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EXPLICIT REPRESENTATION OF COMPACT CONFORMALLY FLAT HYPERSURFACES

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Abstract. We explicitly construct compact conformally flat hypersurfaces in a simply connected, $(n+1)$-dimensional space form, where n is greater than 3. We may assume that the ambient space is the standard $(n+1)$-sphere by a conformal diffeomorphism of a simply connected space form into the sphere. From this viewpoint we give a global parameterization of compact conformally flat hypersurfaces, and we establish relation between two types of hypersurfaces; one has umbilic points and the other has not. It is known that each compact conformally flat hypersurface in a space form is conformally equivalent to a classical Schottky manifold. In order to determine the conformal types of our hypersurfaces, we explicitly represent conformal diffeomorphism of these hypersurfaces to corresponding Schottky manifolds. In particular, we clarify the relation between our results and Pinkall's results.

1. Introduction. In this paper we explicitly construct compact conformally flat hypersurfaces in a simply connected, $(n+1)$-dimensional space form with $n\geq 4$. Furthermore, we give explicit conformal correspondences between these hypersurfaces and Schottky manifolds. We study mainly a special but basic class of such hypersurfaces, namely, conformally flat hypersurfaces diffeomorphic to the torus $S^{n-1}\times S^1$, where S^m is the m-dimensional standard sphere in Euclidean space $\boldsymbol{R}^{m+1}$.

Compact conformally flat hypersurfaces in a simply connected space form can be assumed to be the ones in $\boldsymbol{R}^{n+1}$ by appropriate conformal transformations. Diffeomorphism types of such hypersurfaces was studied by do Carmo, Dajczer and Mercuri (cf. [2]). Let $S^{n-1}\times_n S^1$ denote the n-dimensional Klein bottle, where n stands for an orientation reversing isometry of S^{n-1}. Their result may be stated as follows:

THEOREM A. *Let (M^n, g) be a compact conformally flat manifold with $n\geq 4$. Then M can be immersed as a conformally flat hypersurface $\Phi: M\to\boldsymbol{R}^{n+1}$ if and only if M is diffeomorphic to one of the following manifolds*:

(1) *The standard sphere S^n.*

For some $k\geq 1$

(2) *the connected sum of k copies of $S^{n-1}\times S^1$ if M is orientable,*

(3) *the connected sum of $(k-1)$ copies of $S^{n-1}\times S^1$ and $S^{n-1}\times_n S^1$ if M is non-orientable.*

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On the other hand, the intrinsic conformal geometry of compact conformally flat hypersurfaces was studied by Pinkall (cf. [6]):

THEOREM B. *Every compact conformally flat hypersurface in* $\boldsymbol{R}^{n+1}$ *with* $n \geq 4$ *is conformally equivalent to the standard sphere or a classical Schottky manifold.*

Which classical Schottky manifolds really admit a conformal immersion as a hypersurface in $\boldsymbol{R}^{n+1}$? In the same paper, Pinkall also gave an implicit representation of a class of conformally flat tori without umbilic points in $\boldsymbol{R}^{n+1}$, and conformal correspondences of them to Schottky manifolds. His method is very interesting, but the hypersurfaces which he constructed have singularities in general.

In this paper, we take up the viewpoint that hypersurfaces are conformally immersed in S^{n+1} rather than in $\boldsymbol{R}^{n+1}$. In this situation we can simply represent a class of conformally flat tori without umbilic points in S^{n+1} by parameters on $S^{n-1} \times S^1$. Furthermore, we can give more clearly the correspondences between the hypersurfaces and Schottky manifolds.

We denote by $S^{n\pm 1}$ and S^n the unit spheres with the origin as center. We denote by $[0, l]/\sim$ the circle obtained by the identification with 0 and l. Let $q_1(t)$, $t \in [0, l]/\sim$, be a smooth loop in S^{n+1} with the arc-length parameter t. Let $q_2(t)$ and $(\nabla q_2)(t)$ be the velocity vector field and the acceleration vector field on S^{n+1} of $q_1(t)$, respectively. Set $\|\nabla q_2\| = \max\{\|\nabla q_2\|(t) \mid t \in [0, l]/\sim\}$. Note that the function $\|\nabla q_2\|(t)$ on $[0, l]/\sim$ is the curvature of loop q_1 in S^{n+1}. We regard $q_2(t)$ and ∇q_2 as vectors issuing from the origin in $\boldsymbol{R}^{n+2}$. For the loop q_1, we have an orthonormal basis $\{q_1(t), q_2(t), q_3(t), \dots, q_{n+2}(t)\}$ of $\boldsymbol{R}^{n+2}$ for each $t \in [0, l]$, and functions $\lambda_i(t)$ $(3 \leq i \leq n+2)$ on $[0, l]$ such that

$$\text{(1.1)} \qquad \frac{dq_i}{dt}(t) = -\lambda_i(t) q_2(t) \quad (3 \leq i \leq n+2) \quad \text{and} \quad (\nabla q_2)(t) = \sum_{i=3}^{n+2} \lambda_i(t) q_i(t)\,,$$

(cf. §3, Lemma 1). We take smooth functions $a(t)$, $b(t)$ and $c(t)$ on $[0, l]/\sim$ satisfying $(a^2 + b^2 + c^2)(t) \equiv 1$ and $c(t) > 0$. We define a hypersurface Φ in S^{n+1} as an S^{n-1}-bundle over the loop $(aq_1 + bq_2)(t)$. Namely, a mapping $\Phi: S^{n-1} \times [0, l] \to S^{n+1}$ is defined by

$$\text{(1.2)} \qquad \Phi(y, t) = a(t) q_1(t) + b(t) q_2(t) + c(t) \sum_{i=3}^{n+2} y^i q_i(t)\,,$$

where $y = (y^3, \dots, y^{n+2}) \in S^{n-1}$. The mapping Φ is determined by loop q_1 and functions a, b, c, namely, $\Phi = \Phi(q_1, a, b, c)$. Our first problem is to find the condition on the loop $q_1(t)$ and functions $a(t)$, $b(t)$ and $c(t)$ in order that the mappings Φ define conformally flat hypersurfaces (cf. §3).

THEOREM 1. *We give a smooth loop* $q_1 : [0, l_1 + l_2]/\sim \to S^{n+1}$ *satisfies the following conditions*:

(1) $l_1 > 0$ *and the restricted curve* $q_1|_{[0, l_1]}$ *is a geodesic in* S^{n+1}.

(2) *In the case* $\|\nabla q_2\| \geq 1$, *let* α *and* β *be positive real numbers such that* $\alpha^2 + \beta^2 =$

1 *and* $\beta/\alpha = \|\nabla q_2\|$. *Then the length* l_2 *of the curve* $q_1|_{[l_1, l_1+l_2]}$ *satisfies* $l_2 < \alpha[\pi + 2\sin^{-1}(1/\|\nabla q_2\|)]$.

Then there exist smooth functions $a(t)$, $b(t)$ *and* $c(t)$ *on* $[0, l_1+l_2]/\sim$ *such that the mapping* Φ *gives a compact conformally flat hypersurface without umbilic points.*

If $q_1(t)$ is a geodesic in S^{n+1}, then $(\nabla q_2)(t) \equiv 0$ holds. Therefore we may take loops q_1 satisfying the assumption in Theorem 1 by slight deformation of geodesic loops in S^{n+1}. When $(\nabla q_2)(t) \neq 0$ for some t in hypersurfaces Φ, we obtain various types of conformally flat tori in S^{n+1}. From the viewpoint of our construction of hypersurfaces, Pinkall in [6] discussed only the case that the functions $a(t)$, $b(t)$ and $c(t)$ satisfy $a'(t) = b(t)$ at any t. In this case we have to find these functions on $[0, l]/\sim$ under strongly restricted conditions for a mapping to become an immersion (cf. §3, Lemma 3). However these conditions are necessary only for the part with $(\nabla q_2)(t) \neq 0$ of the hypersurfaces. We can take these functions more freely in the part with $(\nabla q_2)(t) = 0$.

In the construction of conformal correspondences between the conformally flat hypersurfaces Φ and Schottky manifolds, it is crucial to construct a conformal diffeomorphism of an open submanifold with $(\nabla q_2)(t) \neq 0$ of Φ into S^n. For this construction we use Pinkall's method in a local form (cf. §4).

For each conformally flat hypersurface Φ given by (1.1), (1.2) and Theorem 1, there exists an orthonormal basis $\{\boldsymbol{p}_2(t), \ldots, \boldsymbol{p}_{n+2}(t)\}$ in $\boldsymbol{R}^{n+1}$ for each $t \in [l_1, l_1+l_2]$ such that

$$\frac{d\boldsymbol{p}_i}{dt}(t) = -\lambda_i(t)\boldsymbol{p}_2(t) \quad (3 \leq i \leq n+2) \quad \text{and} \quad \frac{d\boldsymbol{p}_2}{dt}(t) = \sum_{i=3}^{n+2} \lambda_i(t)\boldsymbol{p}_i(t)\,. \tag{1.3}$$

Let ψ_t, $l_1 \leq t \leq l_1 + l_2$, be a one-parameter family of Möbius transformations without rotations acting on S^n given by

$$\frac{d\psi_t(x)}{dt} = -\frac{a}{\sqrt{b^2+c^2}}(t)\{\boldsymbol{p}_2(t) - \langle \boldsymbol{p}_2(t), \psi_t(x)\rangle \psi_t(x)\}\,, \qquad \psi_{l_1}(x) = x\,.$$

We define a mapping $\Psi : S^{n-1} \times [l_1, l_1+l_2] \to S^n$ by

$$\Psi(y, t) = \psi_t^{-1}\left[\left\{b(t)\boldsymbol{p}_2(t) + c(t)\sum_{i=3}^{n+2} y^i \boldsymbol{p}_i(t)\right\}\Big/\sqrt{b^2+c^2(t)}\right].$$

THEOREM 2. *We have the following*:

(1) *The mapping* $\Psi : S^{n-1} \times [l_1, l_1+l_2] \to S^n$ *is a diffeomorphism into* S^n.

