq k_0<\cdots<k_i\leq n$. Then $\sigma_i:\mathscr{P}^{i+1}\to\mathscr{P}^i$ of $\mathscr{P}^i$ is defined as the blowing-up of $\mathscr{P}^i$ along the subscheme $S^i$ of $\mathscr{P}^i$. Moreover, we set $X^i:=\sigma(i-1)^{-1}[X(p)]$ for $i\geq 1$ and $X^0:=X(p)$.

Then we have:

(1) $D(k_0,\dots,k_i)$ is a singular subvariety in $X^i$. $SD(k_0,\dots,k_i)$ is an $(n-1)$-dimensional smooth subvariety and the defining ideal for its restriction on each of the affine open sets in $\mathscr{P}^i$ is generated by $u_{k_0},\dots,u_{k_i},v_{k_{i+1}},\dots,v_{k_n}$ if it is non-empty.

(2) $X^i$ is a smooth irreducible divisor in $\mathscr{P}^i$.

(3) $\sigma(i-1)^{-1}[D_k]$ is a smooth divisor in $X^i$ and the divisors $\sigma(i-1)^{-1}[D_{k_j}]$ $(j=0,1,\dots,i-1)$ intersect transversally.

If $i=n-1$, then $D(0,\dots,n-1)$ is of dimension $n-1$ and is defined by the ideal $(u_0,\dots,u_{n-1},v_n^p)$ in $A(\check{u}_n,\check{v}_{n-1})$. Thus it is non-reduced and therefore the support of $D(0,\dots,n-1)$ coincides with that of $SD(0,\dots,n-1)$.

Finally we take the blowing-up $\sigma_{n-1}:\mathscr{P}^n\to\mathscr{P}^{n-1}$ of $\mathscr{P}^{n-1}$ along the reduced part of $SD(0,\dots,n-1)$.

Thus letting $\bar{\sigma}:X^n\to X(p)$ the restriction of $\sigma(n-1):\mathscr{P}^n\to\mathscr{P}$ to $X^n$, we have:

PROPOSITION 4.10. *Let the situation be as in* (4.0). *Assume that* $m=n$. *Then for each* $i$, *we have the following*:

(1) $X^n$ *is a smooth irreducible divisor in* $\mathscr{P}^n$.

(2) $(\bar{\sigma})^{-1}[D_i]$ *is a smooth irreducible divisor in* $X^n$.

(3) $(\bar{\sigma})^{-1}[D_i]\cap(\bar{\sigma})^{-1}[D_j]$ *is empty for any pair* $i$ *and* $j$.

*Finally* $(\bar{\sigma})^{-1}(\bigcup_{i=0}^{n}D_i)$ *is set-theoretically a normal crossing in* $X^n$.

If we begin the above argument in (4.9) with divisors $D_1,\dots,D_m$ of (4.0) instead of the above divisors $D_i$'s, we can naturally get a blowing-up $\sigma_i:\mathscr{P}^{i+1}\to\mathscr{P}^i$ $(i=0,\dots,n-1)$ in the same way as in 4.9. Therefore we immediately obtain an answer to Problem 4.1.

COROLLARY 4.11. *Let the situation be as in* (4.0), $\sigma_i:\mathscr{P}^{i+1}\to\mathscr{P}^i$ *the blowing-up* $(i=0,\dots,n-1)$ *in the same manner as in* (4.9) *and* $\sigma(n-1):=\sigma_{n-1}\sigma_{n-2}\cdots\sigma_0:\mathscr{P}^n\to\mathscr{P}$.

*Then the proper transform* $X^n:=\sigma(n-1)^{-1}[X(p)]$ *of* $X(p)$ *via* $\sigma(n-1)$ *is a smooth irreducible divisor in* $\mathscr{P}^n$. *Moreover, let* $\bar{\sigma}: X^n \to X(p)$ *be the restriction of* $\sigma(n-1): \mathscr{P}^n \to \mathscr{P}$ *to* $X^n$. *Then* $(\bar{\sigma})^{-1}(\bigcup_{i=0}^{m} D_i)$ *is set-theoretically a normal crossing in* $X^n$.

*Furthermore, if* $X(p)$ *in the above condition is replaced by* $X(q)$ *with* $q=p^r$, *the same conclusion as above holds.*

**5. The proof of the Theorem.** We construct a smooth projective variety as follows:

(1) Take $d[n]$ which is divisible by $d[n]-s_n$ $(=(p+1)^a)$, choose $d[n]$ linear forms $L_1, \ldots, L_{d[n]}$ with respect to the variables $Y_0, \ldots, Y_n$ which are in general position as stated in Corollary 2.4 and consider a hypersurface $S$ of degree $d[n]$ in $\boldsymbol{P}^n$ defined by the equation $\prod_{i=1}^{d[n]} L_i=0$.