(2) *The mapping of* $\Psi(y, t)$ *to* $\Phi(y, t)$ *for* $(y, t) \in S^{n-1} \times [l_1, l_1+l_2]$ *is a local conformal diffeomorphism between* $\Psi(S^{n-1} \times [l_1, l_1+l_2])$ *and* $\Phi(S^{n-1} \times [l_1, l_1+l_2])$.

Theorem 2 gives a conformal correspondence between closed domains of hypersurface Φ and S^n. Schottky manifolds are defined from closed domains of S^n (cf. §2). Since hypersurface $\Phi : S^{n-1} \times [0, l_1] \to S^{n+1}$ conformally corresponds to a closed domain of S^n between two parallel hyperplanes in $\boldsymbol{R}^{n+1}$ in a natural way, the correspondence given in Theorem 2 is essential to determine a Schottky manifold equivalent to the hypersurface (cf. §4).

Finally, in §5 we construct conformally flat hypersurfaces in a general case. Conformally flat hypersurfaces with umblic points are obtained by replacing a part $\Phi(S^{n-1}\times[\varepsilon, l_1-\varepsilon])$ ($\varepsilon>0$) of the hypersurface given in Theorem 1 by $S^n-\bigcup_{i=1}^2 D_i$, where D_i ($i=1, 2$) are disjoint open round disks. From two conformally flat hypersurfaces with umbilic points, we have other conformally flat hypersurface by a connected sum. Furthermore, we construct conformally flat hypersurfaces which have umbilic points and are diffeomorphic to the Klein bottle.

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2. Schottky manifolds and conformally flat hypersurfaces. In this section we recall some basic facts on classical Schottky manifolds from [3], [6], and the extrinsic geometry of compact conformally flat hypersurfaces from [1], [2].

The classical Schottky manifolds are constructed as follows: We start with the standard sphere (S^n, g_{can}) and consider (i) closed round $2k$ balls $B_1, \dots, B_k$, and $\tilde{B}_1, \dots, \tilde{B}_k$ ($k\geq 1$) in S^n which are pairwise disjoint, and (ii) k Möbius transformations $\tau_1, \dots, \tau_k: S^n\to S^n$ such that $\tau_i(\mathring{B}_i)=S^n-\tilde{B}_i$. Then the quotient space M^n obtained from $S^n-\bigcup_{i=1}^k(\mathring{B}_i\cup\mathring{\tilde{B}})$ by indentifying ∂B_i with $\partial\tilde{B}_i$ via τ_i carries in a canonical way the structure of a compact conformally flat manifold.

Assume, for simplicity, that the τ_i's are orientation preserving. Then M is diffeomorphic to a connected sum of k copies of $S^{n-1}\times S^1$. In particular, we shall call an orientable Schottky manifold with $k=1$ a conformally flat torus. We see the conformal equivalence classes of the tori more closely by using parameters $r\in\boldsymbol{R}$, $0<r<1$, and $\tau\in SO(n)$ as follows: Let $S_r^{n-1}=\{x\in S^n \mid x^1=r\}$ be an $(n-1)$-dimensional sphere in S^n. Then conformally flat tori are described, by a change of the domain by Möbius transformation without rotation, as

$$M_{(r,\tau)}^n=\{x\in S^n \mid -r\leq x^1\leq r\}/\tau \tag{2.1}$$

where the quotient means the identification of S_{-r}^{n-1} with S_r^{n-1} by τ, since all transformations τ_i in (ii) are hyperbolic elements (cf. [4]). Which conformally flat tori admit conformal immersions into S^{n+1}? Roughly speaking, our result says that a *conformally flat torus with τ in a "neighborhood" of the identity in $SO(n)$ can be immersed conformally into S^{n+1}.*

Let (M^n, g) be a compact conformally flat manifold with $n \geq 4$.

(1) M is a conformally flat hypersurface in S^{n+1} if and only if there exists a principal curvature λ of multiplicity n or $n-1$ for each $p \in M$ (cf. [5]).

Let $\Phi: M \to S^{n+1}$ be a conformal immersion. Let $U \subset M$ be the set of non-umbilic points, i.e., the set where λ has multiplicity $n-1$. For $p \in U$ let $D_p \subset T_pM$ denote the eigenspace of the second fundamental form which corresponds to λ. The following facts are known:

(2) The distribution $p \mapsto D_p$ on U is integrable, and therefore defines a foliation of U by the so-called curvature leaves (cf. [1], [5]).

(3) All curvature leaves are mapped by Φ diffeomorphically onto round $(n-1)$-spheres in S^{n+1} (cf. [7]).

3. Conformal immersions of conformally flat tori. In this section we construct conformal immersions of conformally flat tori into S^{n+1} without umbilic points. Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2}$ be a canonical orthonormal basis of $\boldsymbol{R}^{n+2}$. We denote by $\langle \boldsymbol{e}_i, \ldots, \boldsymbol{e}_{n+2} \rangle$ the linear subspace in $\boldsymbol{R}^{n+2}$ spanned by vectors $\boldsymbol{e}_i, \ldots, \boldsymbol{e}_{n+2}$. Let $S^{n+1} \subset \boldsymbol{R}^{n+2}$, $S^n \subset \boldsymbol{R}^{n+1} = \langle \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2} \rangle$ and $S^{n-1} \subset \boldsymbol{R}^n = \langle \boldsymbol{e}_3, \ldots, \boldsymbol{e}_{n+2} \rangle$ be the unit spheres. We start with a basic example.

EXAMPLE 1. Conformally flat tori in S^{n+1} with $\tau = $identity.

Let $t \in [0, l]/\sim$, $l = 2\pi m$ (m: integer) and $y = (y^3, \ldots, y^{n+2}) \in S^{n-1}$. For $a > 0$ and $c > 0$ with $a^2 + c^2 = 1$, we define $\Phi: S^{n-1} \times [0, l]/\sim \to S^{n+1}$ by $\Phi(y, t) = a(\cos t\boldsymbol{e}_1 + \sin t\boldsymbol{e}_2) + c\sum_{i=3}^{n+2} y^i \boldsymbol{e}_i)$. Mapping $\Phi(S^{n-1} \times [0, l]/\sim)$ becomes a compact conformally flat hypersurface. The curvature leaves corresponding to the principal curvature with multiplicity $n-1$ are the $(n-1)$-spheres $\Phi(S^{n-1} \times \{t\})$ for all t.

We define a mapping Ψ of $S^{n-1} \times [0, l]$ into the n-sphere S^n by $\Psi(y, t) = \sin u(t)\boldsymbol{e}_2 + \cos u(t)(\sum_{i=3}^{n+2} y^i \boldsymbol{e}_i)$, where $u(t)$ is defined by $du/dt = (a/c)\cos u(t)$. The Schottky manifold $\Psi(S^{n-1} \times [0, l])/\sim$ obtained from $\Psi(S^{n-1} \times [0, l])$ by identifying $\Psi(y, 0)$ with $\Psi(y, l)$ for $y \in S^{n-1}$ is conformally equivalent to the hypersurface $\Phi(S^{n-1} \times [0, l]/\sim)$. Indeed, the mapping $\Psi(S^{n-1} \times [0, l])/\sim \ni \Psi(y, t) \to \Phi(y, t) \in \Phi(S^{n-1} \times [0, l]/\sim)$ is a conformal immersion in S^{n+1}. q.e.d.

We construct conformal immersions of conformal flat tori for general $[\tau] \subset SO(n)$, where $[\tau]$ is a conjugacy class of τ. We consider the following situation:

(1) Let $q_1(t)$, $t \in [0, l]/\sim$, be a smooth loop in S^{n+1} with the arc-length parameter t.

(2) *Let* $q_2(t)$ be the velocity vector field of q_1, i.e., $q_2(t) = (dq_1/dt)(t)$.

(3) We take smooth vector fields $q_i(t)$ $(i = 3, \ldots, n+2)$ along q_1, i.e., $q_i(t) \in T_{q_1(t)}(S^{n+1})$, satisfying $\langle q_i, q_j \rangle(t) = \delta_{ij}$ $(2 \leq i, j \leq n+2)$ and $dq_i/dt \equiv 0 \mod q_2(t)$.

LEMMA 1. *There exist vector fields* $q_i(t)$ $(i = 3, \ldots, n+2)$ *for* $t \in [0, l]$ *satisfying the condition* (3).