(2) Take a modification $\bar{\sigma}: \bar{X}^n \to X(q)$ to make $(b')^{-1}(S)$ of $X(q)$ having normal crossing as in Corollary 4.11.

(3) Let $M:=(\bar{\sigma}b')^*\mathcal{O}_{\boldsymbol{P}^n}(S)$ be a line bundle on $\bar{X}^n$, which yields a canonical effective divisor $B=\bigcup a_i D_i$, where each $D_i$ is a component of the set-theoretic pull-back of effective divisors $(\bar{\sigma}b')^{-1}(S)$ via $\bar{\sigma}b'$ and $a_i$ is a positive integer. Note that $\bigcup D_i$ has normal crossing.

(4) Take the branched cyclic covering $\theta: X' \to \bar{X}^n$ of degree $d[n]-s_n$ along the locus $B$. Then $X'$ is canonically contained in $V(\check{M})$ as a divisor and let $\pi: \tilde{X} \to X'$ be the normalization of $X'$.

Then we have:

CLAIM. $\tilde{X}$ is a toroidal embedding without self-intersection.

Let $U^*=X(q)-(b')^{-1}(S)$ and $U=(\pi\theta)^{-1}(U^*)$. As shown in § 4 $\bar{X}^n$ is a hypersurface in the smooth projective variety $\tilde{P}$ which is obtained by blowing up $\boldsymbol{P}^n \times \boldsymbol{P}^n$ along smooth subschemes succesively and is covered by affine open sets $\boldsymbol{A}_\lambda$ each of which is isomorphic to the $2n$-dimensional affine space. Moreover note that $\bar{X}^n$ is covered by Zariski open subsets $V_\lambda$ which are closed smooth hypersurfaces in $\boldsymbol{A}_\lambda$ respectively. Now in view of (4.9) and (4.10), the defining equation of $\theta^{-1}(V_\lambda)$ is locally as follows: letting $w, x_1, x_2, \ldots, x_{2n}$ be a local coordinate of $\boldsymbol{A}^{2n+1}$,

$$f:=w^{(p+1)^a}-x_1^{m_1}x_2^{m_2}\cdot\cdots\cdot x_s^{m_s}=0$$

where $1 \leq s \leq 2n$ and $m_i$ is a non-negative integer. Furthermore we see that for each $\lambda$, there exists $i$ so that $m_i$ is $l$ or $p$ by (4.8.0), which implies that $f$ is irreducible for each $\lambda$. Since $\pi: \theta^{-1}(U^*) \to \bar{X}^n$ is étale, $\theta^{-1}(U^*)$ is a smooth open set in $X'$, hence equals $U$. Let $\tilde{V}:=(\pi\theta)^{-1}(V)$. To show that $U \subset \tilde{X}$ is a toroidal embedding without self-intersection, it suffices to show that

(1) $(U \cap \tilde{V}) \subset \tilde{V}$ is a torus embedding.

(2) $\tilde{V}-(\tilde{V} \cap U)$ is a union of normal irreducible divisors in $\tilde{V}$.

Let $Y$ and $Y'$ be hypersurfaces in $(k^\times)^{2n}$ and $\boldsymbol{A}^{2n}$ defined by the above polynomial

$f$, respectively. Then $Y$ is an open set in $Y'$. Moreover, $Y$ is an algebraic group and $Y'$ is invariant under the coordinatewise multiplication of the elements of $Y$. Since $f$ is irreducible, $Y'$ contains the open set $Y$ which is canonically a $(2n-1)$-dimensional algebraic torus $T^{2n-1}$. Thus $T^{2n-1}$ acts canonically on the normalization $\tilde{Y}$ of $Y'$. Hence we see that $Y\subset\tilde{Y}$ is a torus embedding and $\tilde{Y}-Y$ is a union of normal divisors in $\tilde{Y}$ by virtue of the theory of torus embedding. Therefore we can show that $U\subset\tilde{X}$ is a toroidal embedding without self-intersection. Finally by virtue of Theorem 11* in [KKMS] we get a projective birational morphism $\gamma: X\to\tilde{X}$ from a smooth complete variety $X$ to the projective variety $\tilde{X}$, where the induced morphism $\gamma:\gamma^{-1}(U)\to U$ is an isomorphism. Thus we have constructed a smooth projective variety.

In the sequel we will show that this $X$ is a variety which we want.