PROOF. We regard the loop $q_1(t)$, $t\in[0, l]/\sim$, as a one-dimensional submanifold in S^{n+1}. The normal bundle $\nu(q_1)$ of q_1 is generated by vectors in $T_{q_1(t)}(S^{n+1})$ orthogonal to $q_2(t)$ for each t. We take normal vector fields $q_i(t)$ $(i=3,\dots,n+2)$ parallel along $q_1(t)$ with respect to the normal connection under the initial conditions $\langle q_i, q_j\rangle(0)=\delta_{ij}$. Then these vector fields q_i's satisfy the condition (3). q.e.d.

By (1), (2) and (3) there exist smooth functions $\lambda_i(t)$ $(i=3,\dots,n+2)$ for $t\in[0, l]$, and $\eta\in SO(n)$ such that

$$(3.1)\qquad \begin{cases} \dfrac{dq_2}{dt}(t)=-q_1(t)+\sum_{i=3}^{n+2}\lambda_i(t)q_i(t) \\ \dfrac{dq_i}{dt}(t)=-\lambda_i(t)q_2(t)\quad(3\le i\le n+2) \qquad \text{for}\quad t\in[0, l]\,, \\ [q_3(l),\dots,q_{n+2}(l)]=[q_3(0),\dots,q_{n+2}(0)]\eta\,. \end{cases}$$

We use the following notation:

$$\boldsymbol{\lambda}(t)=(\lambda_3(t),\dots,\lambda_{n+2}(t))\,,\quad \|\boldsymbol{\lambda}\|(t)=\sqrt{(\lambda_3^2+\cdots+\lambda_{n+2}^2)(t)}\,,$$
$$\|\boldsymbol{\lambda}\|=\max\{\|\boldsymbol{\lambda}\|(t)\,|\,t\in[0, l]\}\,.$$

From now on we regard $q_i(t)$ $(i=2,\dots,n+2)$ as vectors issuing from the origin in $\boldsymbol{R}^{n+2}$. We take smooth functions $a(t)$, $b(t)$ and $c(t)$ on $[0, l]/\sim$ such that $(a^2+b^2+c^2)(t)\equiv 1$ and $c(t)>0$. Let $(S^{n-1}\times[0, l])/\eta$ be the torus obtained by identification with $(y, 0)$ and $(\eta^{-1}y, l)$ for each $y\in S^{n-1}$. Then we define a mapping $\Phi:(S^{n-1}\times[0, l])/\eta\to S^{n+1}$ by

$$(3.2)\qquad \Phi(y, t)=a(t)q_1(t)+b(t)q_2(t)+c(t)\left(\sum_{i=3}^{n+2}y^i q_i(t)\right),$$

where $y=(y^3,\dots,y^{n+2})\in S^{n-1}$. Then $\Phi((S^{n-1}\times[0, l])/\eta)$ is a compact hypersurface.

REMARK. In general, the conjugacy class $[\tau]$ is not equal to $[\eta]$ (or $[\eta^{-1}]$) in (3.1) (cf. §4, Ex. 3).

LEMMA 2. *Assume that the mapping Φ is an immersion. Then every $(n-1)$-sphere $\Phi(S^{n-1}\times\{t\})$ for $t\in[0, l]$ is a curvature leaf in $\Phi((S^{n-1}\times[0, l])/\eta)$ corresponding to the principal curvature with multiplicity $(n-1)$ if and only if the mapping Φ locally satisfies at least one of the conditions $\boldsymbol{\lambda}(t)=0$ and $a'(t)=b(t)$.*

PROOF. First assume $(a^2+b^2)(t)\neq 0$. (A) We define vector fields V_1 and V_2 along $\Phi((S^{n-1}\times[0, l])/\eta)$ on S^{n+1} by

$$\begin{cases} (V_1)_{\Phi(y,t)}=c(t)(aq_1+bq_2)(t)-(a^2+b^2)(t)\left(\sum_{i=3}^{n+2}y^i q_i(t)\right), \\ (V_2)_{\Phi(y,t)}=b(t)q_1(t)-a(t)q_2(t)\,. \end{cases}$$

A vector field N normal to $\Phi((S^{n-1}\times[0, l])/\eta)$ on S^{n+1} is given by

$$N_{\Phi(y,t)}=\langle V_2, d\Phi(\partial/\partial t)\rangle_{\Phi(y,t)}(V_1)_{\Phi(y,t)}-\langle V_1, d\Phi(\partial/\partial t)\rangle_{\Phi(y,t)}(V_2)_{\Phi(y,t)}\,.$$

We have

$$d\Phi(\partial/\partial t)=(a'-b)q_1+\left[a+b'-c\left(\sum_{i=3}^{n+2} y^i\lambda_i\right)\right]q_2+b\sum_{i=3}^{n+2}\lambda_i q_i+c'\left(\sum_{i=3}^{n+2} y^i q_i\right), \tag{3.3}$$

$$\begin{aligned}N_{\Phi(y,t)}=&\left[-(a^2+b^2)+a'b-ab'+ac\left(\sum_{i=3}^{n+2} y^i\lambda_i\right)\right](V_1)_{\Phi(y,t)}\\&+\left[c'+b\left(\sum_{i=3}^{n+2} y^i\lambda_i\right)\right](V_2)_{\Phi(y,t)}\,.\end{aligned} \tag{3.4}$$

(B) Each submanifold $\Phi(S^{n-1}\times\{t\})$ is a curvature leaf corresponding to the curvature with multiplicity $(n-1)$ if and only if we have

$$(\nabla_{d\Phi(Y)}N)_{\Phi(y,t)}=P_1(Y, y, t)N_{\Phi(y,t)}+P_2(y, t)d\Phi(Y) \qquad \text{for} \quad Y\in\Gamma[T(S^{n-1}\times\{t\})]\,, \tag{3.5}$$

where P_1 and P_2 are suitable functions. Therefore, we have to obtain the condition on functions a, b and c so that the equality (3.5) holds. We have by (3.4)

$$\begin{aligned}\nabla_{d\Phi(Y)}N=&\left(\sum_{i=3}^{n+2} Y^i\lambda_i\right)(acV_1+bV_2)_{\Phi(y,t)}\\&-(a^2+b^2)\left[-(a^2+b^2)+a'b-ab'+ac\left(\sum_{i=3}^{n+2} y^i\lambda_i\right)\right]\Big/ cd\Phi(Y)\,,\end{aligned}$$

where $Y=\sum_{i=3}^{n+2} Y^i\partial/\partial y_i$. Therefore, by (3.4) and (3.5) we get $\lambda(t)=0$ or

$$\frac{c'+b\left(\sum_{i=3}^{n+2} y^i\lambda_i\right)}{-(a^2+b^2)+a'b-ab'+ac\left(\sum_{i=3}^{n+2} y^i\lambda_i\right)}=\frac{b}{ac}\,. \tag{3.6}$$

The equality (3.6) is equal to $a'(t)=b(t)$ by $aa'+bb'+cc'=0$ and $(a^2+b^2)(t)\neq 0$.

(C) Now, let us assume that $(a^2+b^2)(t)=0$ holds on some interval (α, β). Then

$$d\Phi(\partial/\partial t)=-\left(\sum_{i=3}^{n+2} y^i\lambda_i(t)\right)q_2(t)=-\langle y, \lambda(t)\rangle q_2(t)$$

using $a(t)=a'(t)=b(t)=b'(t)=c'(t)=0$, $c(t)=1$ and (3.3). This implies that there exist points $y\in S^{n-1}$ such that $\|d\Phi(\partial/\partial t)\|_{\Phi(y,t)}=0$, which is a contradiction, because $\Phi((S^{n-1}\times[0, l])/\eta)$ has no singularities. q.e.d.

We are interested in the case $\lambda(t)\not\equiv 0$. So we assume $\lambda\neq 0$ on some interval (α, β). Then we have $a'(t)=b(t)$ on (α, β) by Lemma 2.

LEMMA 3. *We assume $\lambda(t)\neq 0$ on some interval (α, β). Then the mapping $\Phi: S^{n-1}\times$*

$(\alpha, \beta) \to S^{n+1}$ *is an immersion if and only if* $|c'/b|(t) > \|\boldsymbol{\lambda}\|(t)$ *on* (α, β).

REMARK. In the statement of Lemma 3, if we have $b(t_0)=0$ for some t_0, then $|c'/b|(t_0) > \|\boldsymbol{\lambda}\|(t_0)$ means $\lim_{t\to t_0}|c'/b|(t) > \|\boldsymbol{\lambda}\|(t_0)$.

PROOF. We denote by $[d\Phi(\partial/\partial t)]^{\perp}_{\Phi(y,t)}$ the normal component of $d\Phi(\partial/\partial t)$ to the $(n-1)$-sphere $\Phi(S^{n-1}\times\{t\})$. Since

$$d\Phi\left(\frac{\partial}{\partial t}\right) = -c\left[\frac{c'}{b} + \left(\sum_{i=3}^{n+2} y^i\lambda_i\right)\right]q_2 + b\sum_{i=3}^{n+2}\lambda_i q_i + c'\left(\sum_{i=3}^{n+2} y^i q_i\right)$$

by (3.3) and $a'(a+b') = aa' + bb' = -c'c$, we have

$$\left[d\Phi\left(\frac{\partial}{\partial t}\right)\right]^{\perp} = \left[\frac{c'}{b} + \left(\sum_{i=3}^{n+2} y^i\lambda_i\right)\right]\left[-cq_2 + b\left(\sum_{i=3}^{n+2} y^i q_i\right)\right] \quad \text{and}$$

$$\left\|\left[d\Phi\left(\frac{\partial}{\partial t}\right)\right]^{\perp}\right\| = \sqrt{b^2+c^2}\left|\frac{c'}{b} + \sum_{i=3}^{n+2} y^i\lambda_i\right|.$$

Therefore the inequality $\|[d\Phi(\partial/\partial t)]^{\perp}\| > 0$ is equal to $|c'/b|(t) > \|\boldsymbol{\lambda}\|(t)$. q.e.d.

If the loop $q_1(t)$ is not a geodesic on some interval (α, β), then the functions $a(t)$, $b(t)$ and $c(t)$ on (α, β) have to satisfy the following conditions:

$$(3.7) \qquad \begin{cases} \text{(i)} \quad (a^2+b^2+c^2)(t) \equiv 1, & \text{(ii)} \quad c(t) > 0, \\ \text{(iii)} \quad a'(t) = b(t), & \text{(vi)} \quad |c'/b|(t) > \|\boldsymbol{\lambda}\|(t). \end{cases}$$

On the other hand, if the loop $q_1(t)$ is a geodesic on some interval (α, β), then the functions $a(t)$, $b(t)$ and $c(t)$ on (α, β) only have to satisfy the following conditions:

$$(3.8) \qquad \begin{cases} \text{(i)} \quad (a^2+b^2+c^2)(t) \equiv 1, \qquad \text{(ii)} \quad c(t) > 0, \\ \text{(iii)} \quad \|d\Phi(\partial/\partial t)\|^2 = (a+b')^2 + (a'-b)^2 + (c')^2 > 0. \end{cases}$$

THEOREM 1. *Suppose that we have a smooth loop* $q_1 : [0, l_1+l_2]/\sim \to S^{n+1}$ *satisfying the following conditions*:

(1) $l_1 > 0$ *and the restricted curve* $q_1|_{[0,l_1]}$ *is a geodesic in* S^{n+1}.

(2) *In the case* $\|\boldsymbol{\lambda}\| \geq 1$, *let* α *and* β *be positive real numbers such that* $\alpha^2+\beta^2=1$ *and* $\beta/\alpha = \|\boldsymbol{\lambda}\|$. *Then the length* l_2 *of the curve* $q_1|_{[l_1,l_1+l_2]}$ *satisfies* $l_2 < \alpha[\pi + 2\sin^{-1}(1/\|\boldsymbol{\lambda}\|)]$.

Then there exist smooth functions $a(t)$, $b(t)$ *and* $c(t)$ *on* $[0, l_1+l_2]/\sim$ *such that* $\Phi((S^{n+1}\times[0, l_1+l_2])/\eta)$ *becomes a compact conformally flat hypersurface without umbilic points.*

PROOF. (A) First we show the required functions $a(t)$, $b(t)$ and $c(t)$ defined only on $[l_1, l_1+l_2]$ exist. We define functions $A(u)$, $B(u)$ and $C(u)$ on $\boldsymbol{R}$ by $A(u) = \{\bar{\alpha}\sin(u/\bar{\alpha}) - \bar{\beta}\}/\sqrt{2}$, $B(u) = \cos(u/\bar{\alpha})/\sqrt{2}$ and $C(u) = \{\bar{\beta}\sin(u/\bar{\alpha}) + \bar{\alpha}\}/\sqrt{2}$, where $\bar{\alpha}^2+\bar{\beta}^2=1$, $\bar{\alpha}>0$ and $\bar{\beta}>0$. Then we have $(A^2+B^2+C^2)(u) \equiv 1$, $A'(u) = B(u)$ and $|C'/B|(u) = \bar{\beta}/\bar{\alpha}$.

The case $\|\boldsymbol{\lambda}\|<1$. Let us fix $\bar{\alpha}$ and $\bar{\beta}$ with $1>\bar{\beta}/\bar{\alpha}>\|\boldsymbol{\lambda}\|$. Then we have $|C'/B|(u)>\|\boldsymbol{\lambda}\|$ and $C(u)>0$ on $\boldsymbol{R}$. In this case we put $a(t)=A(t)$, $b(t)=B(t)$ and $c(t)=C(t)$. In particular, we can take any interval with length l_2 as a domain of the variable t.

The case $\|\boldsymbol{\lambda}\|\geq 1$. Let us fix $\bar{\alpha}$ and $\bar{\beta}$ and $\bar{\beta}/\bar{\alpha}>\|\boldsymbol{\lambda}\|$ and $l_2<\bar{\alpha}[\pi+2\sin^{-1}(\bar{\alpha}/\bar{\beta})]<\bar{\alpha}[\pi+2\sin^{-1}(1/\|\boldsymbol{\lambda}\|)]$. Then we have $C(u)>0$ for $u\in(-\bar{\alpha}\sin^{-1}(\bar{\alpha}/\bar{\beta}), \bar{\alpha}\pi+\bar{\alpha}\sin^{-1}(\bar{\alpha}/\bar{\beta}))$. Therefore, there exists $u_0\in\boldsymbol{R}$ such that $l_2=\bar{\alpha}(\pi+2u_0)$ and $C(u)>0$ for $u\in[-\bar{\alpha}u_0, \bar{\alpha}(\pi+u_0)]$. In this case, we define

$$a(t)=A(t-l_1-\bar{\alpha}u_0)\,,\quad b(t)=B(t-l_1-\bar{\alpha}u_0)\,,\quad c(t)=C(t-l_1-\bar{\alpha}u_0)$$

for $t\in[l_1, l_1+l_2]$.

(B) Now we prolong functions $a(t)$, $b(t)$ and $c(t)$ on $[l_1, l_1+l_2]$ given in (A) smoothly to the domain $[0, l_1+l_2]/\sim$. These functions have to satisfy the condition (3.8) on $[0, l_1]$. Here we only consider the case $\|\boldsymbol{\lambda}\|\geq 1$ and $u_0>0$.

Let us take $c(t)$ satisfying $0<c(t)=c_0<c(l_1+l_2)=c(l_1)$ on $[\varepsilon, l_1-\varepsilon]$ for small $\varepsilon>0$ and $c'(t)\neq 0$ for $t\in[0, \varepsilon)\cup(l_1-\varepsilon, l_1]$. If $c'(t)\neq 0$, then $\|d\Phi(\partial/\partial t)\|>0$. In the case $c'(t)=0$, if $a'\neq b$ or $b'\neq -a$, then $\|d\Phi(\partial/\partial t)\|>0$. So, we take $p(\neq 1)$ so that $p(l_1-2\varepsilon)<\pi$ and $b(0)=b(l_1+l_2)<\sqrt{1-c_0^2}\sin[p(\varepsilon-l_1/2)]$. Then we define $b(t)=\sqrt{1-c_0^2}\sin[p(t-l_1/2)]$ for $t\in[\varepsilon, l_1-\varepsilon]$. We prolong the function $b(t)$ on $[\varepsilon, l_1-\varepsilon]\cup[l_1, l_1+l_2]$ smoothly to the circle $[0, l_1+l_2]/\sim$ so that $b(0)\leq b(t)\leq b(\varepsilon)$ for $t\in(0, \varepsilon)$ and $b(l_1-\varepsilon)\leq b(t)\leq b(l_1)$ for $t\in(l_1-\varepsilon, l_1)$. We define $a(t)$ by $a^2(t)=1-b^2(t)-c^2(t)$ and $a(t)<0$ on $[0, l_1]$. Then we have $a(t)=-\sqrt{1-c_0^2}\cos[p(t-l_1/2)]$ on $[\varepsilon, l_1-\varepsilon]$. Therefore, we get $a'(t)=pb(t)$ and $b'(t)=-pa(t)$ on $[\varepsilon, l_1-\varepsilon]$.

(C) Take ε_1 so that $l_2+2\varepsilon_1<\bar{\alpha}[\pi+2\sin^{-1}(\bar{\alpha}/\bar{\beta})]$ and $0<\varepsilon_1<\varepsilon$. When we prolong the functions to $[0, l_1+l_2]/\sim$ in (B), we define the functions $a(t)$, $b(t)$ and $c(t)$ on $[0, \varepsilon_1)\cup(l_1-\varepsilon_1, l_1]$ as the direct prolongation of them given in (A). Then, these functions $a(t)$, $b(t)$ and $c(t)$ on $[0, l_1+l_2]/\sim$ satisfy (3.7) on $[0, \varepsilon_1)\cup(l_1-\varepsilon_1, l_1+l_2]$, and (3.8) on $[0, l_1]$. Therefore, the mapping $\Phi=\Phi(q_1, a, b, c)$ gives an immersed conformally flat hypersurface. q.e.d.