Consider a non-uniruled hypersurface $Y$ of degree $d[n]$ in $\boldsymbol{P}^{n+1}$ defined by the equation $Y_0^{s_n}W^{d[n]-s_n}=\prod_{i=1}^{d[n]}L_i$ as shown in Corollary 2.4 and a dominant rational map $g: Y\dashrightarrow\boldsymbol{P}^n$ obtained by the projection $(W: Y_0:,\dots,:Y_n)\mapsto(Y_0:,\dots,:Y_n)$. The rational map $g$ is separable, generically finite and defined except at the point $(1:0:,\dots,:0)$. Let $Y^0:=Y-(1:0:,\dots,:0)$, $g_0=g_{|Y^0}$ and let $\bar{g}: X(q)\times_{\boldsymbol{P}^n}Y^0\to X(q)$ be the canonical morphism induced by $g_0$. Then there is a canonical birational map $f: X\dashrightarrow X(q)\times_{\boldsymbol{P}_2^n}Y^0$.

Thus we have the following diagram:

$$\begin{array}{ccc}
Y^0 & \xrightarrow{g_0} & \boldsymbol{P}_2^n \\
\bar{b}\downarrow & & \downarrow b' \\
X(q)\times_{\boldsymbol{P}_n}Y^0 & \xrightarrow{\bar{g}} & X(q) \\
f\uparrow & & \uparrow\bar{\sigma} \\
X & \xrightarrow{\gamma\pi\theta} & \bar{X}
\end{array}$$

Letting $\bar{S}$ to be the total transform $\bar{\sigma}^{-1}((b')^{-1}(S))$ of $(b')^{-1}(S)$ via $\bar{\sigma}$, we note that $X-(\gamma\pi\theta)^{-1}(\bar{S})$ is canonically contained in both $X$ and $X(q)\times_{\boldsymbol{P}^n}Y^0$ under the birational map $f$. Hence we infer that a general rational curve on $X$ is contained in some fiber of $\bar{b}$ by the non-uniruledness of $Y$.

Now since $X$ is uniruled, there exist a variety $T$ with $\dim T+1=\dim X$ and a dominant rational map $\phi:\boldsymbol{P}^1\times T\dashrightarrow X$.

In order to complete the proof of the theorem, it suffices to show the following two steps.

STEP 1. $q=1$, *if $X$ is separably uniruled.*

PROOF. By taking a small smooth open set $\bar{Y}$ in $g_0^{-1}(\boldsymbol{P}_2^n-S)$, we see that the restriction of the morphism $g_0:\bar{Y}\to\boldsymbol{P}_2^n$ to $\bar{Y}$ is an étale morphism and $X_0:=X(q)\times_{\boldsymbol{P}^n}\bar{Y}$ is a smooth open subscheme in $X(q)\times_{\boldsymbol{P}^n}Y^0$ which is canonically contained in $\bar{X}$.

Thus there is a smooth open subset $T_1$ in $T$ where $\phi$ is defined as a morphism on $\boldsymbol{P}^1 \times T_1$ and $\phi(\boldsymbol{P}^1 \times T_1)$ is contained in $X_0$. Since $\phi$ is separable, the claim in Proposition 1.1 asserts that there exist a point $y$ in $\bar{Y}$ and a rational curve $C \subset X_0$ in the fiber of $(\bar{b})^{-1}(y)$ such that $\varphi^* T_{X_0}$ is generated by global sections with the normalization $\varphi: \boldsymbol{P}^1 \to C$. On the other hand, noting that $\bar{g}$ is étale on $X_0$ we have a natural isomorphism $T_{X_0} \simeq \bar{g}^*(T_{X(q)})_{|X_0}$. Consequently $\varphi^* T_{X_0} \simeq \bar{\varphi}^* T_{X(q)}$, where $\bar{\varphi}: \boldsymbol{P}^1 \to \bar{g}(C)$ is the normalization of $\bar{g}(C)$ induced by $\varphi$. Thus we infer that $\bar{\varphi}^* T_{X(q)}$ is generated by global sections, and therefore $q=1$ by Proposition 3.7.

STEP 2. *Let $L$ be a line in the fiber $\bar{b}^{-1}(y)$ for $y \in \bar{Y}$. Then we have* $-(K_{\bar{X}} \cdot L) = n - p^r + 1$.

PROOF. Since $\bar{g}_{|X_0}: X_0 \to X(q)$ is étale and $L \subset \bar{b}^{-1}(\bar{Y})$, we get $(K_X \cdot L) = \deg K_{X_0|L} = (\bar{g}^* K_{X(q)} \cdot L) = (K_{X(q)} \cdot \bar{g}(L)) \deg \bar{g}_{|L}$. Then note that $\bar{g}_{|L}$ is an isomorphism and $\bar{g}(L)$ is a line in $(b')^{-1}(g_0(y))$. Thus we infer that $-(K_X \cdot L) = n - p^r + 1$ by Proposition 3.5.

q.e.d.

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