4. Correspondence between conformally flat hypersurfaces and Schottky manifolds. In this section we give the conformal correspondences between the conformally flat hypersurfaces $\Phi((S^{n-1}\times[0, l_1+l_2])/\eta)$ given in §3 and Schottky manifolds. First we study a relation between our hypersurfaces $\Phi((S^{n-1}\times[0, l_1+l_2])/\eta)$ and Pinkall's construction (Lemma 4). Then we see that Pinkall's hypersurfaces satisfy $a'(t)=b(t)$ everywhere on $[0, l_1+l_2]/\sim$. In order to construct a conformal correspondence between our hypersurface and Schottky manifold, Pinkall's method is useful. Therefore, we next give a local conformal diffeomorphism $\hat{\Phi}$ of $\Phi(S^{n-1}\times[l_1, l_1+l_2])$ into S^n following Pinkall's method (Lemma 5). From now on, we write $J=[l_1, l_1+l_2]$. Finally, we represent the conformal diffeomorphism $\hat{\Phi}$ and the image of $\hat{\Phi}$ in S^n exactly by using parameters (y, t) on $\Phi(S^{n-1}\times J)$ (Theorem 2).

We define a mapping $f: S^n\times J\to S^{n+1}$ by

$$f(x,t)=f\left(\sum_{i=2}^{n+2} x^i e_i, t\right)=a(t)q_1(t)+\sqrt{b^2+c^2}(t)\left(\sum_{i=2}^{n+2} x^i q_i(t)\right).$$

We define $(n-1)$-spheres $M_t \subset S^n \times J$ for each $t \in J$ by

$$M_t=\left\{\left(\frac{b}{\sqrt{b^2+c^2}}(t)e_2+\frac{c}{\sqrt{b^2+c^2}}(t)\left(\sum_{i=3}^{n+2} y^i e_i\right), t\right) \middle| y \in S^{n-1}\right\}.$$

We denote $M=\bigcup_{t\in J} M_t$.

LEMMA 4. *We have the following*:

(1) $f|_M=\Phi$, *where we indentified*

$$\left(\frac{b}{\sqrt{b^2+c^2}}(t)e_2+\frac{c}{\sqrt{b^2+c^2}}(t)\left(\sum_{i=3}^{n+2} y^i e_i\right), t\right) \quad \text{with} \quad \left(\sum_{i=3}^{n+2} y^i e_i, t\right).$$

(2) $(df)_{(x,t)}(\mathbf{0}, \partial/\partial t) \in df[T_{(x,t)}(S^n \times \{t\})]$ *for* $(x, t) \in M$.

PROOF. (1) follows easily from the definitions f and Φ. As for (2), we have

$$df_{(x,t)}\left(\mathbf{0}, \frac{\partial}{\partial t}\right)=(a'-x^2\sqrt{b^2+c^2})q_1+\left[a-\frac{aa'}{\sqrt{b^2+c^2}}x^2-\sqrt{b^2+c^2}\sum_{i=3}^{n+2}\lambda_i x^i\right]q_2$$
$$-\frac{aa'}{\sqrt{b^2+c^2}}\left(\sum_{i=3}^{n+2} x^i q_i\right)+\sqrt{b^2+c^2}x^2\left(\sum_{i=3}^{n+2}\lambda_i q_i\right),$$

by (3.1) and (3.7). Take a vector field ξ along $f(S^n \times \{t\})$ on S^{n+1} defined by

$$\xi_{f(x,t)}=\sqrt{b^2+c^2}(t)q_1(t)-a(t)\left(\sum_{i=2}^{n+2} x^i q_i(t)\right).$$

Then $\xi_{f(x,t)}$ is a vector field normal to $f(S^n \times \{t\})$. We have

$$\left\langle df_{(x,t)}\left(\mathbf{0}, \frac{\partial}{\partial t}\right), \xi_{f(x,t)}\right\rangle=\frac{a'}{\sqrt{b^2+c^2}}(t)-x^2=\frac{b}{\sqrt{b^2+c^2}}(t)-x^2.$$

Therefore, if $(x, t) \in M$, then $df_{(x,t)}(0, \partial/\partial t) \in df[T_{(x,t)}(S^n \times \{t\})]$. q.e.d.

The definition of f and Lemma 4 imply that our hypersurfaces $\Phi(S^{n-1} \times J)$ are explicit representations of the ones constructed implicitly by Pinkall (cf. [6]).

We define a one-parameter family of Möbius transformations $\varphi_t: S^n \to S^n$ for $t \in J$ as follows:

(1) For each $t \in J$, let Y^t be a vector field on $f(S^n \times \{t\})$ obtained by the orthogonal projection of $df_{(x,t)}(0, \partial/\partial t)$ to $df[T_{(x,t)}(S^n \times \{t\})]$.

(2) Let X^t be a time-dependent vector field on S^n defined by $df_{(x,t)}(X^t_x, 0)=Y^t_{f(x,t)}$. We note that Y^t (resp. X^t) is a conformal vector field on $f(S^n \times \{t\})$ (resp. S^n) (cf. (4.3) and (4.4) below).

(3) We define φ_t by $d\varphi_t(x)/dt = -X^t_{\varphi_t(x)}$ under the initial condition $\varphi_{t_1}(x) = x$.

Define a diffeomorphism $\varphi: S^n \times J \to S^n \times J$ by $\varphi(x, t) = (\varphi_t(x), t)$. Denote $\hat{M} := \varphi^{-1}(M)$ and $\hat{f} := f \circ \varphi$. Then we have $\hat{f}|_{\hat{M}} = \Phi \circ \varphi : \hat{M} \to S^{n+1}$ by Lemma 4. Since φ_t is a Möbius transformation, $\hat{M}_t = \varphi^{-1}(M_t)$ is also an $(n-1)$-sphere for each t. The following lemma is a local form of [6, Lemmas 5 and 6]. Here we give a proof using the parameters (y, t) on $S^{n-1} \times J$.

LEMMA 5. (1) *Let $\pi: S^n \times J \to S^n$ be a canonical projection defined by $\pi(x, t) = x$. Then the mapping $\pi|_{\hat{M}}: \hat{M} \to \pi(\hat{M})$ is a diffeomorphism. Therefore the mapping $\hat{f}|_{\hat{M}}$ defines a mapping $\hat{\Phi}: \pi(\hat{M}) \to S^{n+1}$.*

(2) *The mapping $\hat{\Phi}: \pi(\hat{M}) \to \Phi(S^{n-1} \times J) \subset S^{n+1}$ is a local conformal diffeomorphism.*

PROOF. We prove directly the statement (2). Since $\Phi(S^{n-1} \times J)$ is an immersed hypersurface and $\pi(\hat{M})$ is the union of $(n-1)$-spheres $\pi(\hat{M}_t)$ in S^n, the mapping $\pi: \hat{M} \to \pi(\hat{M})$ is a diffeomorphism. We fix a point $(x_0, t_0) \in M$, and then $(\varphi_{t_0}^{-1}(x_0), t_0) \in \hat{M}$.

(A) We first study the images by $d\hat{f}$ of tangent vectors to $\hat{M}_{t_0}$ at $(\varphi_{t_0}^{-1}(x_0), t_0)$. Let us take a curve $\bar{x}(s) = (x(s), t_0)$, $|s| < \varepsilon$, in $\hat{M}_{t_0}$ with $x(0) = \varphi_{t_0}^{-1}(x_0)$. Then, we have

$$
(4.1) \qquad d\hat{f}_{(\varphi_{t_0}^{-1}(x_0), t_0)}\left[\left.\frac{dx}{ds}\right|_{s=0}, 0\right] = \left.\frac{d\hat{f}(x(s), t_0)}{ds}\right|_{s=0} = \left.\frac{df[\varphi_{t_0}(x(s)), t_0]}{ds}\right|_{s=0}
$$

$$
= df_{(x_0, t_0)}\left[(d\varphi_{t_0})_{\varphi_{t_0}^{-1}(x_0)} \left.\frac{dx}{ds}\right|_{s=0}, 0\right].
$$

By the definition of $\hat{\Phi}$, we have $d\hat{\Phi}_{\varphi_{t_0}^{-1}(x_0)}(dx/ds|_{s=0}) = d\hat{f}_{(\varphi_{t_0}^{-1}(x_0), t_0)}[dx/ds|_{s=0}, 0]$.

(B) We now study the images by $d\hat{f}$ of transversal vectors of $\hat{M}_{t_0}$ at $(\varphi_{t_0}^{-1}(x_0), t_0)$. Let us take a curve $(x(t), t)$, $|t - t_0| < \varepsilon$, in M given by

$$
x(t) = \left\{ b(t)e_2 + c(t)\left(\sum_{i=3}^{n+2} y_0^i e_i\right)\right\} \Big/ \sqrt{b^2 + c^2}(t),
$$

where $x(0) = x_0 = \{b(t_0)e_2 + c(t_0)(\sum_{i=3}^{n+2} y_0^i e_i)\}/\sqrt{b^2 + c^2}(t_0)$. Denote $\gamma(t) = \varphi_t^{-1}[x(t)]$. Then $(\gamma(t), t) \subset \hat{M}$. Since $\hat{f}(\gamma(t), t) = f(\varphi_t(\gamma(t)), t) = f(x(t), t) = \Phi(y_0, t)$, we have

$$
\left.\frac{d\hat{f}(\gamma(t), t)}{dt}\right|_{t=t_0} = d\hat{f}_{(\gamma(t_0), t_0)}\left[\left.\frac{d\gamma}{dt}\right|_{t=t_0}, 0\right] + d\hat{f}_{(\gamma(t_0), t_0)}\left[\mathbf{0}, \frac{\partial}{\partial t}\right] = d\Phi_{(y_0, t_0)}\left(\mathbf{0}, \frac{\partial}{\partial t}\right).
$$

We note that the vector $[d\Phi_{(y_0, t_0)}(\mathbf{0}, \partial/\partial t)]^\perp$ does not vanish from our construction of Φ (cf. Lemma 3 and its proof). On the other hand, we have

$$
d\hat{f}_{(\gamma(t_0), t_0)}\left[\left.\frac{d\gamma}{dt}\right|_{t=t_0}, 0\right] = \left.\frac{df(\varphi_{t_0}(\gamma(t)), t_0)}{dt}\right|_{t=t_0} = df_{(x_0, y_0)}\left[(d\varphi_{t_0})_{\gamma(t_0)} \left.\frac{d\gamma}{dt}\right|_{t=t_0}, 0\right],
$$

and

$$d\hat{f}_{(\gamma(t_0),t_0)}\left[0, \frac{\partial}{\partial t}\right] = \frac{df(\varphi_t(\gamma(t_0)), t)}{dt}\bigg|_{t=t_0}$$
$$= df_{(x_0,t_0)}\left[\frac{d\varphi_t(\gamma(t_0))}{dt}\bigg|_{t=t_0}, 0\right] + (df)_{(x_0,t_0)}\left[\mathbf{0}, \frac{\partial}{\partial t}\right] = 0\,.$$

Indeed, the last equality follows from $d\varphi_t(\gamma(t_0))/dt\,|_{t=t_0} = -X^{t_0}_{x_0}$, $df_{(x_0,t_0)}[0, \partial/\partial t] = Y^{t_0}_{f(x_0,t_0)}$, $(x_0, t_0) \in M$, and Lemma 4. Therefore, we have

$$\text{(4.2)}\qquad d\hat{f}_{(\gamma(t_0),t_0)}\left[\frac{d\gamma}{dt}\bigg|_{t=t_0}, \frac{\partial}{\partial t}\right] = df_{(x_0,t_0)}\left[(d\varphi_{t_0})_{\varphi^{-1}_{t_0}(x_0)}\frac{d\gamma}{dt}\bigg|_{t=t_0}, 0\right].$$

Since $d\hat{\Phi}_{\gamma(t_0)}(d\gamma/dt\,|_{t=t_0}) = d\hat{f}_{(\gamma(t_0),t_0)}[d\gamma/dt\,|_{t=t_0}, \partial/\partial t]$ by the definition of $\hat{\Phi}$, the equations (4.1) and (4.2) imply that $\hat{\Phi}$ is a local conformal diffeomorphism. q.e.d.

In Lemma 5 we obtained a new parameter space $\pi(\hat{M}) \subset (S^n, g_{\mathrm{can}})$ of conformally flat hypersurface $\Phi(S^{n-1} \times J)$. Then the space $\pi(\hat{M})$ is diffeomorphic to $S^{n-1} \times J$ and the mapping $\hat{\Phi}: \pi(\hat{M}) \to S^{n+1}$ is a conformal immersion.

To study the domain $\pi(\hat{M})$ in more detail, we represent the Möbius transformations φ_t by rotations and pure Möbius transformations. By the definition and the proof of Lemma 4, we have

$$\text{(4.3)}\qquad Y^t_{f(x,t)} = df_{(x,t)}\left(\frac{\partial}{\partial t}\right) - \left(\frac{a'}{\sqrt{b^2+c^2}}(t) - x^2\right)\xi_{f(x,t)}$$
$$= -\sqrt{b^2+c^2}(t)\left[\left(\sum_{i=3}^{n+2}\lambda_i(t)x^i\right)q_2(t) - x^2\left(\sum_{i=3}^{n+2}\lambda_i(t)q_i(t)\right)\right]$$
$$+ a(t)\left[(1-(x^2)^2)q_2(t) - x^2\left(\sum_{i=3}^{n+2}x^i q_i(t)\right)\right],$$

and

$$\text{(4.4)}\qquad X^t_x = -\left[\left(\sum_{i=3}^{n+2}\lambda_i(t)x^i\right)e_2 - x^2\left(\sum_{i=3}^{n+2}\lambda_i(t)e_i\right)\right]$$
$$+ \frac{a}{\sqrt{b^2+c^2}}(t)\left[(1-(x^2)^2)e_2 - x^2\left(\sum_{i=3}^{n+2}x^i e_i\right)\right].$$

LEMMA 6. *We have the following*:

(1) *For given smooth functions* $\lambda_i(t)$ $(i=3, \dots, n+2)$ *on* $[l_1, l_1+l_2]$, *there exist a curve* $\boldsymbol{p}_2(t)$ *for* $t \in [l_1, l_1+l_2]$ *in* $S^n \subset \boldsymbol{R}^{n+1} = \langle e_2, \dots, e_{n+2}\rangle$ *and vector fields* $\boldsymbol{p}_i(t)$ $(i=3, \dots, n+2)$ *on* S^n *along the curve* $\boldsymbol{p}_2$ *such that* $\langle \boldsymbol{p}_i, \boldsymbol{p}_j\rangle(t) = \delta_{ij}$ $(3 \le i, j \le n+2)$, $\boldsymbol{p}_2'(t) = \sum_{i=3}^{n+2}\lambda_i(t)\boldsymbol{p}_i(t)$ *and* $\boldsymbol{p}_i' = -\lambda_i(t)\boldsymbol{p}_2(t)$ $(3 \le i \le n+2)$.

(2) *We regard vector fields* $\boldsymbol{p}_i(t)$ $(3 \le i \le n+2)$ *as vectors of* $\boldsymbol{R}^{n+1}$ *issuing from the origin. Let us take* $(\boldsymbol{p}_2(l_1), \dots, \boldsymbol{p}_{n+2}(l_1)) = (e_2, \dots, e_{n+2})$. *We define a one-parameter family of Möbius transformations without rotations by*

$$(4.5) \qquad \frac{d\psi_t(x)}{dt} = -\frac{a}{\sqrt{b^2+c^2}}(t)\{\boldsymbol{p}_2(t) - \langle \psi_t(x), \boldsymbol{p}_2(t)\rangle \psi_t(x)\}\,, \qquad \psi_{t_1}(x) = x\,.$$

Then we have $\varphi_t(x) = \sum_{j=2}^{n+2} \langle \psi_t(x), \boldsymbol{p}_j(t)\rangle \boldsymbol{e}_j$.

PROOF. (1) is the result immediate from an ordinary differential equation on $SO(n+1)$. We prove (2). The first (resp. the second) term of X represents an infinitesimal vector field of rotations (resp. of pure Möbius transformations). Therefore, for each t, we change the basis of $\boldsymbol{R}^{n+1}$ from $\{\boldsymbol{e}_2, \ldots, \boldsymbol{e}_{n+2}\}$ to $\{\boldsymbol{p}_2(t), \ldots, \boldsymbol{p}_{n+2}(t)\}$ satisfying (1). When we represent $\varphi_t(x)$ as

$$\varphi_t(x)\left(=\sum_{j=2}^{n+2} \varphi_t^j(x)\boldsymbol{e}_j\right) = \sum_{j=2}^{n+2} \langle \psi_t(x), \boldsymbol{p}_j(t)\rangle \boldsymbol{e}_j\,,$$

where $\psi_t(x)$ is a Möbius transformation for each t defined by (4.5), we have

$$\begin{aligned}\frac{d\varphi_t(x)}{dt} &= \left(\sum_{i=3}^{n+2} \lambda_i(t)\varphi_t^i(x)\right)\boldsymbol{e}_2 - \varphi_t^2(x)\sum_{j=3}^{n+2} \lambda_j(t)\boldsymbol{e}_j \\ &\quad - \frac{a}{\sqrt{b^2+c^2}}(t)\left[(1-(\varphi_t^2(x))^2)\boldsymbol{e}_2 - \varphi_t^2(x)\sum_{j=3}^{n+2} \varphi_t^j(x)\boldsymbol{e}_j\right] = -X^t_{\varphi_t(x)}\,.\end{aligned}$$

q.e.d.

We summarize Lemmas 5 and 6: Let us define an $(n-1)$-sphere $N_t \subset S^n$ for each $t \in J$ by

$$N_t = \left\{\frac{1}{\sqrt{b^2+c^2}}(t)\left[b(t)\boldsymbol{p}_2(t) + c(t)\left(\sum_{i=3}^{n+2} y^i \boldsymbol{p}_i(t)\right)\right]\middle|\, y \in S^{n-1}\right\}.$$

Putting $\hat{N}_t = \psi_t^{-1}(N_t)$ and $\hat{N} = \bigcup_{t\in J} \hat{N}_t$, we have $\partial\hat{N} = N_{l_1} \cup \psi_{l_1+l_2}^{-1}(N_{l_1+l_2})$, and $\pi(\hat{M}) = \hat{N}$. We define a mapping $\Psi: S^{n-1} \times J \to S^n$ by

$$\Psi(y, t) = \psi_t^{-1}\left[\frac{1}{\sqrt{b^2+c^2}}(t)\left\{b(t)\boldsymbol{p}_2(t) + c(t)\left(\sum_{i=3}^{n+2} y^i \boldsymbol{p}_i(t)\right)\right\}\right].$$

THEOREM 2. *In the above notation we have the following*:

(1) *The mapping* $\Psi: S^{n-1} \times J \to S^n$ *is a diffeomorphism into* S^n.

(2) *The mapping* $\hat{\Phi}: \hat{N} \to \Phi(S^{n-1} \times J)$ *corresponding* $\Psi(y, t)$ *to* $\Phi(y, t)$ *for* $(y, t) \in S^{n-1} \times J$ *is a local conformal diffeomorphism.*

To give a global conformal correspondence between a hypersurface $\Phi((S^{n-1} \times [0, l_1+l_2])/\eta)$ and a Schottky manifold, we take a real number $C \in \boldsymbol{R}$ and a closed domain $N' \subset S^n$ satisfying the following conditions (cf. §3, Ex. 1):

(1) N' is either N'_1 or N'_2 below such that $\mathring{\hat{N}} \cap \mathring{N}' = \emptyset$:

$$N_1'=\left\{\sum_{i=2}^{n+2} x^i e_i \in S^n \,\middle|\, C \le x^2 \le \frac{b}{\sqrt{b^2+c^2}}(l_1)\right\},$$

$$N_2'=\left\{\sum_{i=2}^{n+2} x^i e_i \in S^n \,\middle|\, \frac{b}{\sqrt{b^2+c^2}}(l_1) \le x^2 \le C\right\}.$$

(2) There exists a conformal immersion of N' into S^{n+1} such that the image is $\Phi(S^{n-1}\times[0, l_1])$.

Thus we can prolong the immersion $\hat{\Phi}: \hat{N} \to S^{n+1}$ conformally to $\hat{\Phi}: \hat{N} \cup N' \to S^{n+1}$ so that the image is a conformally flat hypersurface $\Phi((S^{n-1}\times [0, l_1+l_2])/\eta)$. We obtain the Schottky manifold from $\hat{N} \cup N'$ by identifying two $(n-1)$-spheres of the boundary $\partial(\hat{N}\cup N')=S_1 \cup S_2$, where $S_1=\{Ce_2+\sqrt{1-C^2}(\sum_{i=3}^{n+2} y^i e_i) \mid y \in S^{n-1}\}$ and $S_2=\psi_{l_1+l_2}^{-1}(N_{l_1+l_2})$. This identification is defined by Φ, and given by

$$Ce_2+\sqrt{1-C^2}\left(\sum_{i=3}^{n+2} y^i e_i\right)=\psi_{l_1+l_2}^{-1}\left(\frac{1}{\sqrt{b^2+c^2}}\left[bp_2+c\left(\sum_{i=3}^{n+2}(\eta^{-1}y)^i p_i\right)\right](l_1+l_2)\right)$$

for each $y \in S^{n-1}$ from (3.1). Since the action of each ψ_t does not include any rotation, the rotation τ in (2.1) of the Schottky manifold determines from the correspondence with N_{l_1} and $N_{l_1+l_2}$ given by

(4.6)

$$\frac{1}{\sqrt{b^2+c^2}}\left\{bp_2+c\left(\sum_{i=3}^{n+2} y^i p_i\right)\right\}(l_1) \leftrightarrow \frac{1}{\sqrt{b^2+c^2}}\left\{bp_2+c\left(\sum_{i=3}^{n+2}(\eta^{-1}y)^i p_i\right)\right\}(l_1+l_2).$$

If $q_1(t)$ is a geodesic in S^{n+1}, then each $p_i(t)$ is a constant by Lemma 6. Since we constructed the mapping $\Phi((S^{n-1}\times[0, l_1+l_2])/\eta)$ in Theorem 1 by swinging a geodesic $q_1(t)$, the rotation $[\tau] \subset SO(n)$ determined by Φ is an element in some neighborhood of the identity (cf. Ex. 3 below).

EXAMPLE 2. We give an example of $\{p_i(t)\}$ corresponding to $\{q_i(t)\}$ given in Lemma 6. For the sake of simplicity we consider a loop $q_1(t)$ in S^3. For $0<\rho<\pi/2$, $p \in \boldsymbol{Z}$ and $r=\sqrt{\cos^2\rho+p^2\sin^2\rho}$, let us define

$$q_1(t)=\cos\rho\{\sin(t/r)e_1+\cos(t/r)e_2\}+\sin\rho\{\sin(pt/r)e_3+\cos(pt/r)e_4\}$$

for $0 \le t \le 2\pi r$. Then we have

$$q_2(t)=(\cos\rho/r)\{\cos(t/r)e_1-\sin(t/r)e_2\}+(p/r)\sin\rho\{\cos(pt/r)e_3-\sin(pt/r)e_4\}.$$

We take

$$\bar{q}_3(t)=-\sin\rho\{\sin(t/r)e_1+\cos(t/r)e_2\}+\cos\rho\{\sin(pt/r)e_3+\cos(pt/r)e_4\},$$

$$\bar{q}_4(t)=-(p/r)\sin\rho\{\cos(t/r)e_1-\sin(t/r)e_2\}+(\cos\rho/r)\{\cos(pt/r)e_3-\sin(pt/r)e_4\}.$$

Then $\bar{q}_3(t)$ and $\bar{q}_4(t)$ satisfy

$$\bar{q}_3' = (p^2-1)/r^2 \sin\rho\cos\rho q_2(t) + (p/r^2)\bar{q}_4(t)\,, \quad \bar{q}_4'(t) = -(p/r^2)\bar{q}_3(t)\,.$$

If we put

$$[q_3(t), q_4(t)] = [\bar{q}_3(t), \bar{q}_4(t)]\begin{bmatrix} \cos(pt/r^2), & \sin(pt/r^2) \\ -\sin(pt/r^2), & \cos(pt/r^2) \end{bmatrix},$$

then we have

$$\begin{cases} q_3'(t) = (p^2-1)/r^2 \sin\rho\cos\rho\cos(pt/r^2)q_2(t)\,, \\ q_4'(t) = (p^2-1)/r^2 \sin\rho\cos\rho\sin(pt/r^2)q_2(t)\,. \end{cases}$$

Therefore, $\|\boldsymbol{\lambda}\| = (p^2-1)/r^2 \sin\rho\cos\rho$ and

$$[q_3(2\pi r), q_4(2\pi r)] = [q_3(0), q_4(0)]\begin{bmatrix} \cos(2\pi p/r), & \sin(2\pi p/r) \\ -\sin(2\pi p/r), & \cos(2\pi p/r) \end{bmatrix}.$$

For $u = \sqrt{(p^2-1)^2\sin^2\rho\cos^2\rho + p^2}$ and φ with $u\sin\varphi = (p^2-1)\sin\rho\cos\rho$ and $u\cos\varphi = p$, we define

$$\boldsymbol{p}_2(t) = \cos\varphi\begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} + \sin\varphi\left\{\sin(ut/r^2)\begin{bmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{bmatrix} + \cos(ut/r^2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\},$$

$$\bar{\boldsymbol{p}}_3(t) = -\cos(ut/r^2)\begin{bmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{bmatrix} + \sin(ut/r^2)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

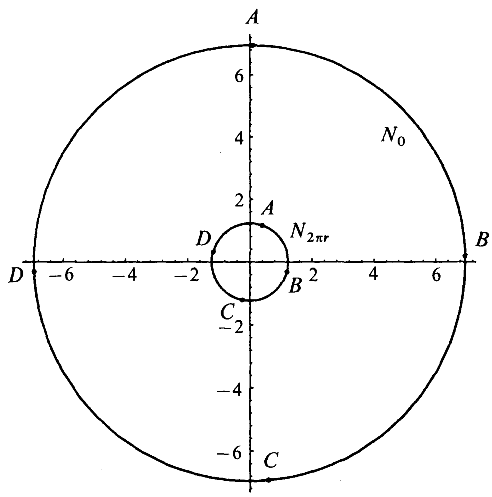

FIGURE 1.

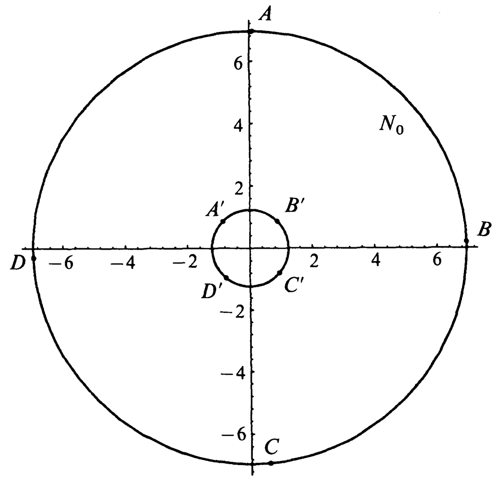

FIGURE 2.

$$\bar{\boldsymbol{p}}_4(t) = -\sin\varphi\begin{bmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{bmatrix} + \cos\varphi\left\{\sin(ut/r^2)\begin{bmatrix}\sin\varphi\\ -\cos\varphi\\ 0\end{bmatrix} + \cos(ut/r^2)\begin{bmatrix}0\\0\\1\end{bmatrix}\right\},$$

for $0 \le t \le 2\pi r$. Then $\bar{\boldsymbol{p}}_3(t)$ and $\bar{\boldsymbol{p}}_4(t)$ satisfy

$$\bar{\boldsymbol{p}}_3'(t) = (p^2-1)/r^2 \sin\rho\cos\rho\boldsymbol{p}_2(t) + (p/r^2)\bar{\boldsymbol{p}}_4(t)\,, \quad \bar{\boldsymbol{p}}_4' = -(p/r^2)\bar{\boldsymbol{p}}_3(t)\,.$$

We define

$$[\boldsymbol{p}_3(t), \boldsymbol{p}_4(t)] = [\bar{\boldsymbol{p}}_3(t), \bar{\boldsymbol{p}}_4(t)]\begin{bmatrix}\cos(pt/r^2), & \sin(pt/r^2)\\ -\sin(pt/r^2), & \cos(pt/r^2)\end{bmatrix}.$$

Then we have

$$\begin{cases}\boldsymbol{p}_3'(t) = (p^2-1)/r^2 \sin\rho\cos\rho\cos(pt/r^2)\boldsymbol{p}_2(t)\,,\\ \boldsymbol{p}_4'(t) = (p^2-1)/r^2 \sin\rho\cos\rho\sin(pt/r^2)\boldsymbol{p}_2(t)\,.\end{cases}$$

We note that $\{\boldsymbol{p}_2(t), \bar{\boldsymbol{p}}_3(t), \bar{\boldsymbol{p}}_4(t)\}$ is a Frenet frame (cf. Ex. 3). q.e.d.

EXAMPLE 3. Let us take $p=2$ and $\rho=\pi/12$ in Example 2. Then we have $\|\boldsymbol{\lambda}\| < 4/5$. We put $a(t) = \{5\sin(\sqrt{41}t/5) - 4\}/\sqrt{82}$, $b(t) = \cos(\sqrt{41}t/5)/\sqrt{2}$ and $c(t) = \{4\sin(\sqrt{41}t/5) + 5\}/\sqrt{82}$. The hypersurface $\Phi(S^{n-1} \times [0, 2\pi r])$ constructed from $\{q_1, q_2, q_3, q_4, a, b, c\}$ does not close at $t=0$ and $t=2\pi r$, but we can study the rotation τ determined from the correspondence between N_0 and $N_{2\pi r}$ given by (4.6). In Figure 1, we illustrate this correspondence with N_0 and $N_{2\pi r}$. Note that $\sum_{i=3}^4 (\eta^{-1}y)^i \boldsymbol{p}_i(2\pi r) = \sum_{i=3}^4 y^i \bar{\boldsymbol{p}}_i(2\pi r)$. Figure 2 also shows the correspondence with N_0 and $N_{2\pi r}$ when we replace by y^i the coefficients $(\eta^{-1}y)^i$ of $\boldsymbol{p}_i(2\pi r)$ $(i=3, 4)$ in (4.6). The circles N_0 and $N_{2\pi r}$ in Figures 1 and 2 have the same center. This is obtained as follows: First we map the surface in S^2 into $\boldsymbol{R}^2$ by stereographic projection from the north pole. Next we map the origin of circle $N_{2\pi r}$ on the one of N_0 by a linear fractional transformation preserving N_0. But these correspondences of N_0 to $N_{2\pi r}$ are not exact rotations, because these include an action of a linear fractional transformation without a rotation. q.e.d.

5. Other conformally flat hypersurfaces. For $0<\alpha<1$ we fix an n-sphere $S_\alpha: \alpha e_1+\beta(\sum_{i=2}^{n+2} x^i e_i)$ in S^{n+1}, where $\alpha^2+\beta^2=1$ and $\sum_{i=2}^{n+2}(x^i)^2=1$. Let D_α be an open round disk in S^{n+1} which includes e_1 and has S_α as the boundary. Let $\bar{\Phi}$ be a hypersurface in S^{n+1} diffeomorphic to $S^{n-1}\times[0, 1]$. Namely, we define $\bar{\Phi}(y, t)=\bar{a}(t)e_1+\bar{b}(t)e_2+\bar{c}(t)(\sum_{i=3}^{n+2} y^i e_i)$, where $t\in[0, 1]$, $\sum_{i=3}^{n+2}(y^i)^2=1$, $(\bar{a}^2+\bar{b}^2+\bar{c}^2)(t)\equiv 1$ and $\bar{c}(t)>0$. Since $\|\bar{\Phi}(y, t)-\alpha e_1\|^2\geq\beta^2$ implies $\bar{a}(t)\leq\alpha$, we have $\bar{\Phi}(S^{n-1}\times\{t\})\subset D_\alpha$ if $\bar{a}(t)>\alpha$, $\bar{\Phi}(S^{n-1}\times\{t\})\subset S^{n+1}\setminus\bar{D}_\alpha$ if $\bar{a}(t)<\alpha$ and $S_\alpha\cap\bar{\Phi}=\{\bar{\Phi}(S^{n-1}\times\{t\})\,|\,\bar{a}(t)=\alpha\}$.

First, we construct a conformally flat torus in S^{n+1} with umbilic points. We can obtain them by slightly deforming hypersurfaces $\Phi((S^{n-1}\times[0, l_1+l_2])/\eta)$ given in Theorem 1 at the part $\Phi(S^{n-1}\times[0, l_1])$. In the definition of the above $\bar{\Phi}$, we replace the domain $[0, 1]$ of $\bar{\Phi}$ by $[0, l_1]$, and represent the curve $\bar{a}(t)e_1+\bar{b}(t)e_2$ by using the geodesic $q_1(t)=\cos(t-l_1/2)e_1+\sin(t-l_1/2)\, e_2$ and the velocity vector field $q_2(t)$. Then there exists a conformally flat hypersurface Φ given in Theorem 1 such that $\bar{\Phi}(y, t)=\Phi(y, t)$ for $t\in[0, l_1]$. In particular, we take the curve $\tau(t)=(aq_1+bq_2)(t)$ with $\langle\tau(l_2/2), e_1\rangle>\alpha$, $\langle\tau(0), e_1\rangle<\alpha$ and $\langle\tau(l_1), e_1\rangle<\alpha$. Then there exist $t_1\in(0, l_1/2)$ and $t_2\in(l_1/2, l_1)$ such that $\Phi(S^{n-1}\times\{t_i\})\cap S_\alpha=\Phi(S^{n-1}\times\{t_i\})$ for $i=1, 2$. Furthermore, we assume $\Phi(S^{n-1}\times\{t_1\})\cap\Phi(S^{n-1}\times\{t_2\})=\varnothing$. We denote by S'_α the domain of S_α sandwiched between $\Phi(S^{n-1}\times\{t_1\})$ and $\Phi(S^{n-1}\times\{t_2\})$. We replace $\Phi((S^{n-1}\times[0, l_1+l_2])/\eta)$ by

$$\Phi(S^{n-1}\times[0, t_1])\cup S'_\alpha\cup\Phi(S^{n-1}\times[t_2, l_1+l_2])\,. \tag{5.1}$$

Furthermore we modify functions $a(t)$, $b(t)$ and $c(t)$ in neighborhoods of $t=t_1$ and t_2 so

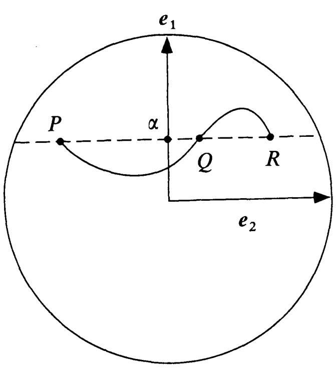

FIGURE 3.

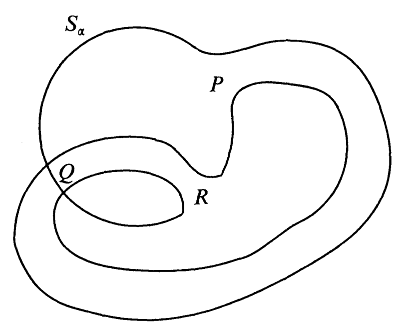

FIGURE 4.

that the hypersurface (5.1) becomes smooth. Thus we have a conformally flat torus in S^{n+1} with umbilic points.

Next we construct conformally flat hypersurfaces diffeomorphic to the Klein bottle $S^{n-1}\times_n S^1$ which always have umbilic points. Take a curve $\bar{\tau}(t)=\bar{a}(t)e_1+\bar{b}(t)e_2$, $0\leq t\leq 1$, as in Figure 3. We construct a hypersurface $\bar{\Phi}$ by inflating the curve $\bar{\tau}$, and connect S_α with $\bar{\Phi}$ as in Figure 4. Thus we have a conformally flat Klein bottle in S^{n+1}.

Naturally, the curve $\bar{\tau}$ can be swung between P and Q in the same way as in Theorem 1. Thus we obtain conformally flat Klein bottles in S^{n+1} corresponding to elements in a neighborhood of $[A]\subset O(n)$, where $A=(a_{ij})$ with $a_{11}=-1$ and $a_{ij}=\delta_{ij}$ if $(i,j)\neq(1,1)$.

Finally, let M_1 and M_2 be two conformally flat tori in S^{n+1} both of which have umbilic points. We assume that a local minimum of the distance between two tori is attained at umbilic points of each torus. Take a minimal geodesic in S^{n+1} connecting the points, and construct a connected sum of two tori by inflating the geodesic. This gives a conformally flat hypersurface diffeomorphic to $(S^{n-1}\times S^1)\sharp(S^{n-1}\times S^1)$.

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